CS 517: Problem Set 2

1. Prove that the following problem is NP-complete. given a set of linear inequalities over a set of variables $x_1, \ldots, x_n$, determine whether there is an assignment of integers to the x_i's that satisfies all inequalities.

   Example:

   $$x_1 + 3x_2 - x_5 \geq 3$$
   $$2x_2 - x_4 \leq x_3$$
   $$x_1 + x_2 + x_3 + x_4 + x_5 \geq 0$$

   is satisfied by $x_1 = 10$; $x_2 = x_3 = x_4 = x_5 = 0$ (among many others).

2. (a) A 3-coloring of a graph is a way to color each vertex red, green, or blue so that no edge in the graph has endpoints with the same color. In class we showed that determining whether a graph is 3-colorable is NP-complete.

   Suppose you have a subroutine that tells you whether the given graph has a 3-coloring (i.e., it responds YES or NO). Show how to use the subroutine to compute a 3-coloring of any 3-colorable graph (i.e., output an actual assignment of colors to the vertices).

   (b) Let $L = \{x \mid \exists w : M(x, w) = 1\}$ be an arbitrary NP language, where M runs in polynomial time. Show that if P = NP, then there is a polynomial-time function f with the following property:

   $$\text{if } x \in L \text{ then } M(x, f(x)) = 1$$

   In other words, if P = NP then it is possible to not only decide NP problems in polynomial time, but also compute witnesses in polynomial time.

   Hint: First show that *"is there an L-witness for x that starts with this particular prefix?"* is in NP, if $L \in$ NP.

3. Consider the following puzzle game. The game board is a grid of $n \times m$ squares. Some of the squares initially contain a token which can be either red or blue. To win the game, you must remove a subset of the tokens so that:

   1. Every row has at least one token remaining.
   2. No column has tokens of different colors remaining.

   Prove that it is NP-complete to decide whether such a puzzle can be solved.

   Examples: