CS 517: Problem Set 2

1. Prove that the following problem is NP-complete. Given a set of linear inequalities over a set of variables \(x_1, \ldots, x_n\), determine whether there is an assignment of integers to the \(x_i\)'s that satisfies all inequalities.

   Example:

   \[
   \begin{align*}
   x_1 + 3x_2 - x_5 & \geq 3 \\
   2x_2 - x_4 & \leq x_3 \\
   x_1 + x_2 + x_3 + x_4 + x_5 & \geq 0
   \end{align*}
   \]

   is satisfied by \(x_1 = 10; x_2 = x_3 = x_4 = x_5 = 0\) (among many others).

2. (a) A 3-coloring of a graph is a way to color each vertex red, green, or blue so that no edge in the graph has endpoints with the same color. In class we showed that determining whether a graph is 3-colorable is NP-complete.

   Suppose you have a subroutine that tells you whether the given graph has a 3-coloring (i.e., it responds \(\text{yes or no}\)). Show how to use the subroutine to compute a 3-coloring of any 3-colorable graph (i.e., output an actual assignment of colors to the vertices).

   (b) Let \(L = \{x \mid \exists w : M(x, w) = 1\}\) be an arbitrary NP language, where \(M\) runs in polynomial time. Show that if \(P = NP\), then there is a polynomial-time function \(f\) with the following property:

   \[
   \text{if } x \in L \text{ then } M(x, f(x)) = 1
   \]

   In other words, if \(P = NP\) then it is possible to not only decide NP problems in polynomial time, but also compute witnesses in polynomial time.

   Hint: First show that “is there an \(L\)-witness for \(x\) that starts with this particular prefix?” is in NP, if \(L \in NP\).

3. Consider the following puzzle game. The game board is a grid of \(n \times m\) squares. Some of the squares initially contain a token which can be either red or blue. To win the game, you must remove a subset of the tokens so that:

   1. Every row has at least one token remaining.
   2. No column has tokens of different colors remaining.

   Prove that it is NP-complete to decide whether such a puzzle can be solved.

   Examples: