CS 517: Homework #7

Due Friday June 3, on TEACH, typeset in \LaTeX.

Please read the guidelines in the syllabus for homework submissions. In summary: justify the correctness and efficiency of your answers.

1. Let $M$ be a probabilistic TM, and let $\text{time}(M, x)$ denote the number of steps taken by $M$ on input $x$. Note that $\text{time}(M, x)$ is a random variable since $M$ is probabilistic.

A probabilistic TM $M$ runs in strict polynomial-time if there is a polynomial $p$ such that for all $x$, $\Pr[\text{time}(M, x) \leq p(|x|)] = 1$. I.e., $M$ always takes at most $p(|x|)$ steps.

A probabilistic TM $M$ runs in expected polynomial-time if there is a polynomial $p$ such that for all $x$, $\mathbb{E}[\text{time}(M, x)] \leq p(|x|)$.

The standard complexity classes PP, BPP, etc., are defined in terms of strict polynomial-time.

Define $\text{ePP}$ to be the same as PP, except we allow the Turing machine to be expected-polynomial-time. That is, $L \in \text{ePP}$ if there exists a an expected-poly-time probabilistic TM $M$ with $L = \{x \mid \Pr[M(x) = 1] > 1/2\}$.

Show that $\text{EXP} \subseteq \text{ePP}$. In fact, it is possible to show that $\text{EXP}$ can be decided (in the PP sense) by expected-constant-time probabilistic TMs.

Hint: If one of the threads does an exponentially long computation, then there should be a bunch more very short threads to bring the average back down to polynomial.

2. Let $L$ be a problem in $\text{RP}$, so that:

\[
\begin{align*}
 x \in L & \Rightarrow \Pr[M(x) = 1] \geq 1/2 \\
 x \notin L & \Rightarrow \Pr[M(x) = 0] = 1
\end{align*}
\]

Show that for each $n$, there exists a set $R$ of random tapes for $M$ with $|R| = n$ and satisfying the following property:

\[
x \in L \iff \exists r \in R : M(x; r) = 1
\]

Show how this gives an alternate proof that $\text{RP} \subseteq \text{P/poly}$.

Hint: Show that there exists a single $r$ that works for at least half of the length-$n$ inputs, then repeat.