Towards Optimal Garbled Circuit Constructions

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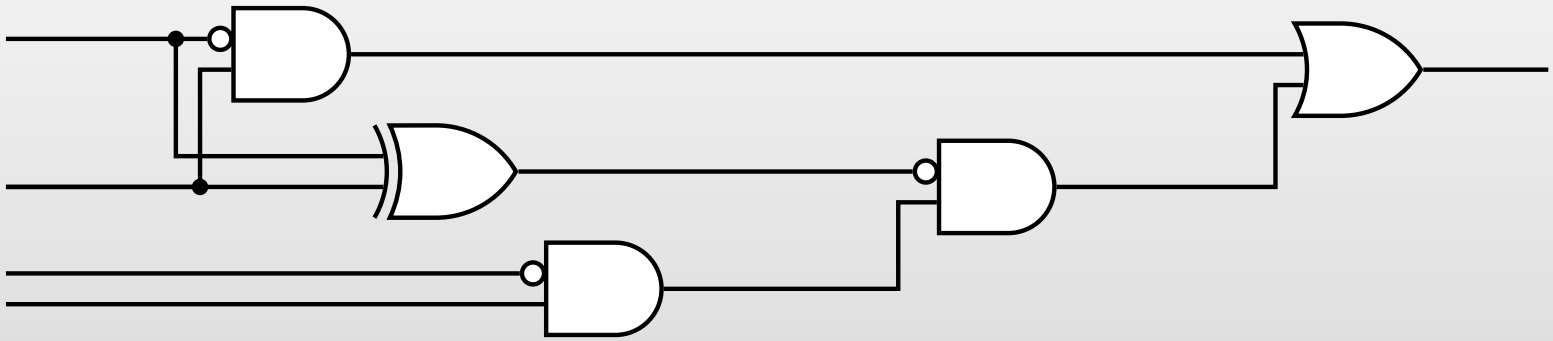
Samee Zahur, Mike Rosulek, David Evans: *Two Halves Make a Whole: Reducing Data Transfer in Garbled Circuits using Half Gates.* Eurocrypt 2015, ia.cr/2014/756

Garbled circuit framework [Yao86]



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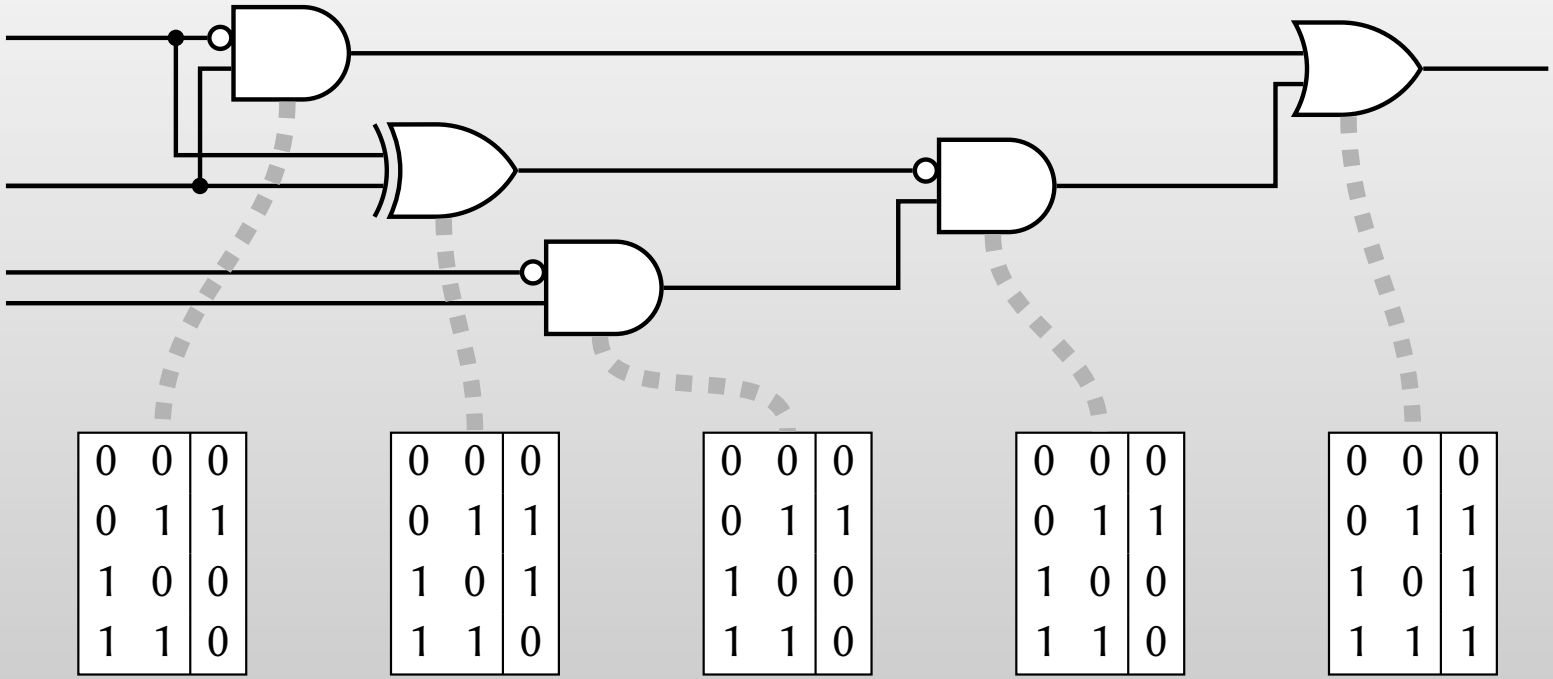

0	0	0
0	1	1
1	0	0
1	1	0

0	0	0
0	1	1
1	0	1
1	1	0

0	0	0
0	1	1
1	0	0
1	1	0

0	0	0
0	1	1
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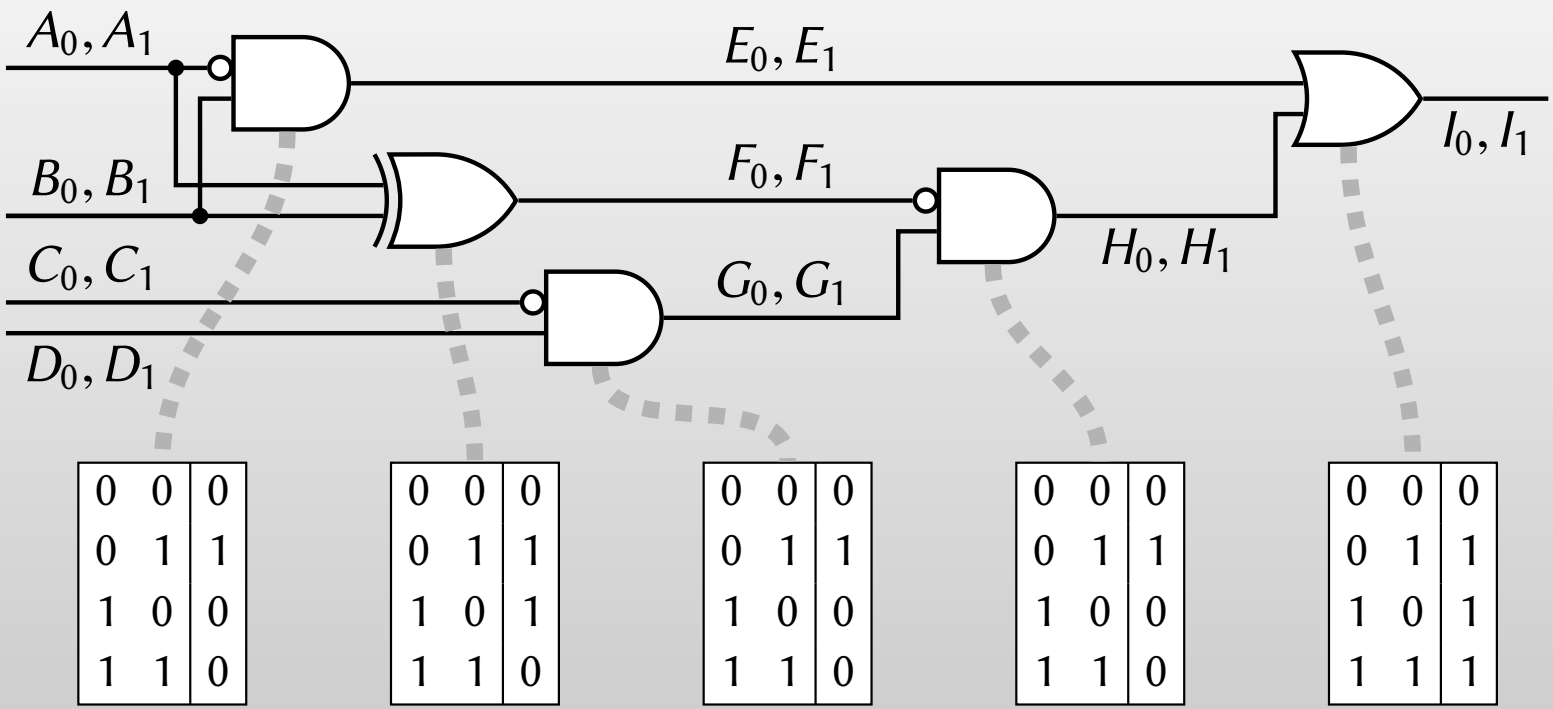

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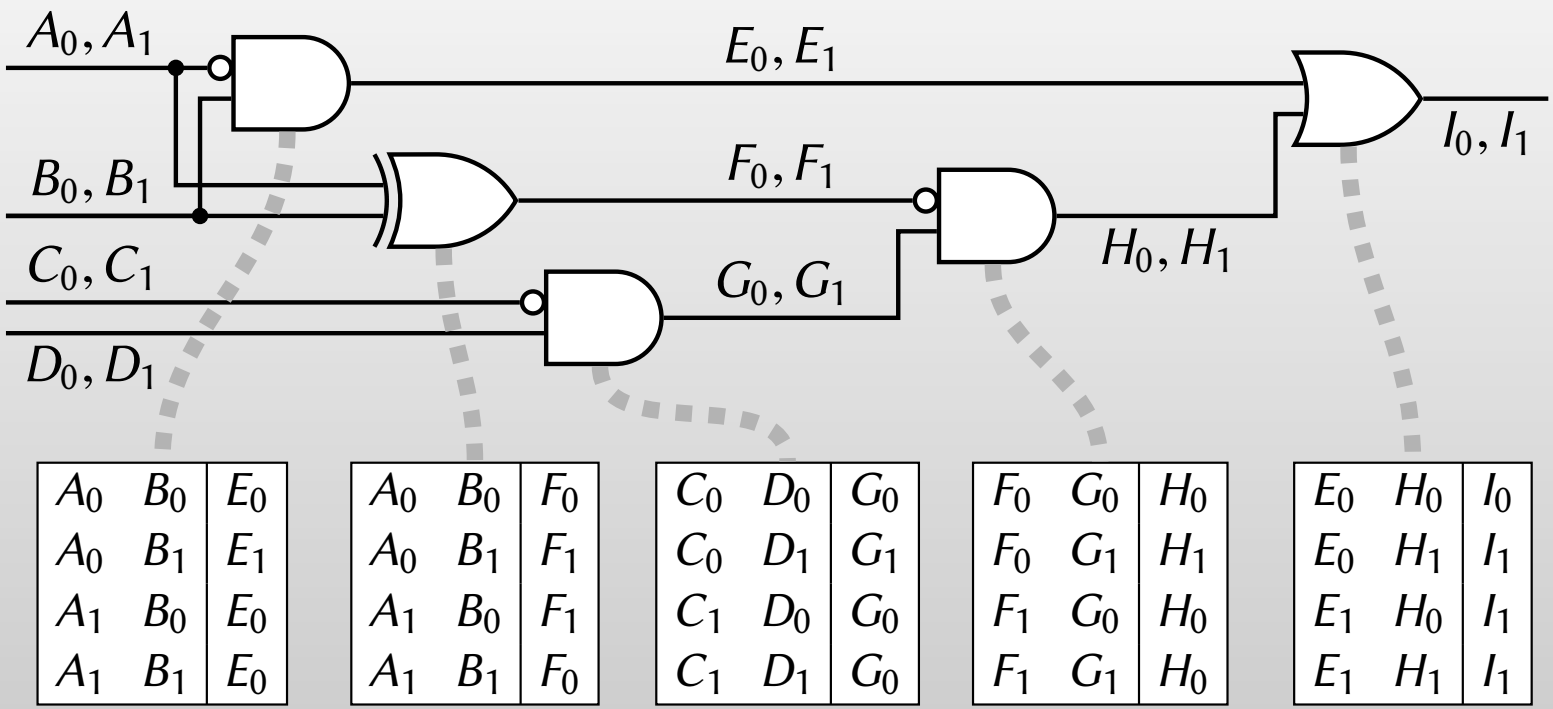

A_0	B_0	E_0
A_0	B_1	E_1
A_1	B_0	E_0
A_1	B_1	E_0

A_0	B_0	F_0
A_0	B_1	F_1
A_1	B_0	F_1
A_1	B_1	F_0

C_0	D_0	G_0
C_0	D_1	G_1
C_1	D_0	G_0
C_1	D_1	G_0

F_0	G_0	H_0
F_0	G_1	H_1
F_1	G_0	H_0
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E_0	H_0	I_0
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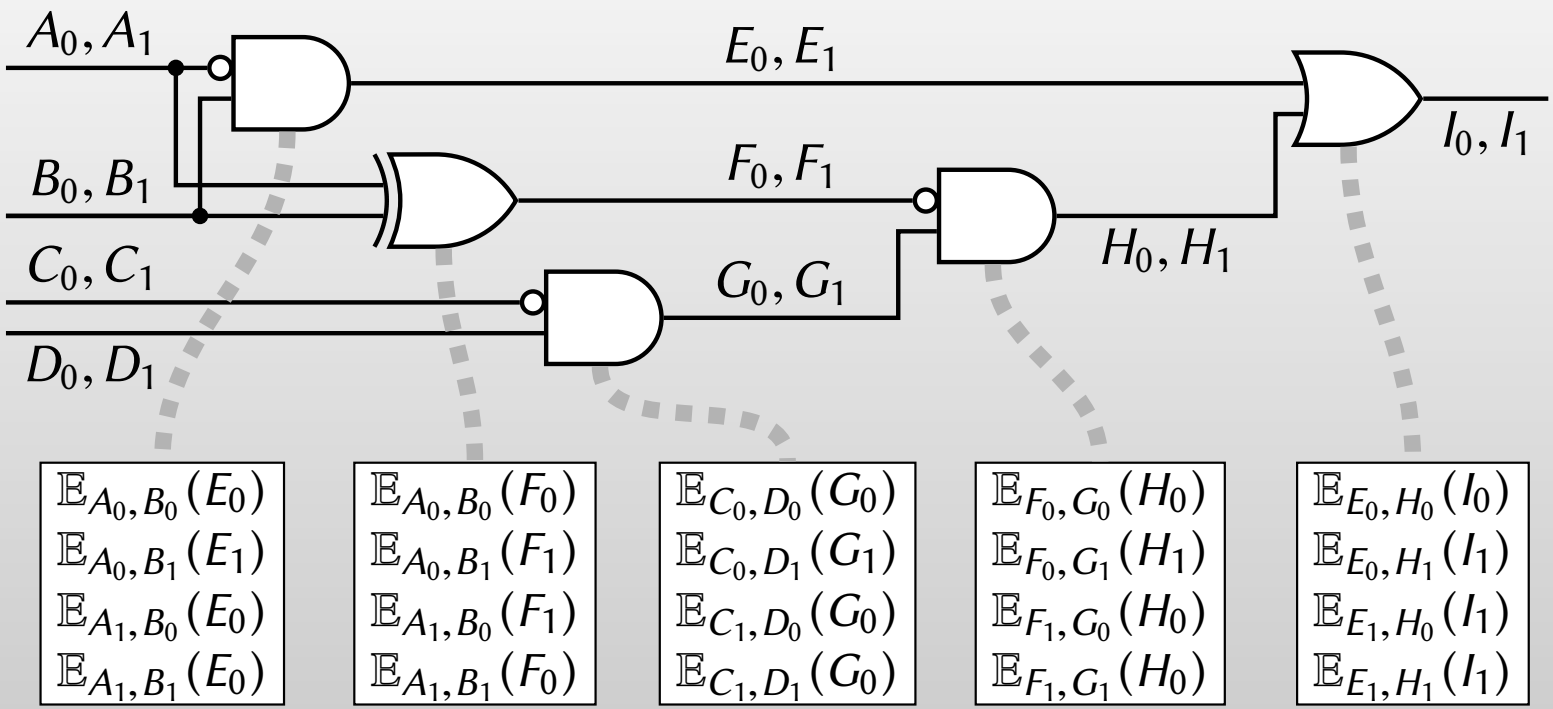

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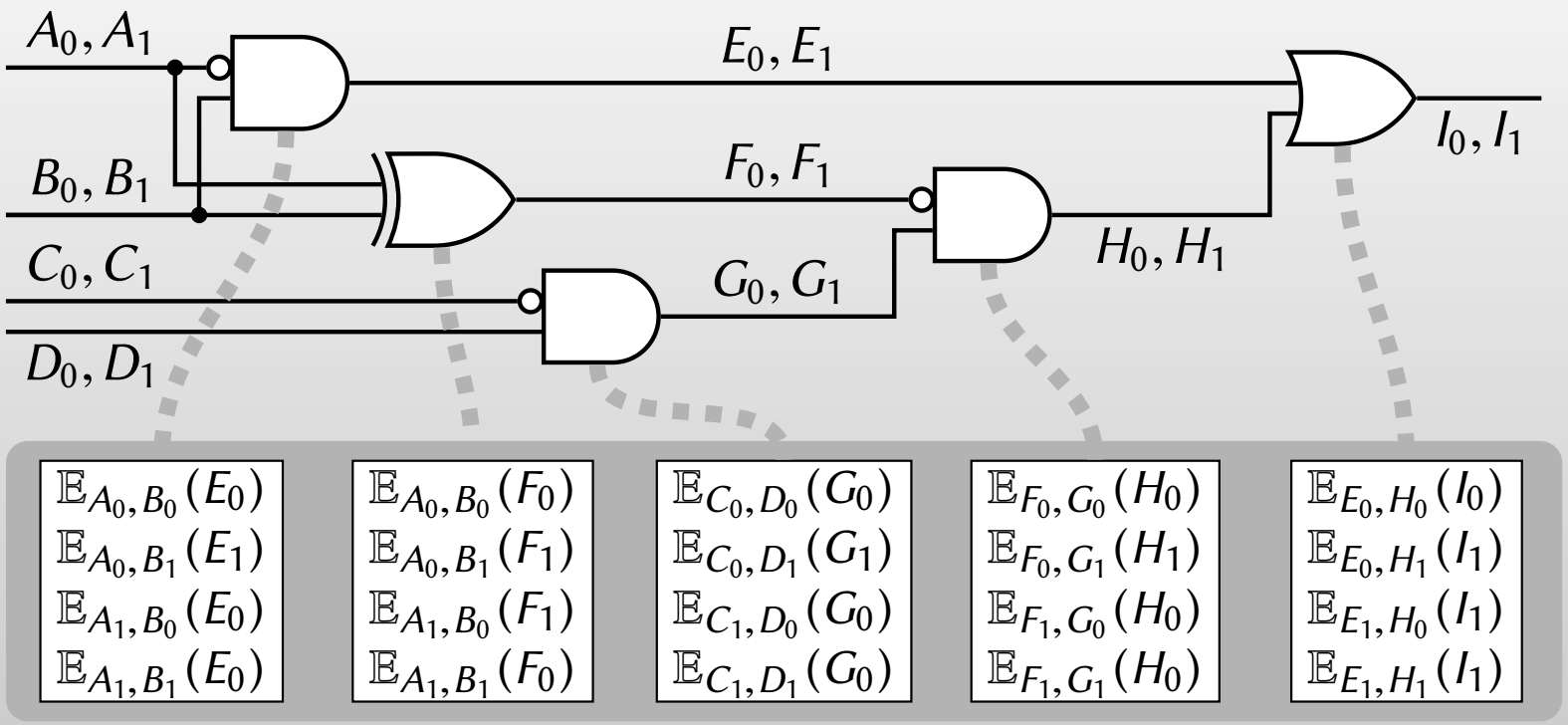

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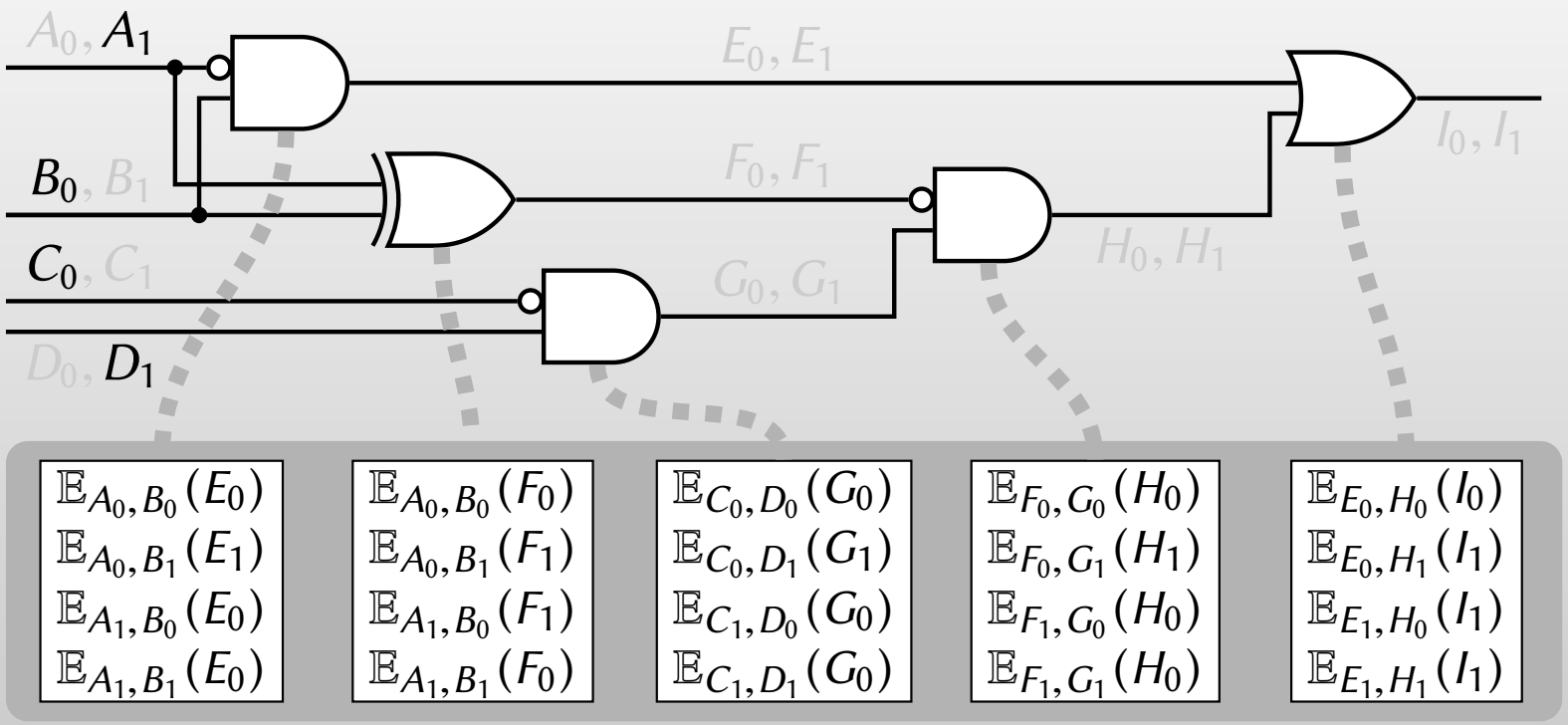

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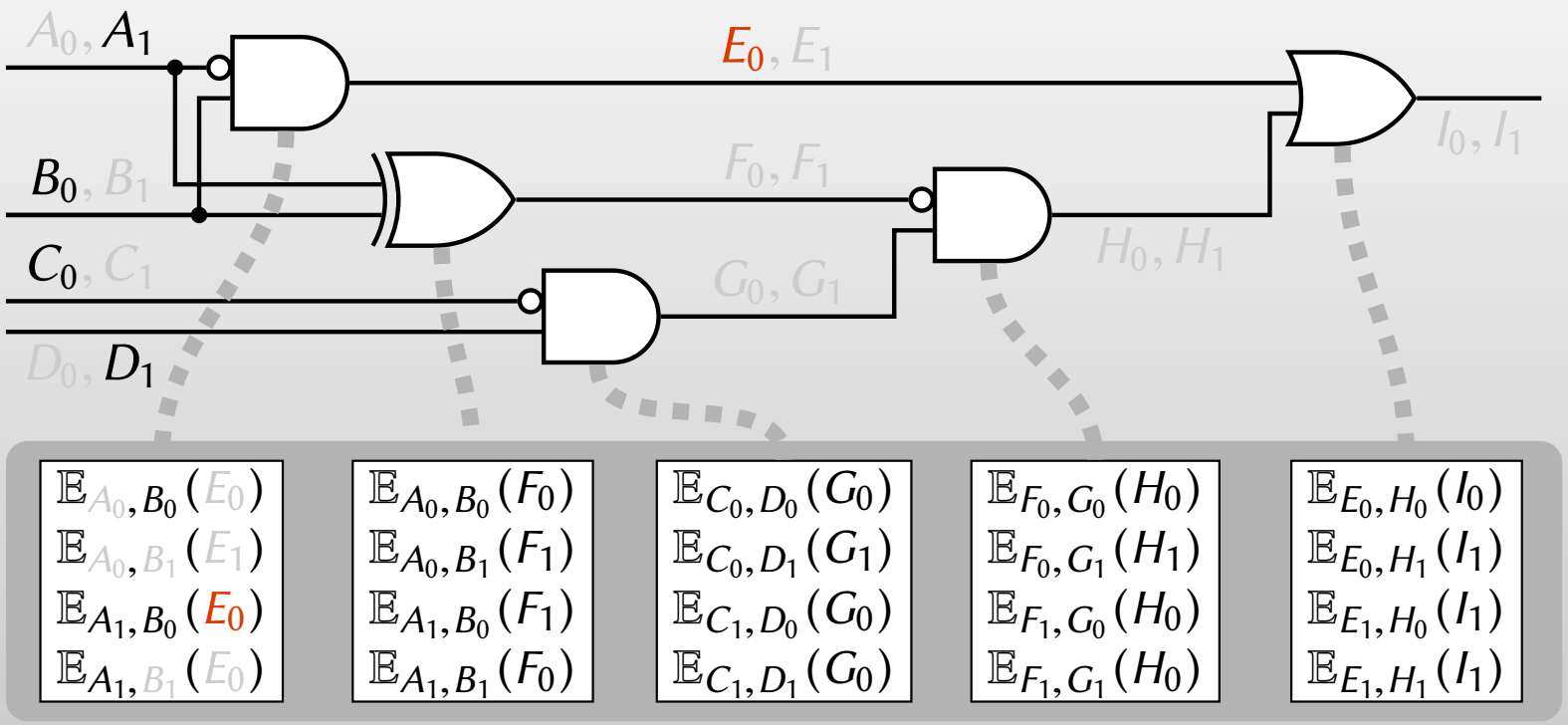

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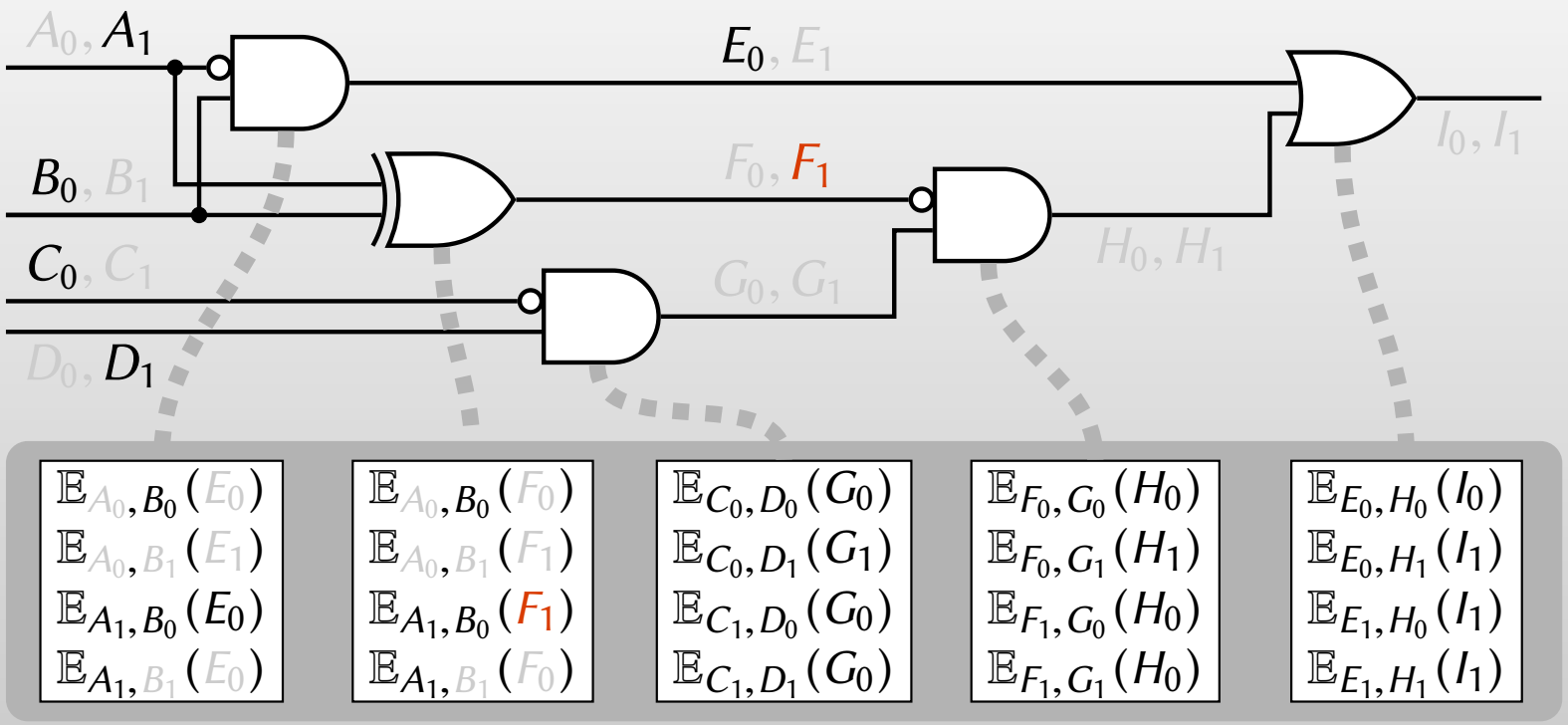

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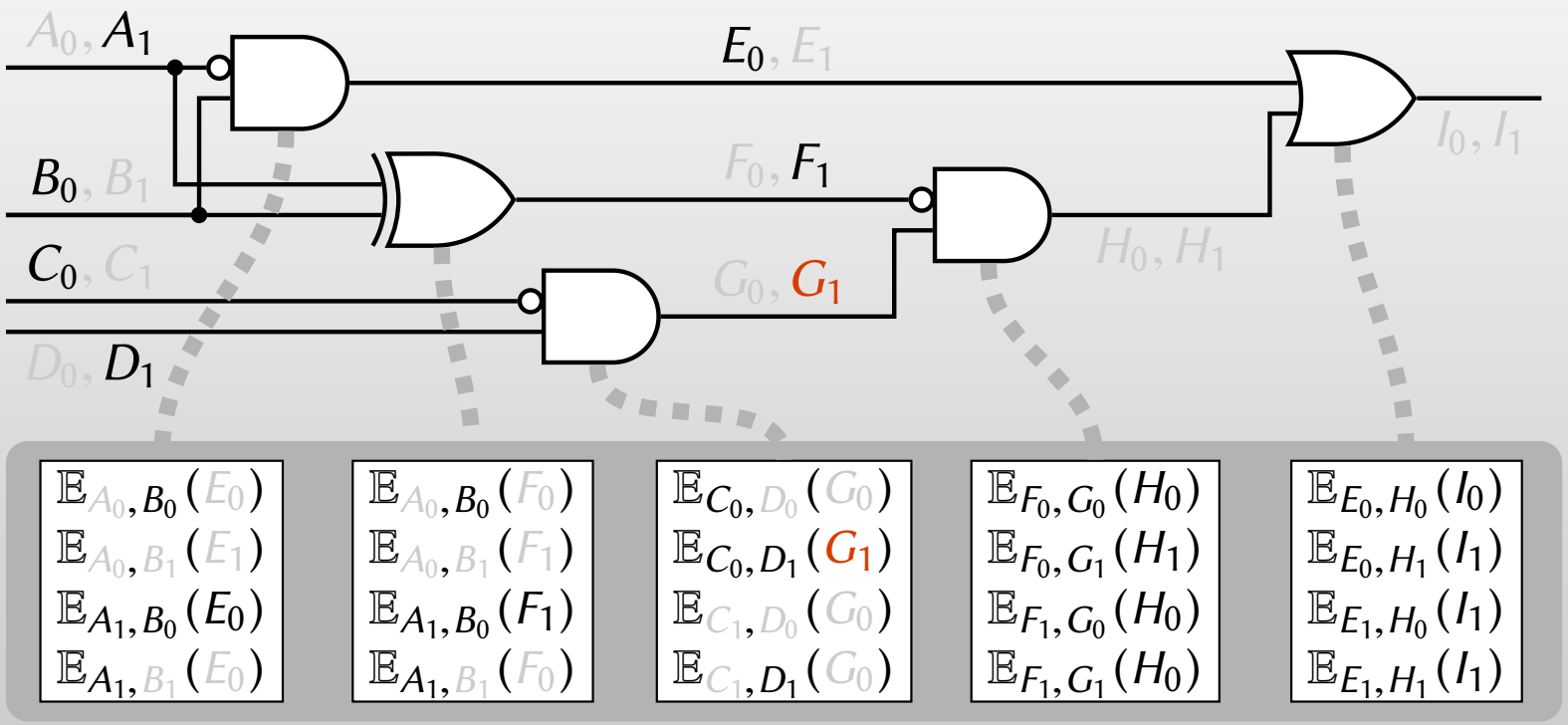

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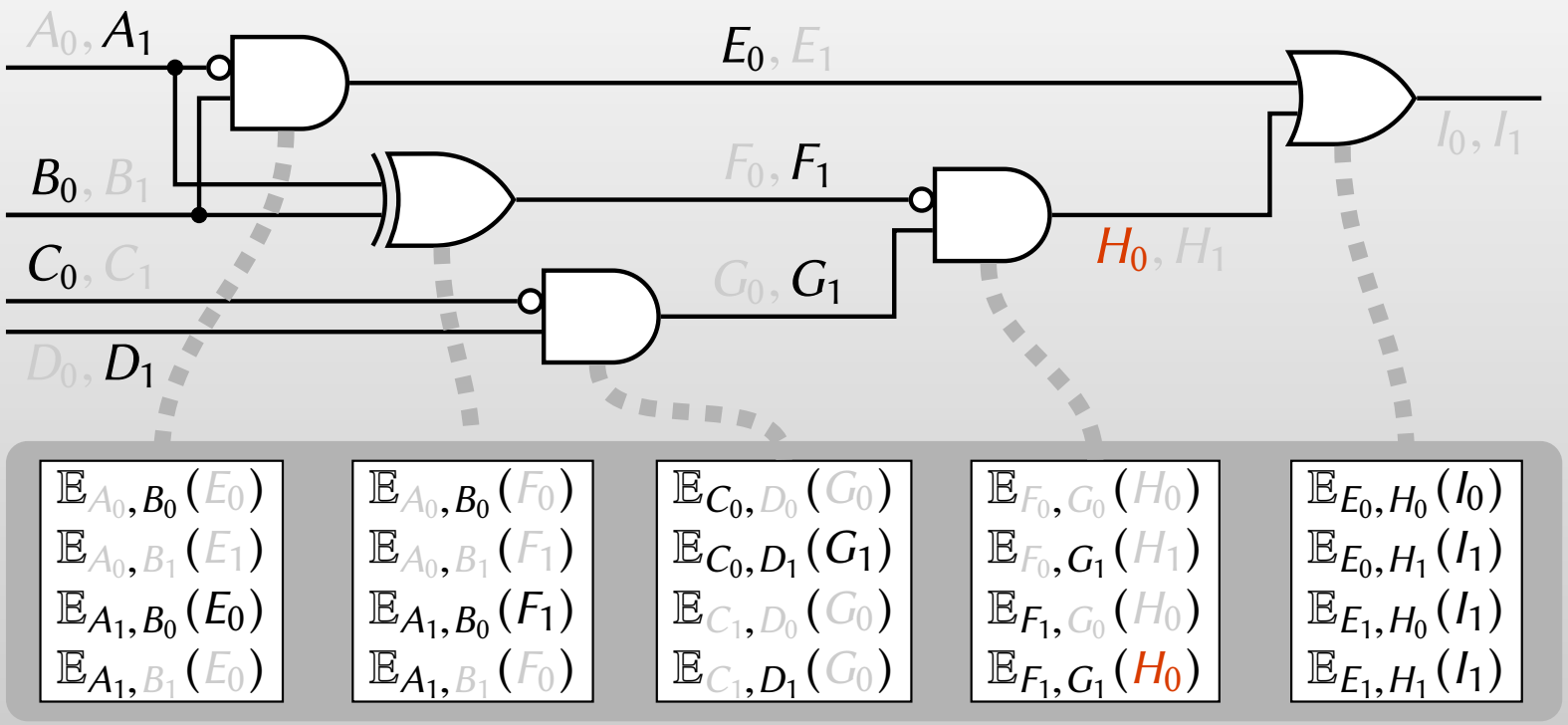

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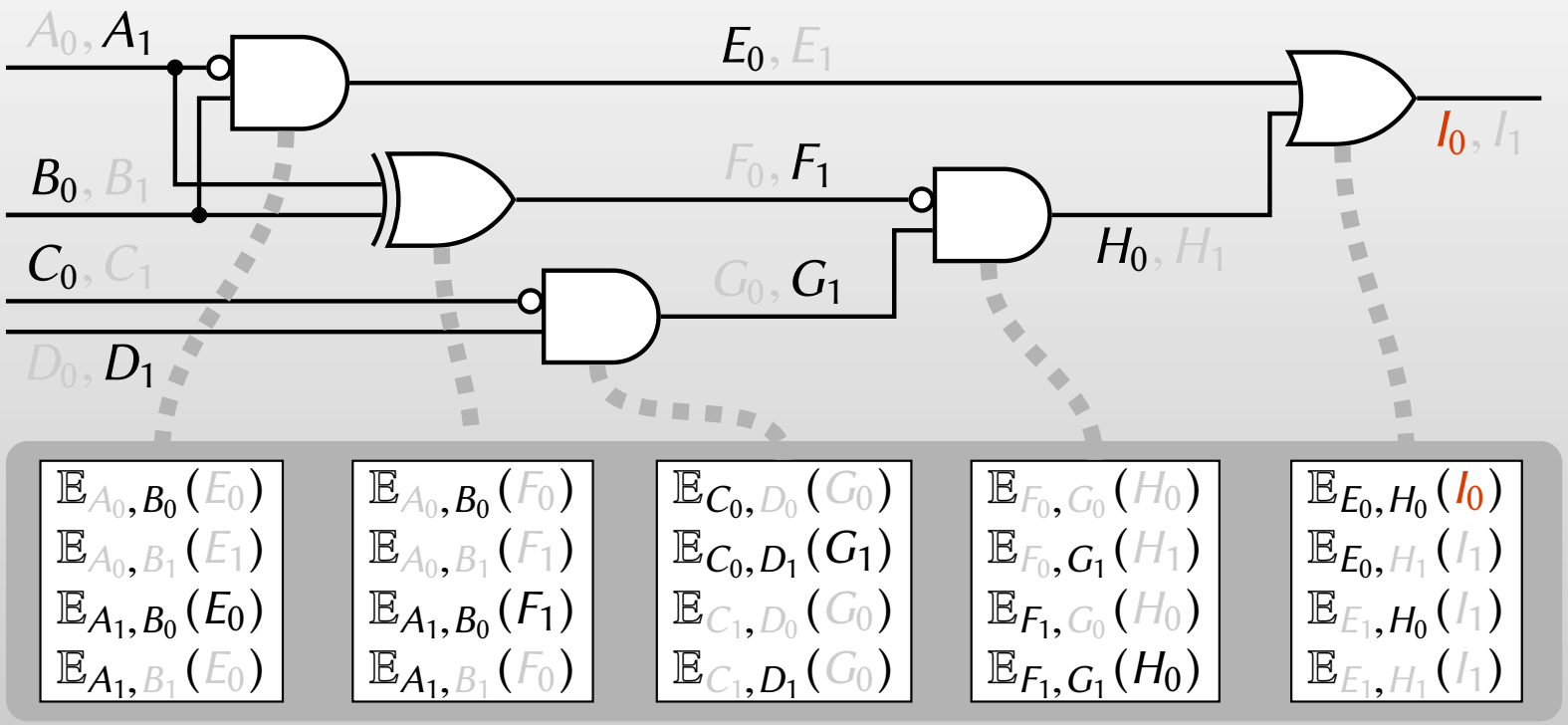

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Applications

secure computation

zero-knowledge proofs

private function evaluation

randomized encodings

secure outsourcing

one-time programs

key-dependent security for encryption

⋮

How small can garbled circuits get?

Garbled circuit size

		size per gate ($\times\lambda$) XOR	AND
Classical	[Yao86,GMW87]	?	?
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GRR2	[PinkasSchneiderSmartWilliams09]	2	2

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FleXOR	[KolesnikovMohasselRosulek14]	$\{0, 1, 2\}$	2
HalfGates	[ZahurRosulekEvans15]	**0**	**2**

Is there a lower bound?

Lower bounds for garbled circuits

Pitfalls:

Want to resolve **optimal constants**

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Theorem ([ZahurEvansRosulek15])

*A **linear gate garbling scheme** requires 2λ bits to garble a single AND gate. Within this model, "half-gates" construction is optimal.*

Rest of talk

1: Restricted model: What is a “linear gate garbling scheme?”

2: The lower bound for AND gates (sketch)

3: Looking forward

“Known, practical techniques”

Symmetric-key cryptography only

- Formalize via computationally unbounded adversaries + random oracle [ImpagliazzoRudich88]
- e.g.: encrypt garbled truth table as $\mathbb{E}_{A,B}(C) = H(A\|B) \oplus C$

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Linear operations exclusively

- Wire labels, garbled gates, RO outputs are field/ring elements (e.g.: $GF(2^\lambda)$)
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One caveat: state of art schemes are *all non-linear* in one specific way . . .

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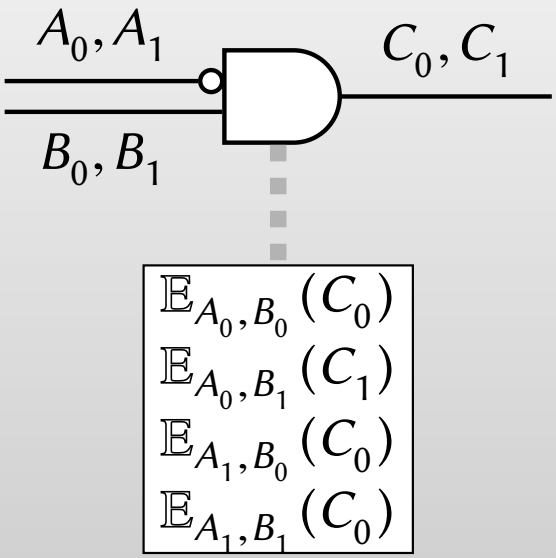

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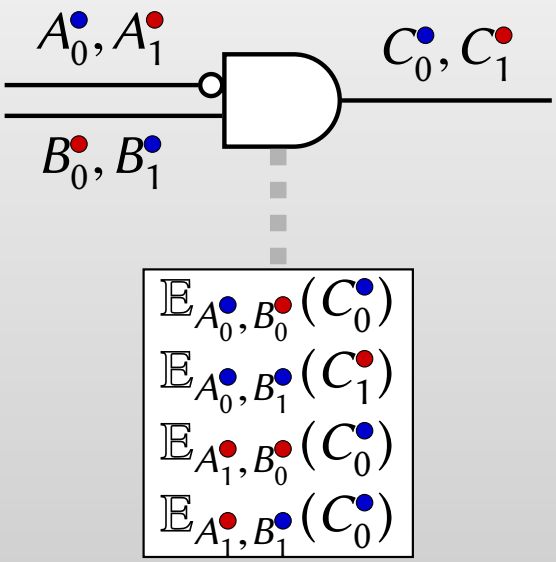

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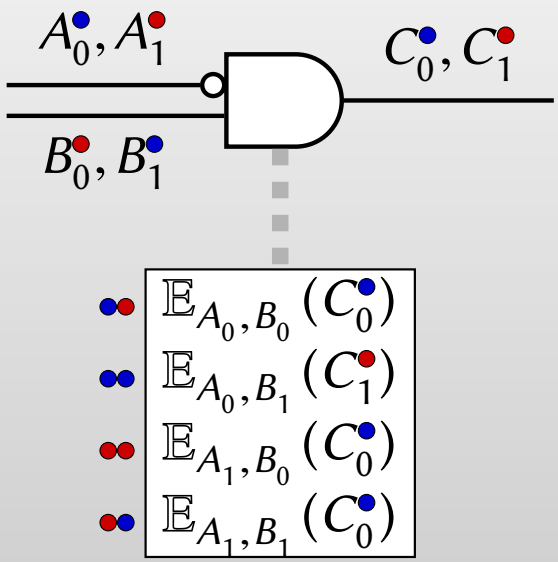

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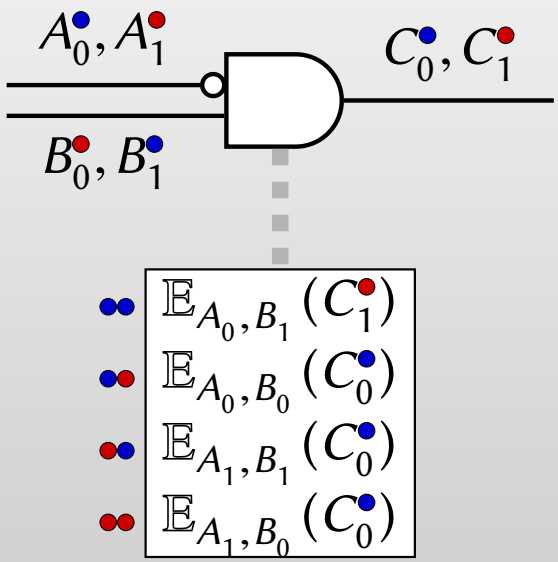

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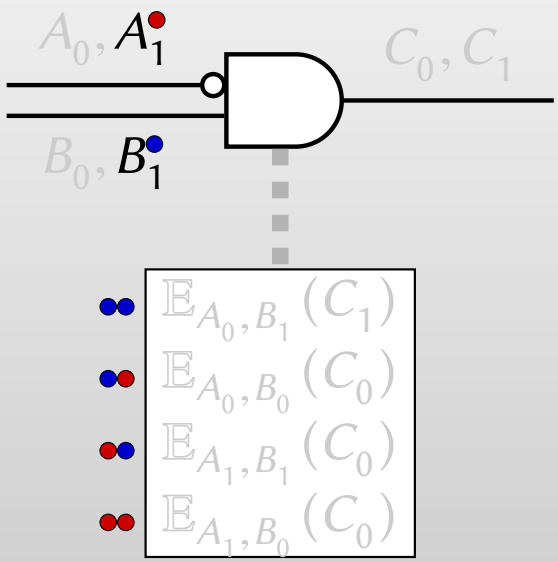

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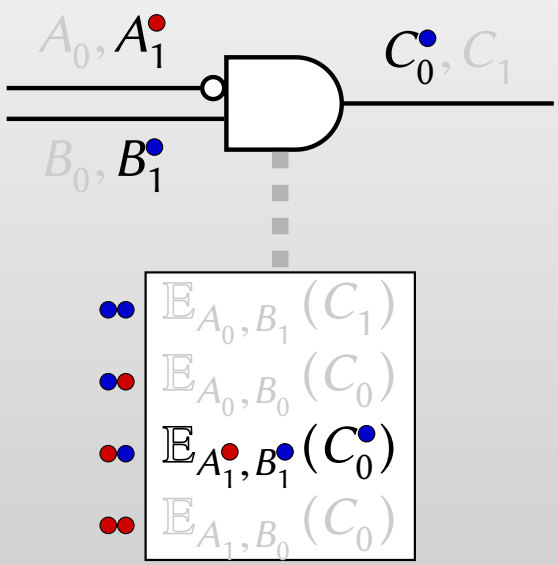

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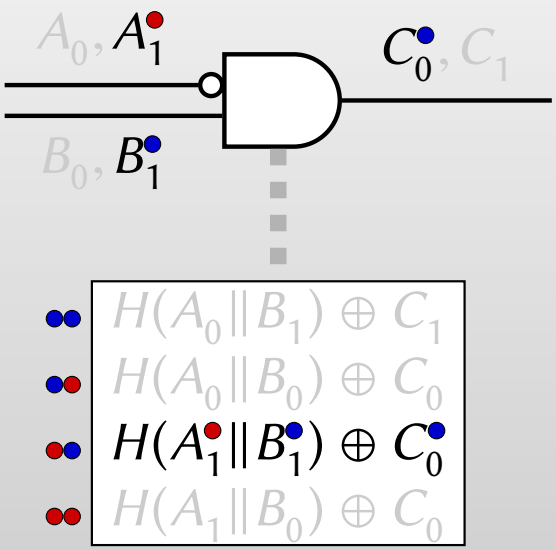

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$\Rightarrow$ smaller garbled circuit, encryption/decryption "linear"

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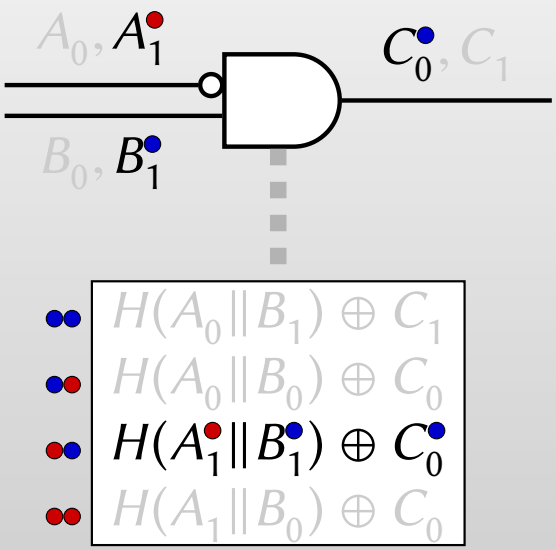

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⇒ smaller garbled circuit, encryption/decryption "linear"

✗ **Non-linear:** evaluator's choice of linear operation depends on color

✗ Optimized GC schemes crucially take advantage of nonlinearity!

Linear (gate) garbling: details

Apart from point-permute, and calls to random oracle, everything is linear.

Important: only the constructions, not adversary, must be linear!

Linear (gate) garbling: details

Garbler:

- samples field elements

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$$\left\langle \begin{array}{l} A^{\bullet} = \text{true} \\ A^{\bullet} = \text{false} \\ B^{\bullet} = \text{false} \\ B^{\bullet} = \text{true} \end{array} \right\rangle$$

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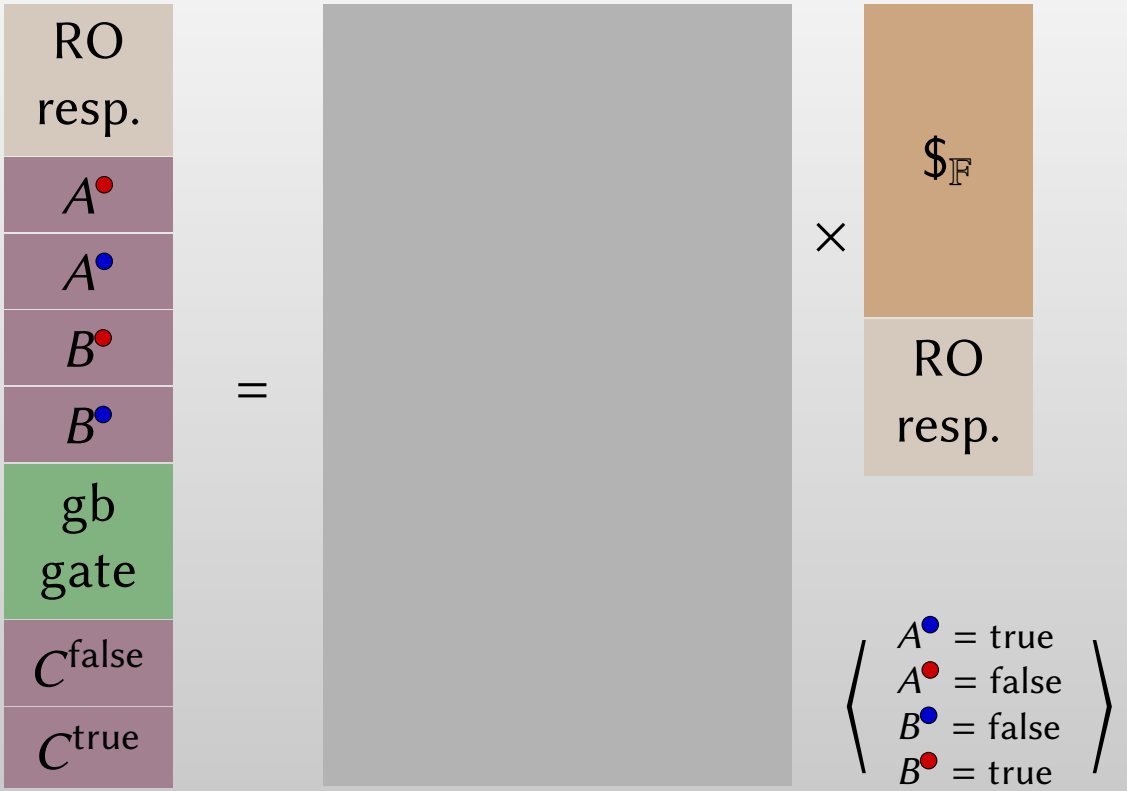

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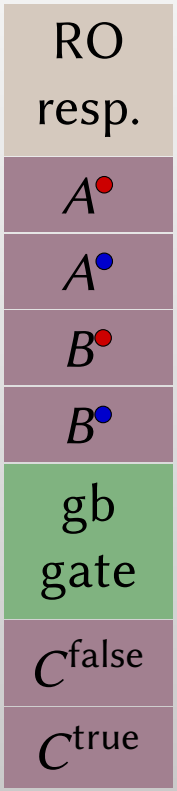

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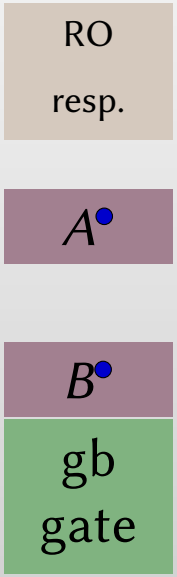

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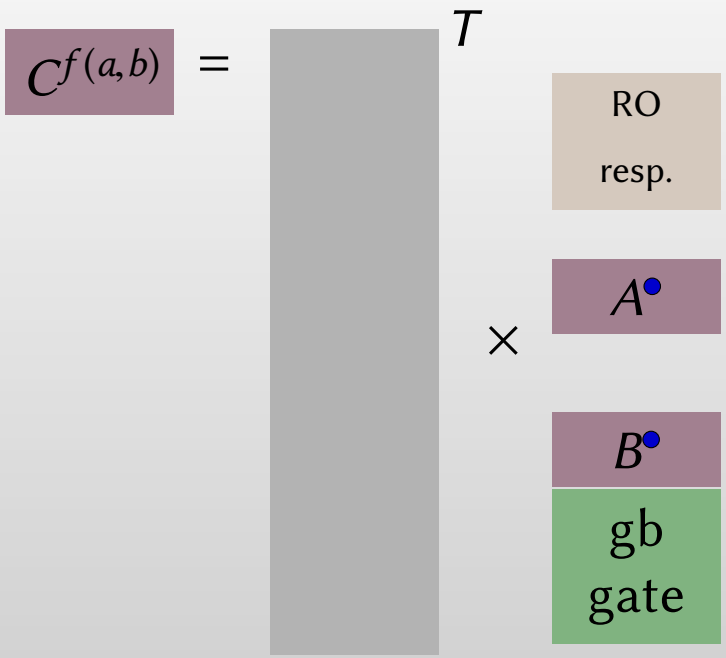

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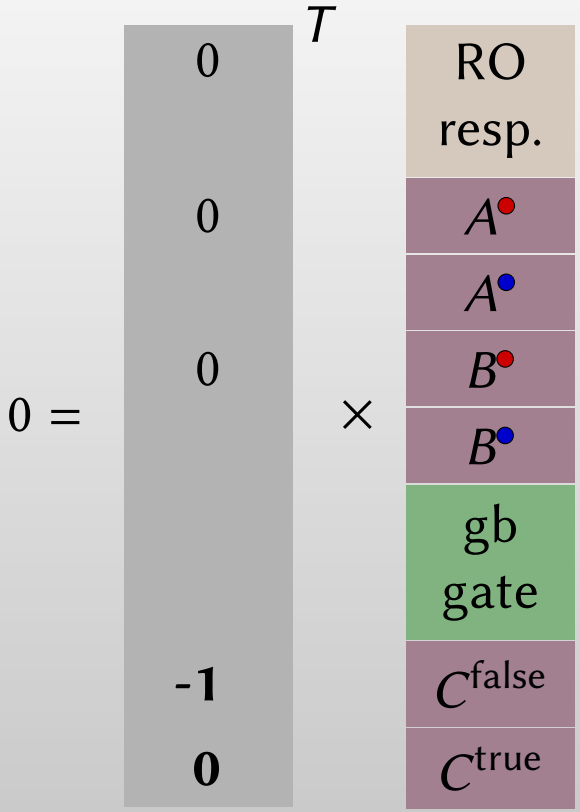

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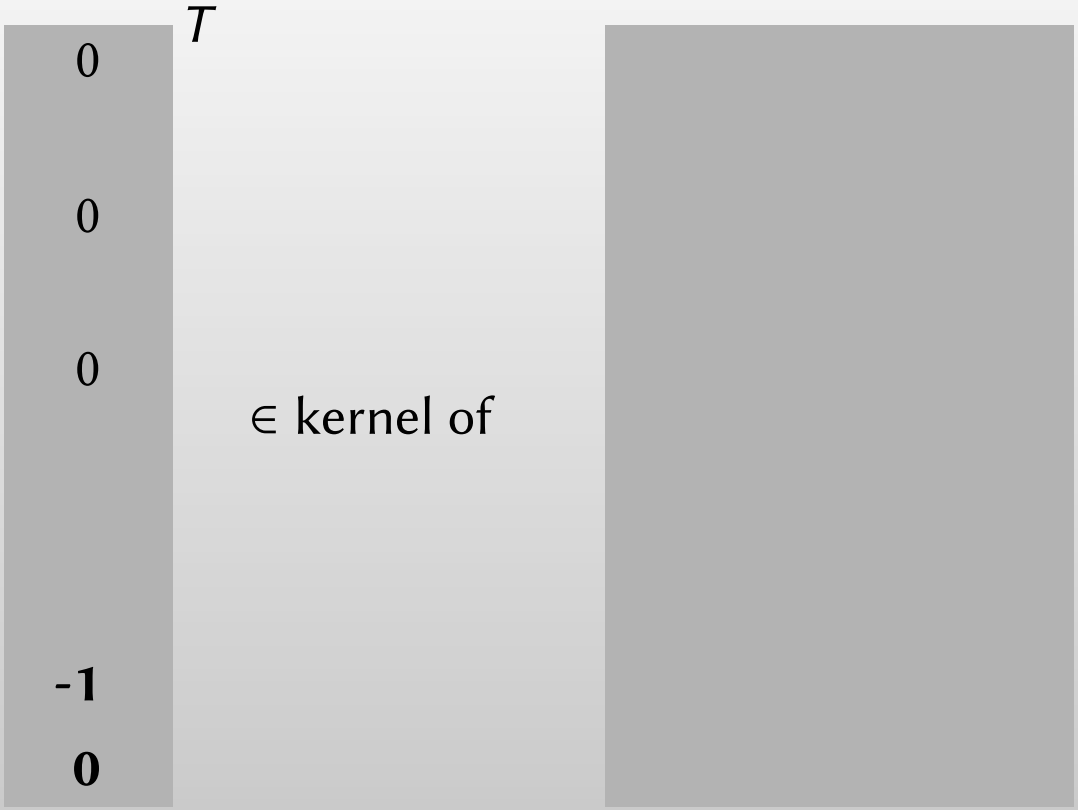

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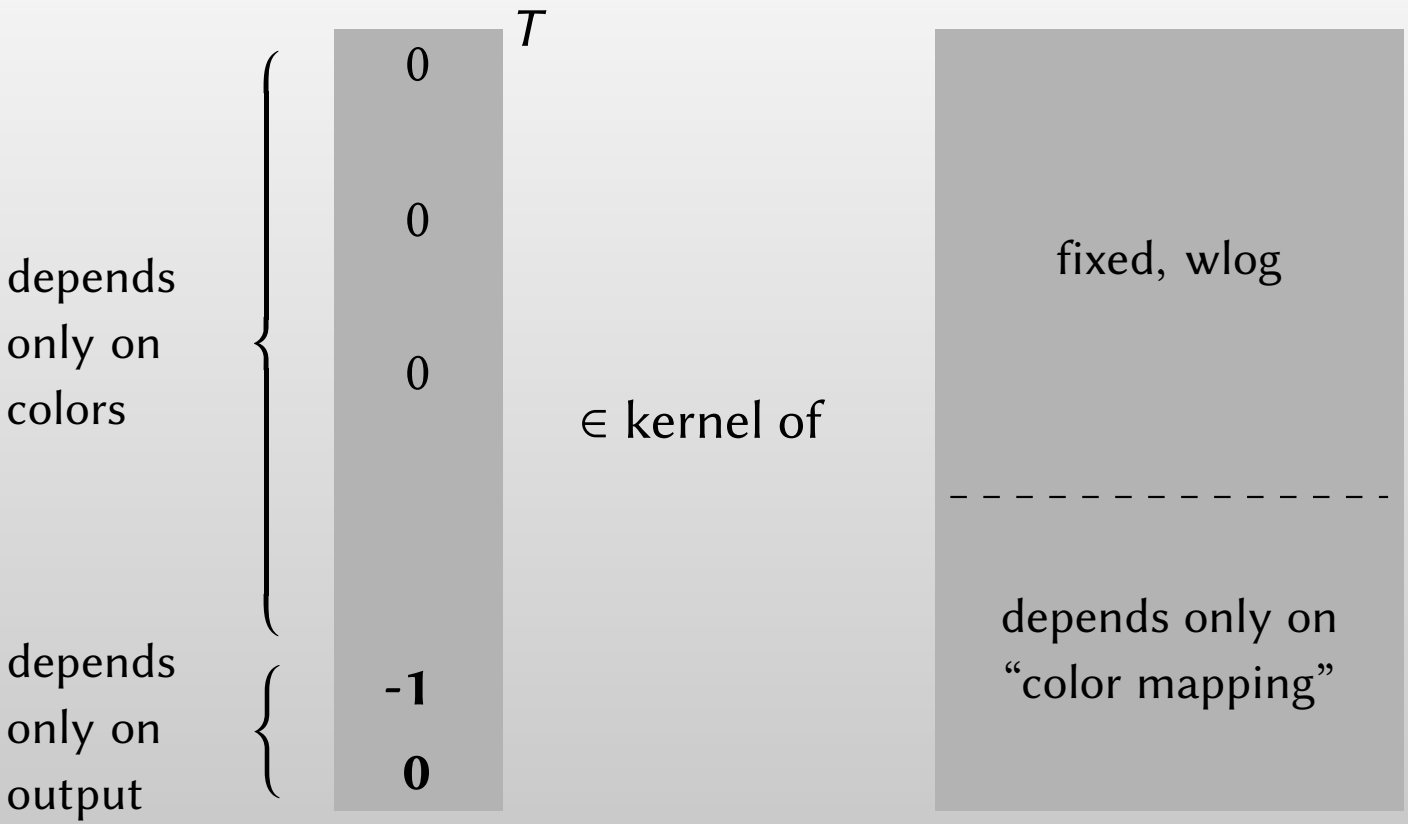

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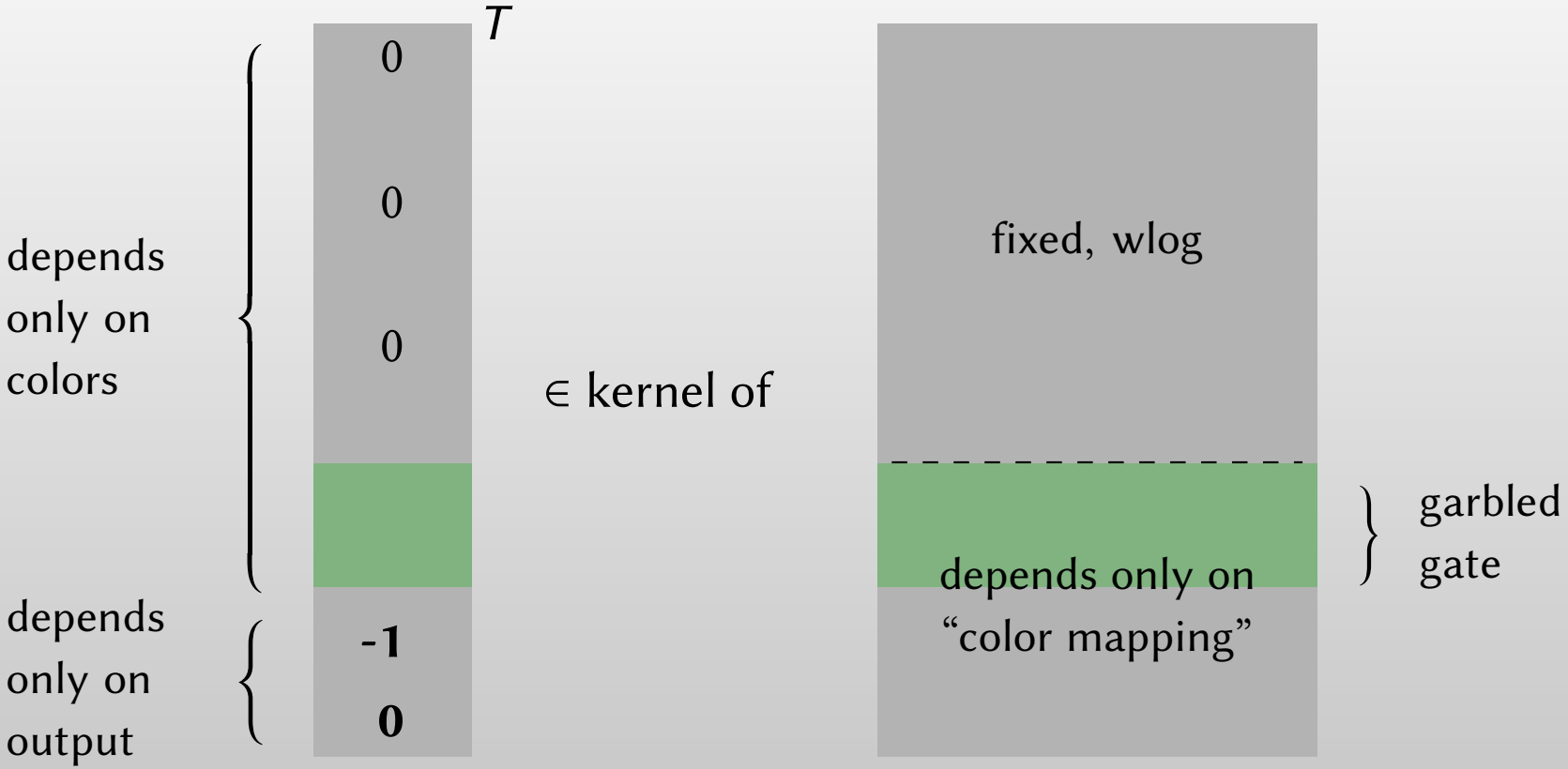

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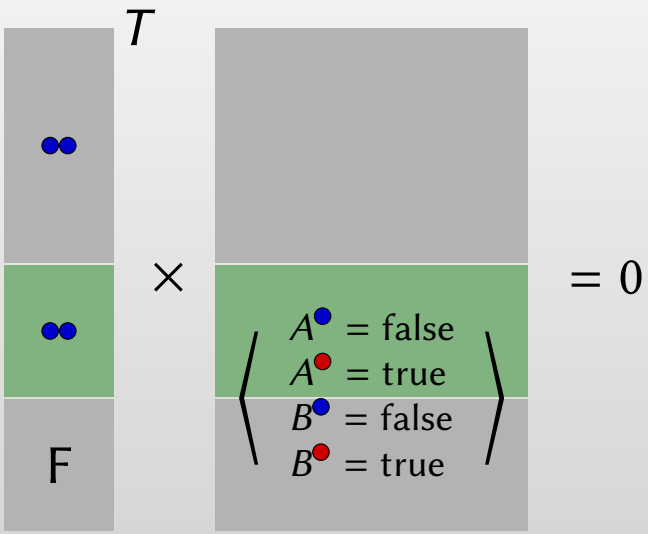

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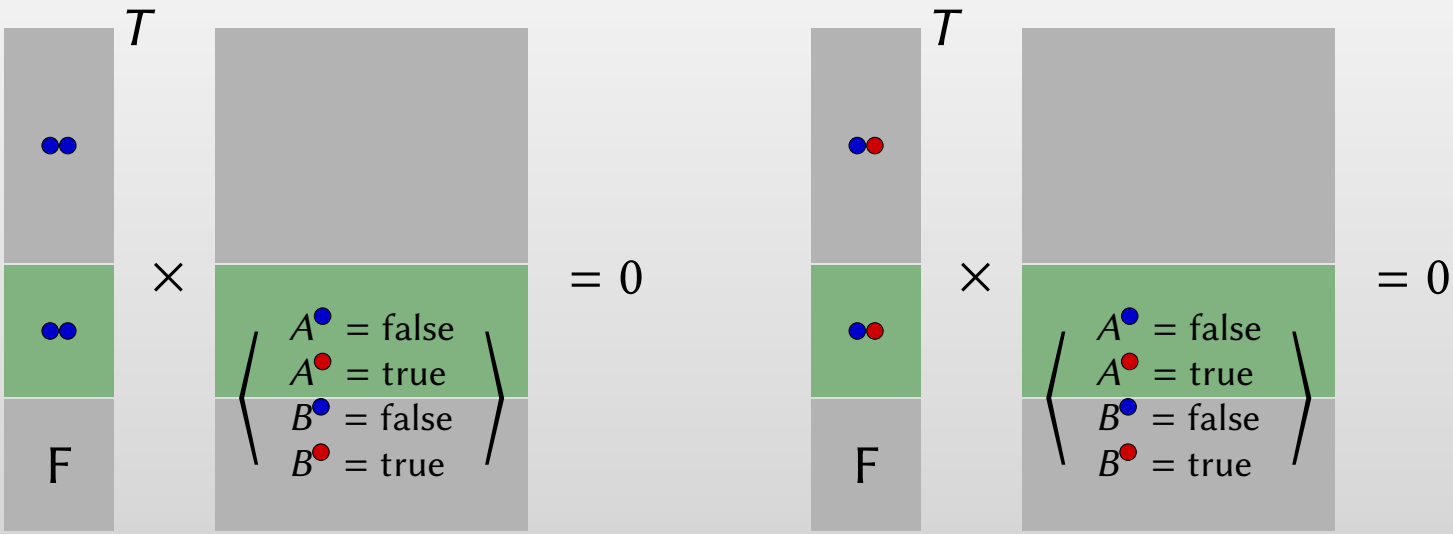

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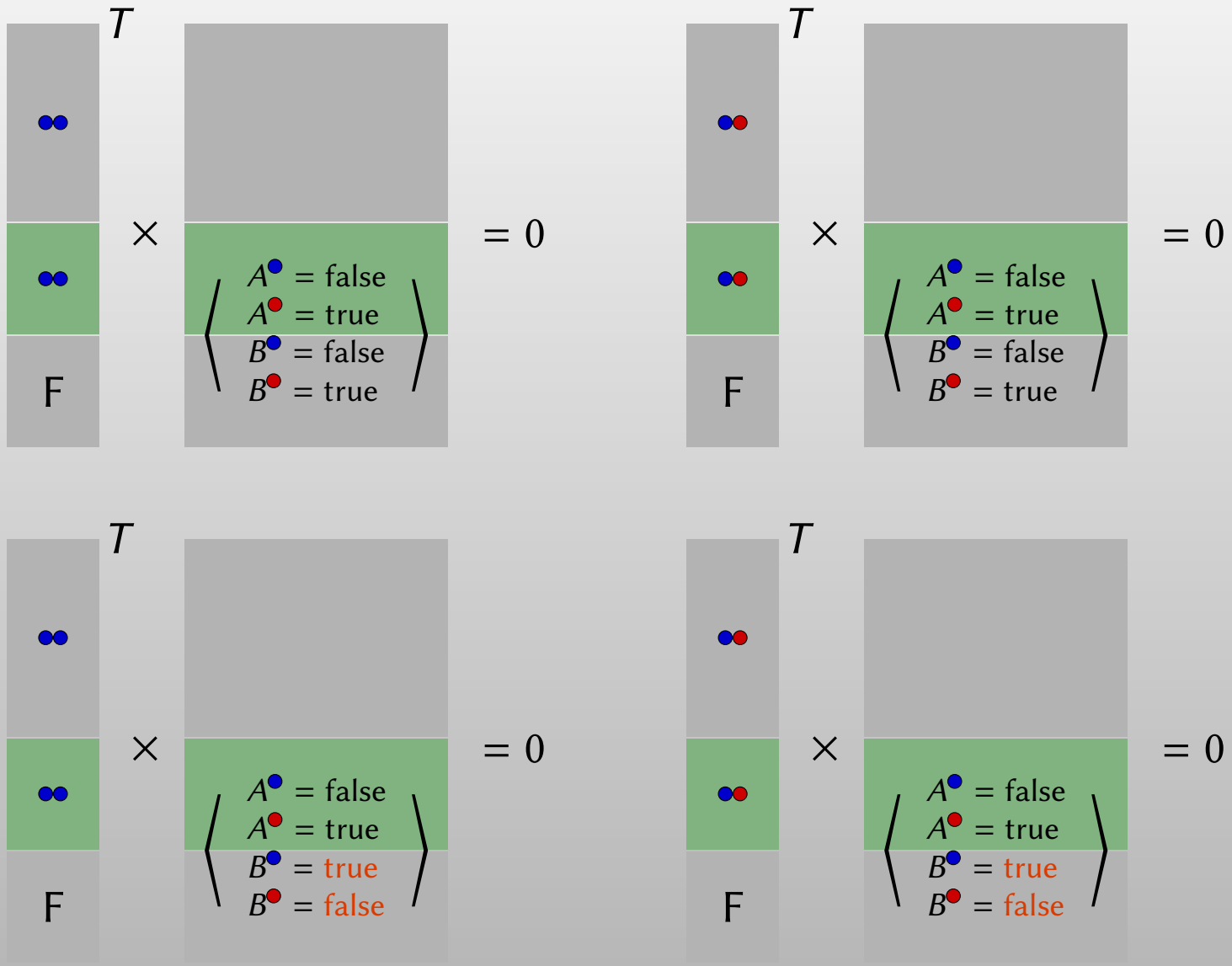

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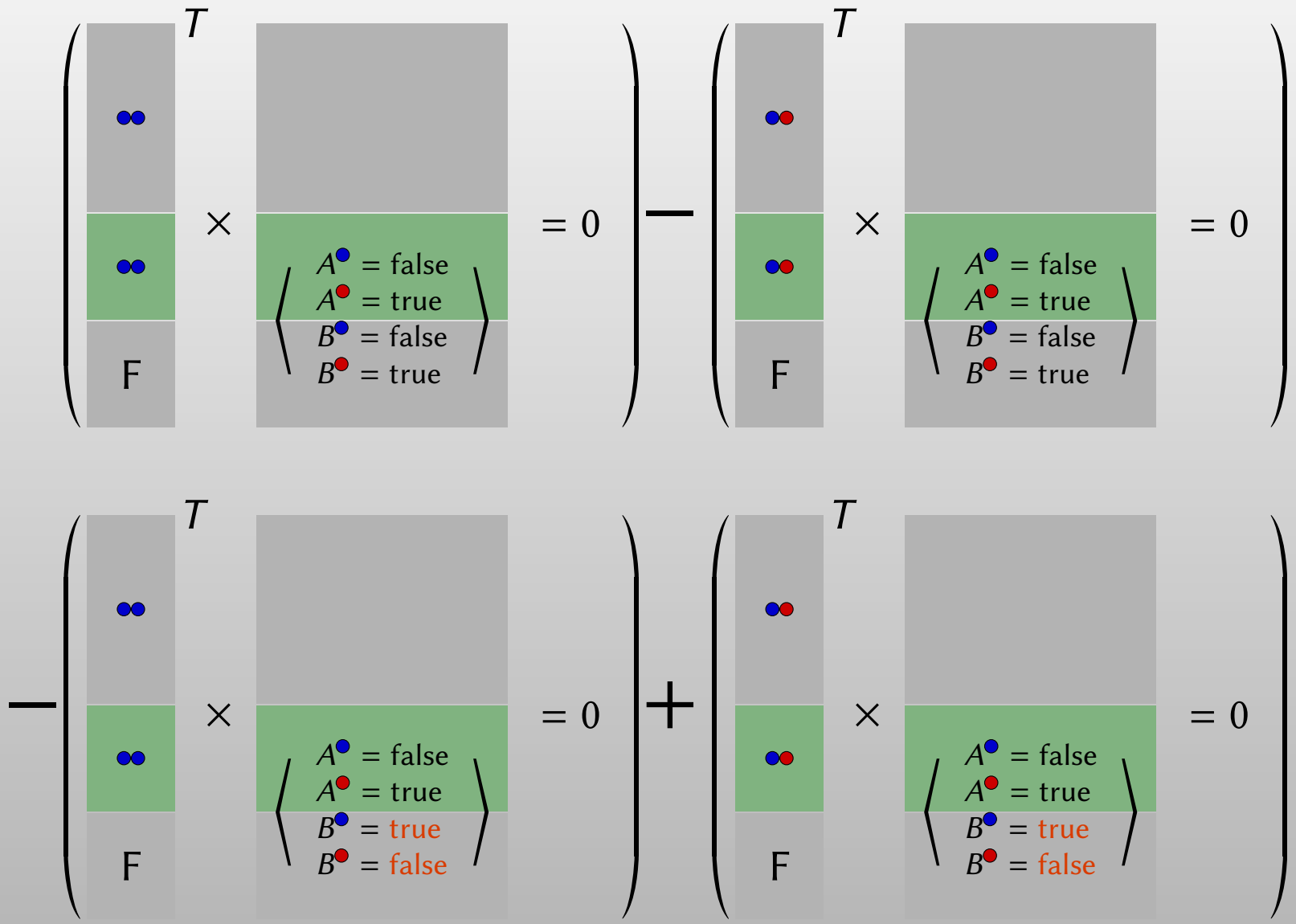

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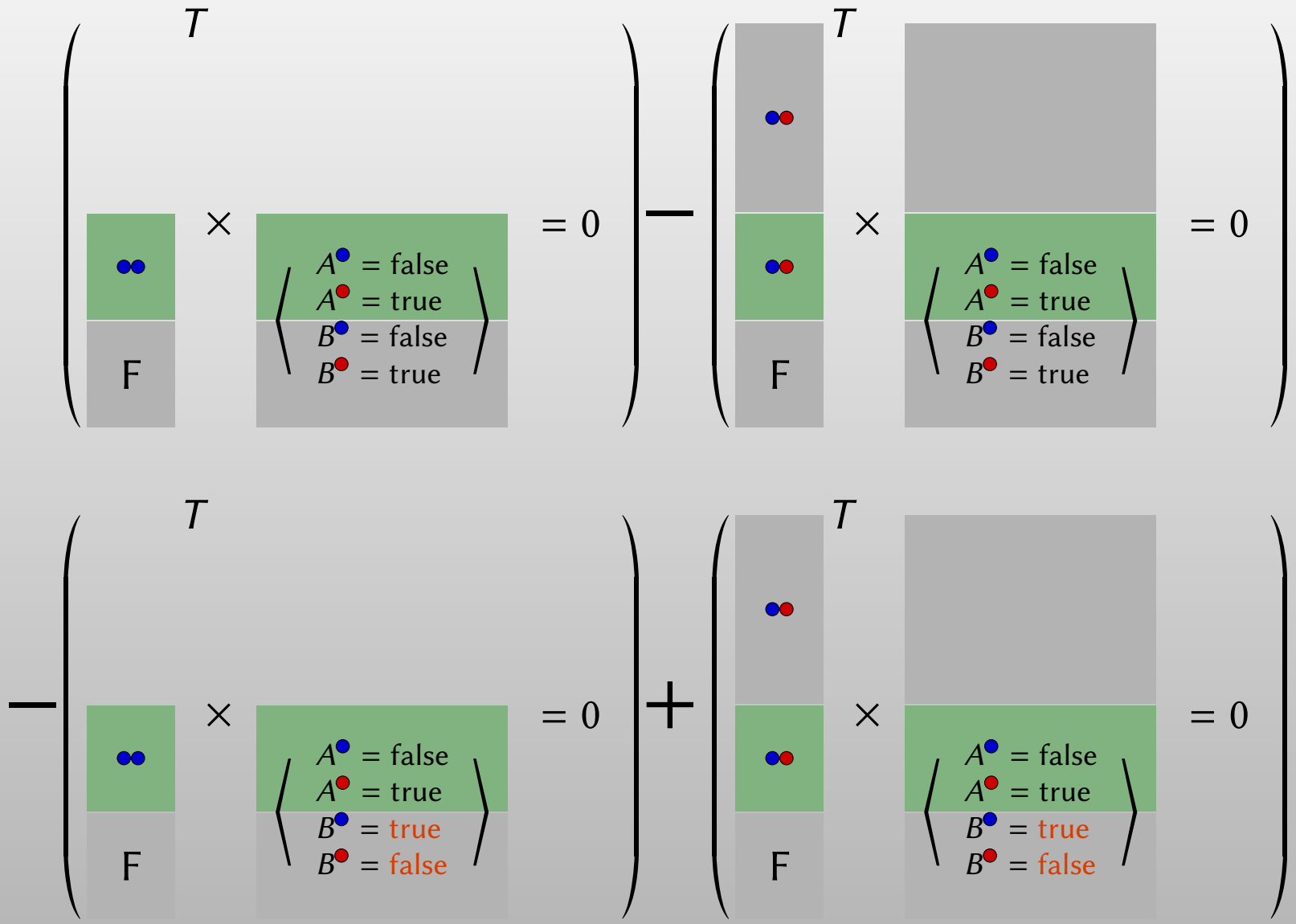

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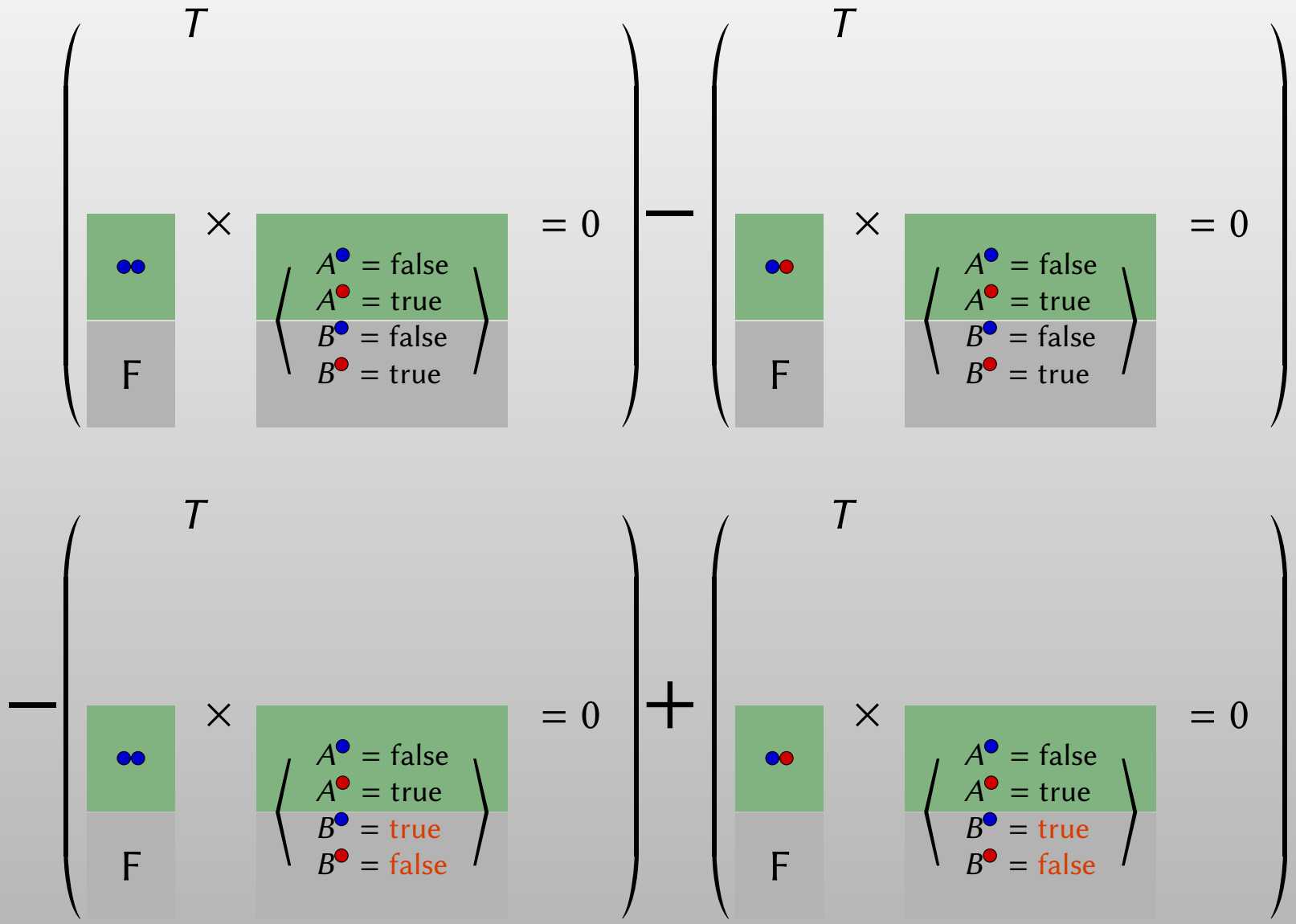

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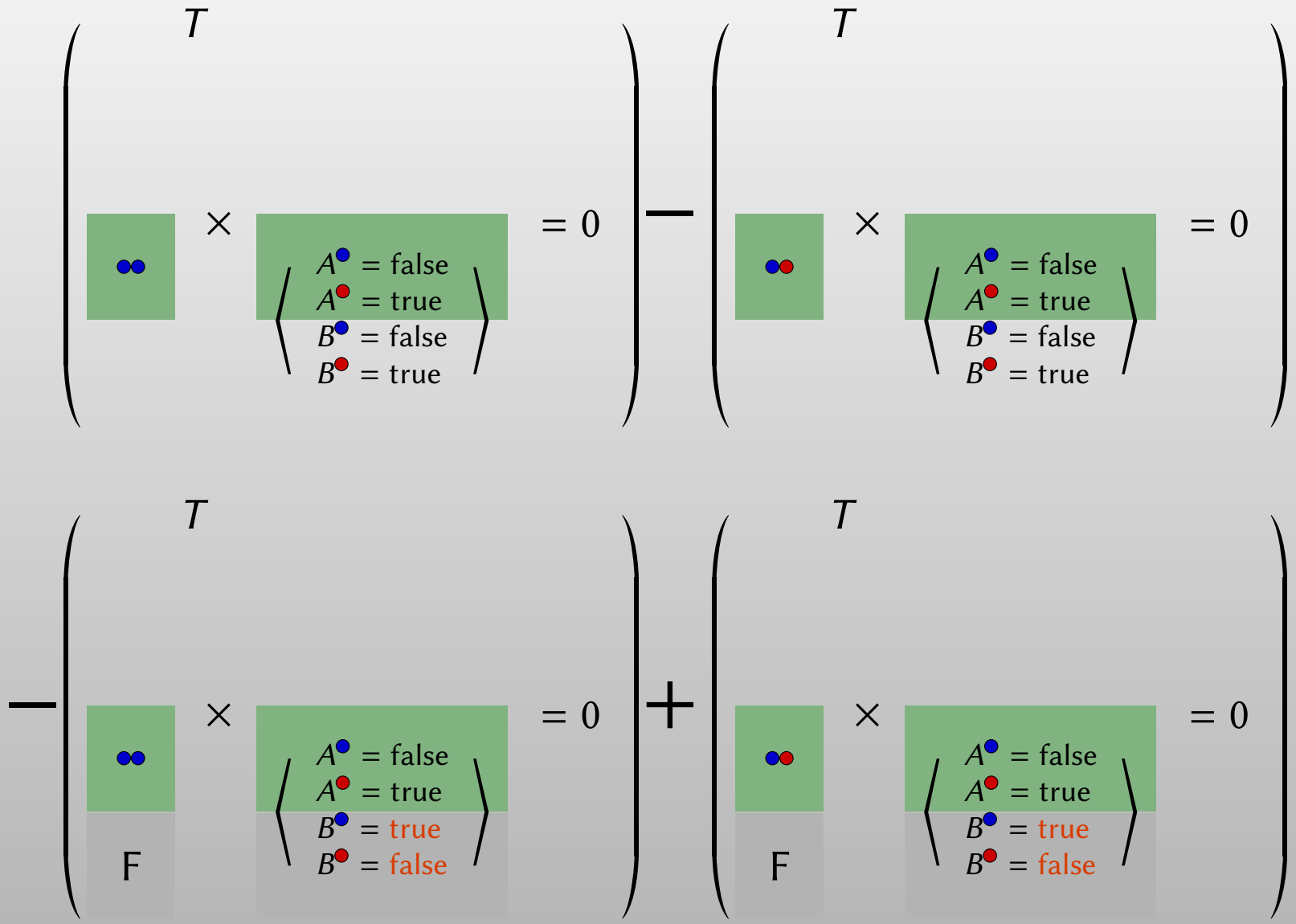

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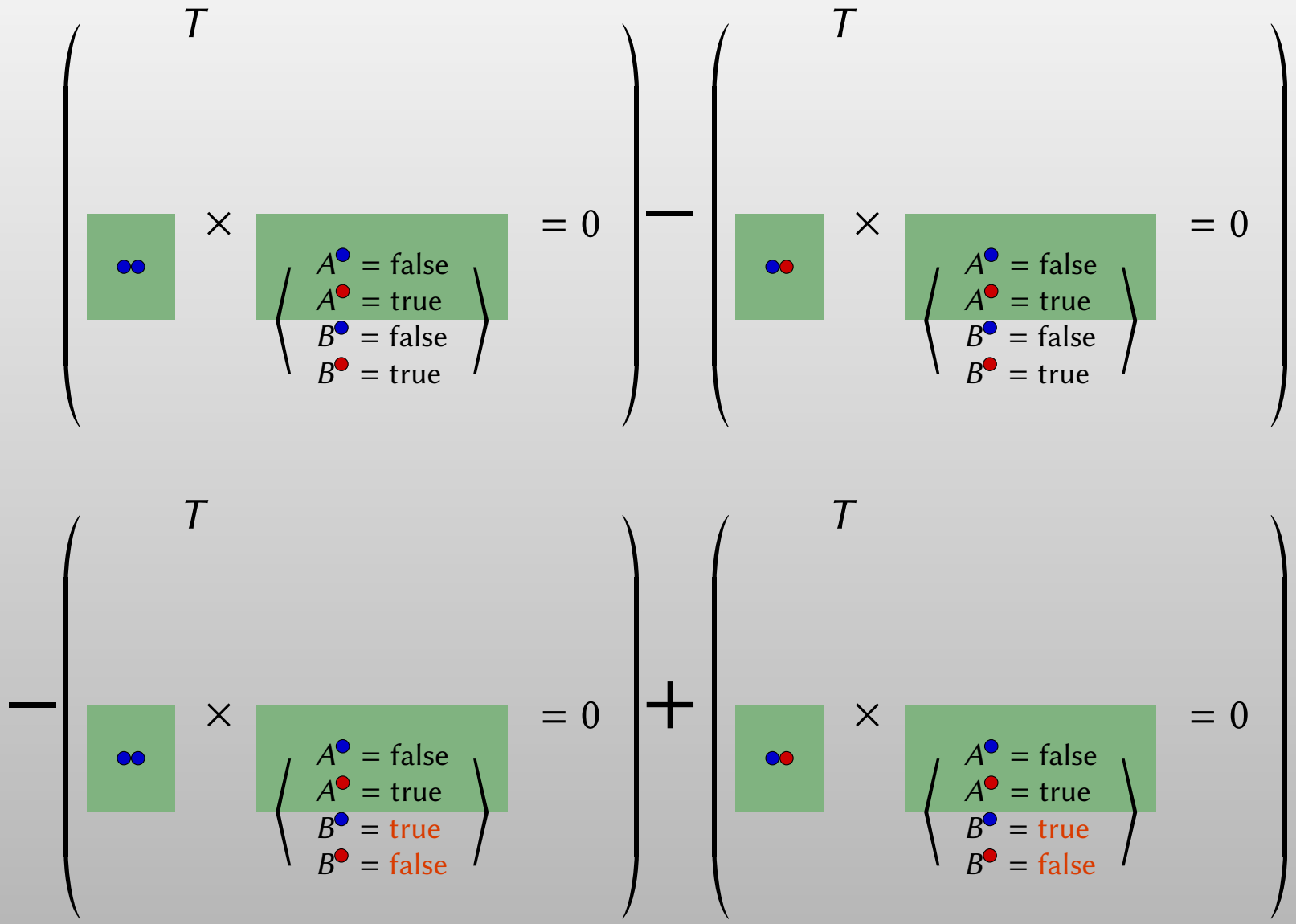

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$\Rightarrow$ M, M' have at least 2 rows

$\Rightarrow$ garbled gate has at least 2λ bits

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Theorem

A linear garbling scheme requires 2λ bits to garble a single AND *gate.*

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Purpose of lower bounds in a restricted model:

- ~~Give up, you can't do any better~~
- Try to do better by avoiding assumptions of the restricted model

Limitations of model

Limitation:
point-and-permute is sole source of non-linearity

Opportunity:
smaller GC using alternatives to point-permute [BallMalkinRosulek16]

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linear!	maybe non-linear (but still practical) techniques lead to smaller GC? e.g., compute $GF(2^\lambda)$-inverse of a wire label

Beyond garbled circuits

General model of "practical symmetric-key crypto" ?

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Similar models have been fruitful for lower bounds / impossibility results.

Generic group model [Shoup97]:

- Algorithm only uses prescribed group operations
- Generic ("structure preserving") signature schemes require 3 group elements, etc. [AbeGHO11,AbeGOT14]

[Black-box] Arithmetic cryptography
[IshaiPrabhakaranSahai09,ApplebaumAvronBrzuska15]

- Algorithm uses *arbitrary* field as black-box
- Asymptotic lower bounds & impossibility results

the end!