Towards Optimal Garbled Circuit Constructions

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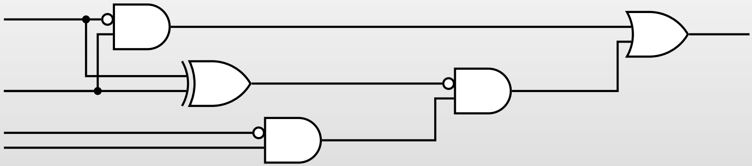
Samee Zahur, Mike Rosulek, David Evans: *Two Halves Make a Whole: Reducing Data Transfer in Garbled Circuits using Half Gates.* Eurocrypt 2015, ia.cr/2014/756

Garbled circuit framework [Yao86]



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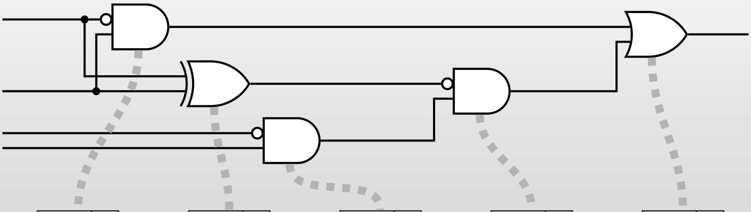

0	0	0
0	1	1
1	0	0
1	1	0

0	0	0
0	1	1
1	0	1
1	1	0

0	0	0
0	1	1
1	0	0
1	1	0

0	0	0
0	1	1
1	0	0
1	1	0

0	0	0
0	1	1
1	0	1
1	1	1

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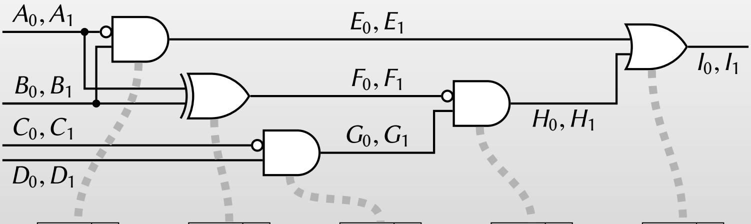

0	0	0
0	1	1
1	0	0
1	1	0

0	0	0
0	1	1
1	0	1
1	1	0

0	0	0
0	1	1
1	0	0
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- Pick random **labels** W_0, W_1 on each wire

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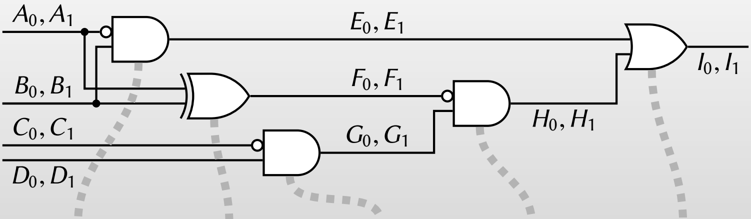

A_0	B_0	E_0
A_0	B_1	E_1
A_1	B_0	E_0
A_1	B_1	E_0

A_0	B_0	F_0
A_0	B_1	F_1
A_1	B_0	F_1
A_1	B_1	F_0

C_0	D_0	G_0
C_0	D_1	G_1
C_1	D_0	G_0
C_1	D_1	G_0

F_0	G_0	H_0
F_0	G_1	H_1
F_1	G_0	H_0
F_1	G_1	H_0

E_0	H_0	I_0
E_0	H_1	I_1
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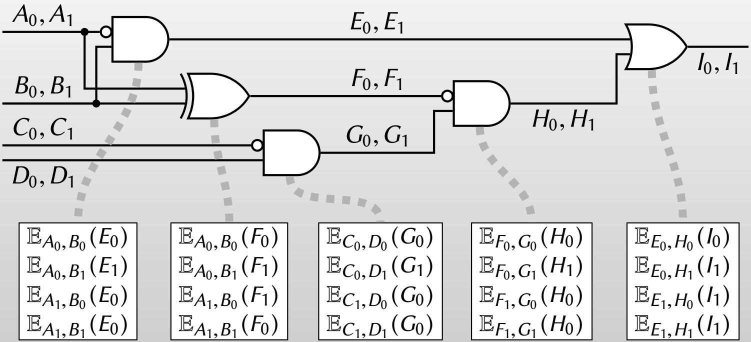

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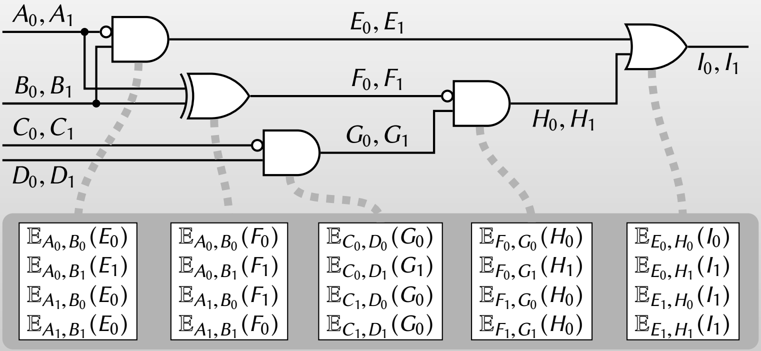

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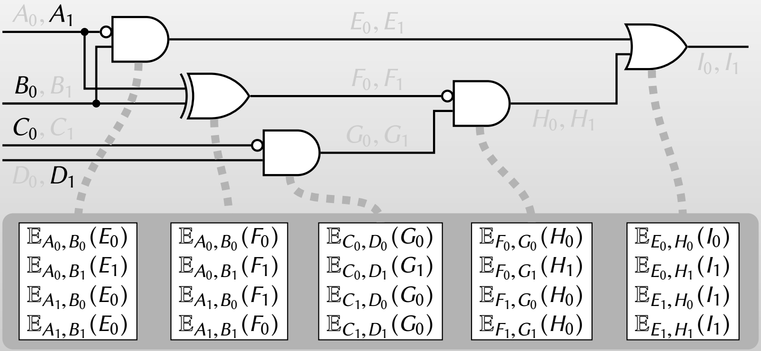

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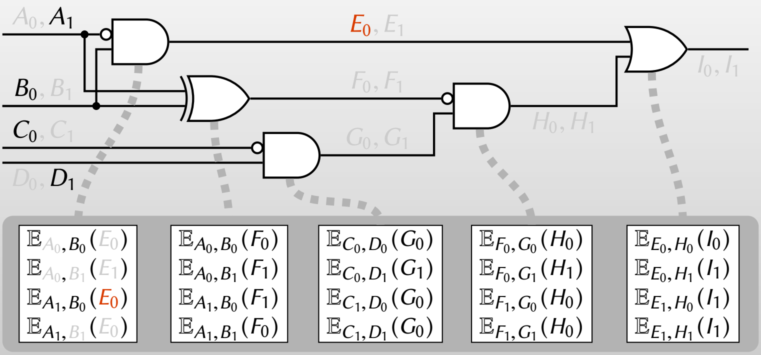

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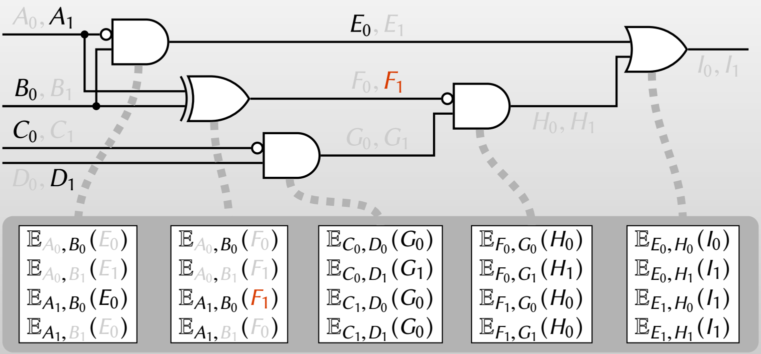

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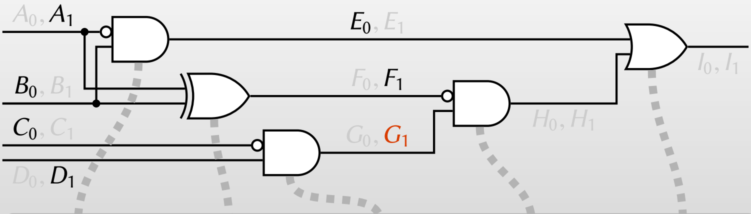

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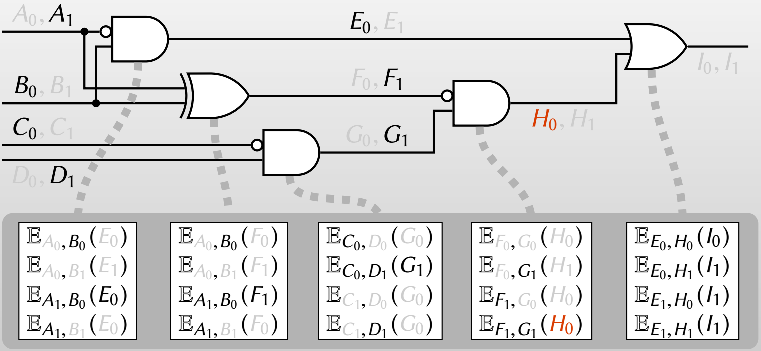

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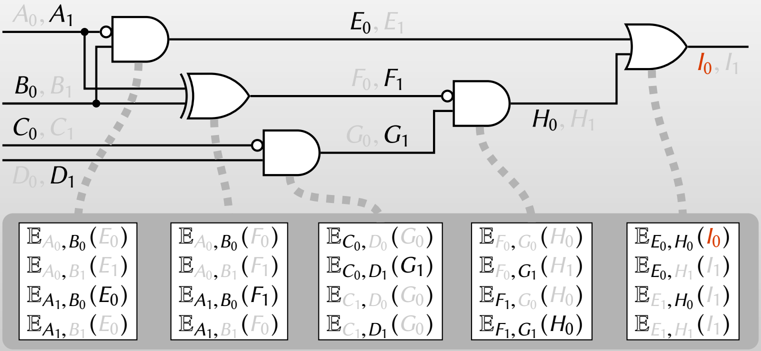

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Applications

secure computation

zero-knowledge proofs

private function evaluation

randomized encodings

secure outsourcing

one-time programs

key-dependent security for encryption

$\vdots$

How small can garbled circuits get?

Garbled circuit size

		size per gate ($\times\lambda$)	
		XOR	AND
Classical	[Yao86,GMW87]	?	?
Point/permute	[BeaverMicaliRogaway90]	4	4

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FleXOR	[KolesnikovMohasselRosulek14]	$\{0, 1, 2\}$	2
HalfGates	[ZahurRosulekEvans15]	**0**	**2**

Is there a lower bound?

Lower bounds for garbled circuits

Pitfalls:

Want to resolve **optimal constants**

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Theorem ([ZahurEvansRosulek15])

A ***linear gate garbling scheme*** *requires 2λ bits to garble a single* AND *gate. Within this model, "half-gates" construction is optimal.*

Rest of talk

1: Restricted model: What is a "linear gate garbling scheme?"

2: The lower bound for AND gates (sketch)

3: Looking forward

"Known, practical techniques"

Symmetric-key cryptography only

- Formalize via computationally unbounded adversaries + random oracle [ImpagliazzoRudich88]
- e.g.: encrypt garbled truth table as $\mathbb{E}_{A,B}(C) = H(A \| B) \oplus C$

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Linear operations exclusively

- Wire labels, garbled gates, RO outputs are field/ring elements (e.g.: $GF(2^\lambda)$)
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One caveat: state of art schemes are *all non-linear* in one specific way . . .

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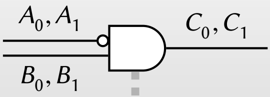

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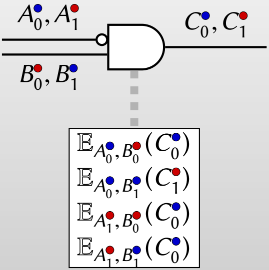

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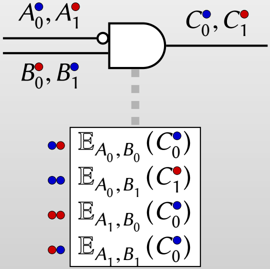

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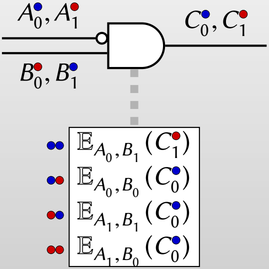

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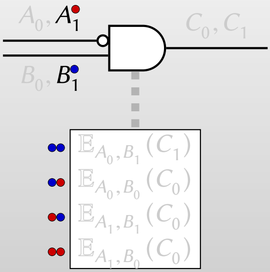

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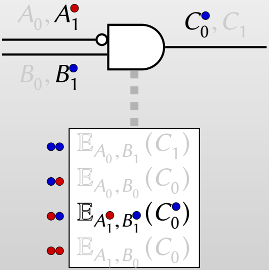

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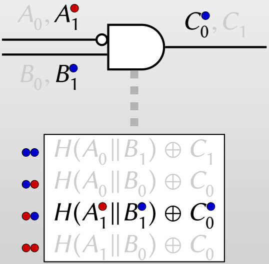

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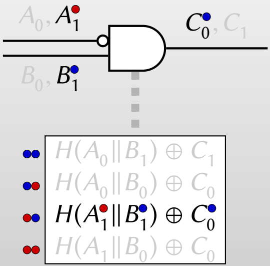

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✗ **Non-linear:** evaluator's choice of linear operation depends on color

✗ Optimized GC schemes crucially take advantage of nonlinearity!

Linear (gate) garbling: details

Apart from point-permute, and calls to random oracle, everything is linear.

Important: only the constructions, not adversary, must be linear!

Linear (gate) garbling: details

$\$_{\mathbb{F}}$

Garbler:

- samples field elements

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$$\left\langle \begin{array}{l} A^{\bullet} = \text{true} \\ A^{\bullet} = \text{false} \\ B^{\bullet} = \text{false} \\ B^{\bullet} = \text{true} \end{array} \right\rangle$$

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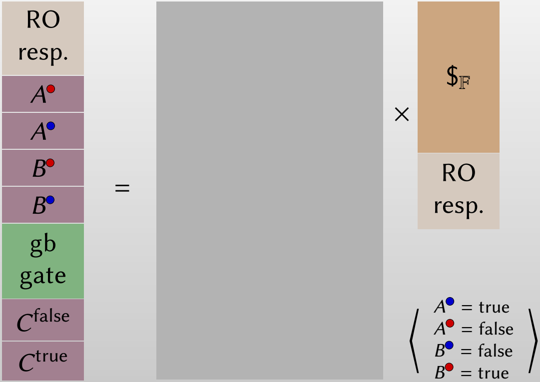

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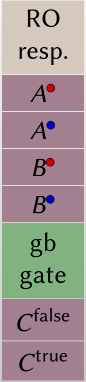

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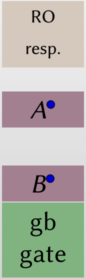

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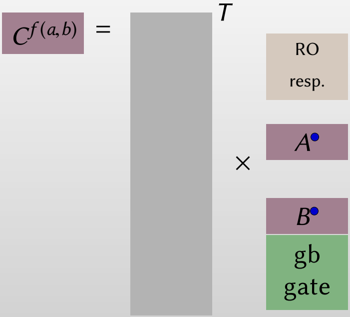

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Linear (gate) garbling: details

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ \mathbf{-1} \\ \mathbf{0} \end{bmatrix}^T \times \begin{bmatrix} \text{RO resp.} \\ A^{\color{red}\bullet} \\ A^{\color{blue}\bullet} \\ B^{\color{red}\bullet} \\ B^{\color{blue}\bullet} \\ \text{gb gate} \\ C^{\text{false}} \\ C^{\text{true}} \end{bmatrix}$$

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$$\begin{array}{r}\text{depends only on colors}\left\{\begin{array}{c}0\\0\\0\\ \\ \end{array}\right.\\ \text{depends only on output}\left\{\begin{array}{c}\mathbf{-1}\\ \mathbf{0}\end{array}\right.\end{array}{}^{T} \in \text{kernel of} \begin{array}{|c|}\hline \text{fixed, wlog}\\ \hline \text{depends only on “color mapping”}\\ \hline\end{array}$$

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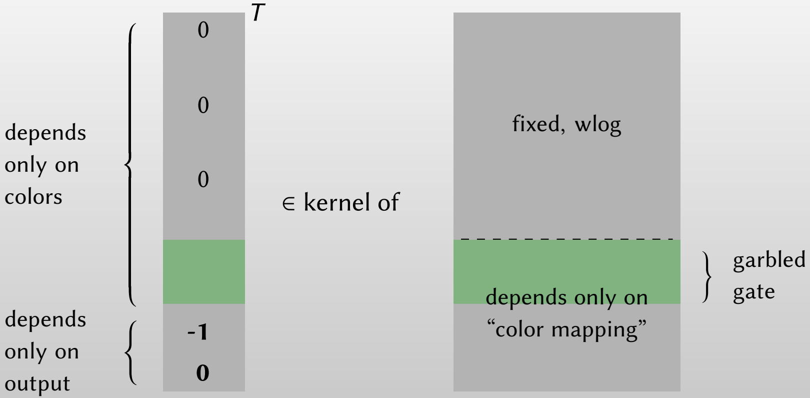

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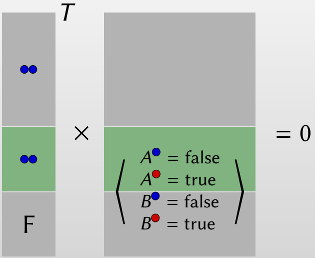

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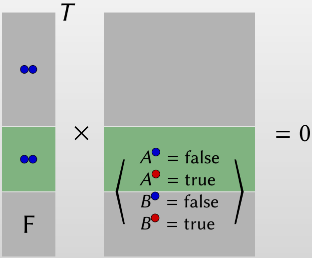

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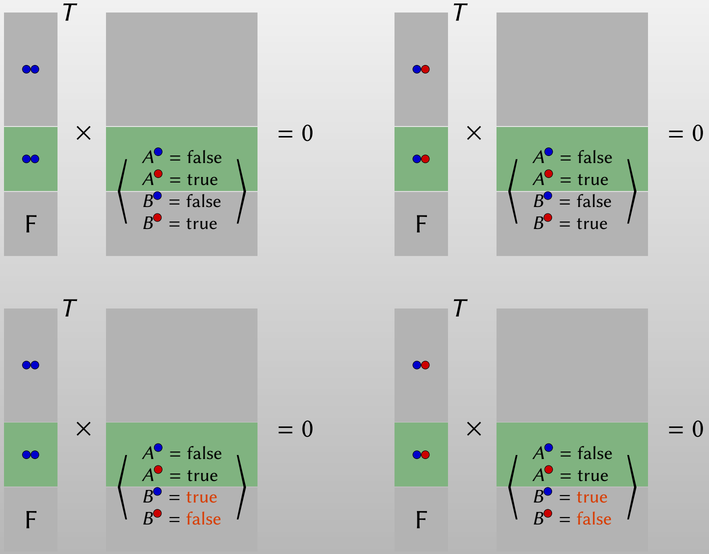

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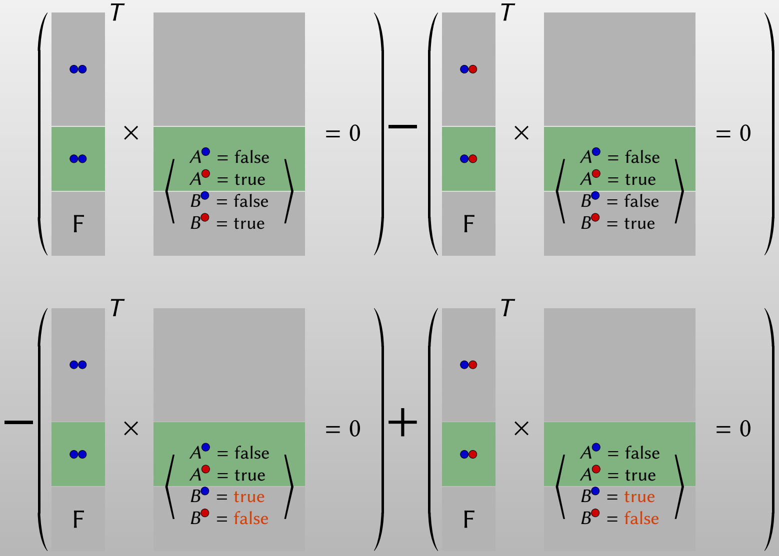

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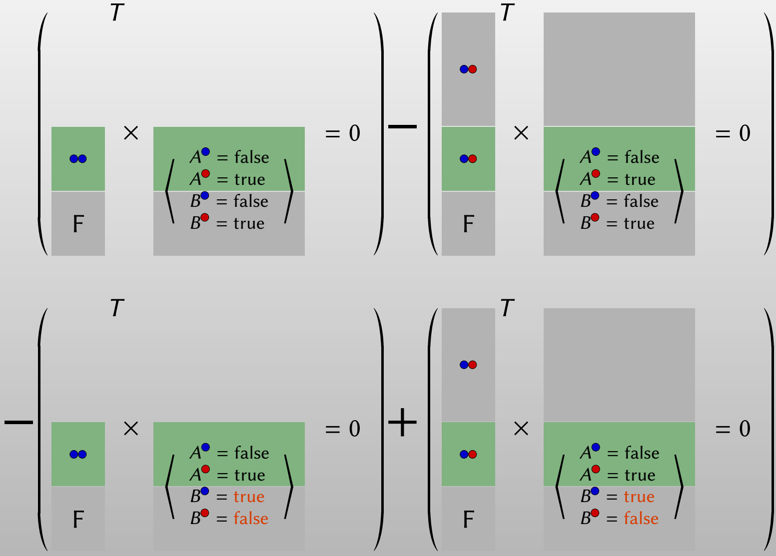

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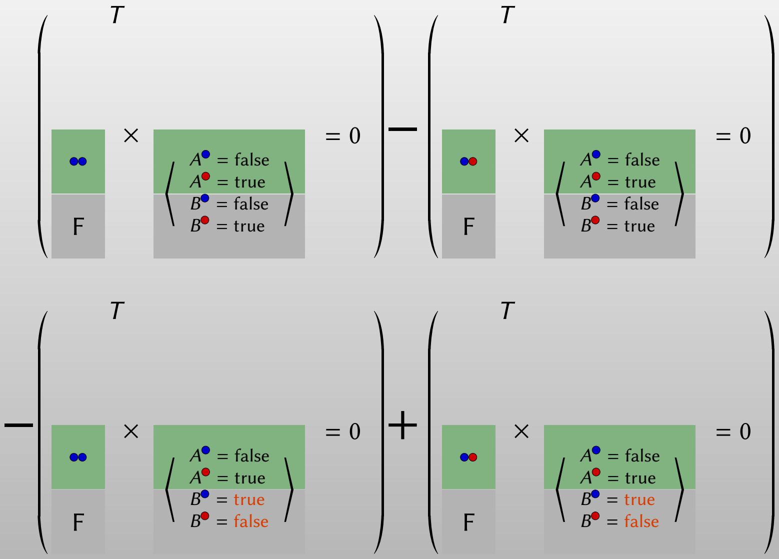

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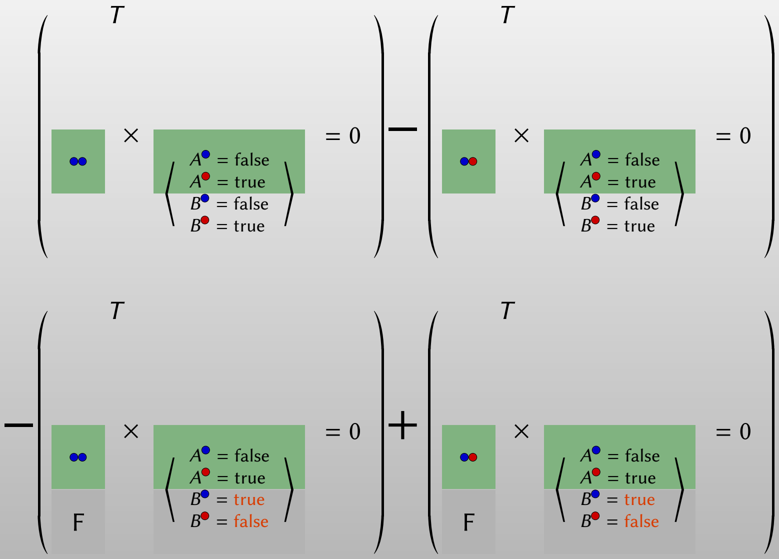

Lower bound (part 1)

$$
\left({\color{blue}\bullet}{\color{blue}\bullet} \times \left\langle \begin{array}{l} A^{\color{blue}\bullet} = \text{false} \\ A^{\color{red}\bullet} = \text{true} \\ B^{\color{blue}\bullet} = \text{false} \\ B^{\color{red}\bullet} = \text{true} \end{array} \right\rangle = 0 \right)^{T}
-
\left({\color{blue}\bullet}{\color{red}\bullet} \times \left\langle \begin{array}{l} A^{\color{blue}\bullet} = \text{false} \\ A^{\color{red}\bullet} = \text{true} \\ B^{\color{blue}\bullet} = \text{false} \\ B^{\color{red}\bullet} = \text{true} \end{array} \right\rangle = 0 \right)^{T}
$$

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+
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$\Rightarrow$ $\dim(\text{rowspace}(M - M')) \geq 1$

$\Rightarrow$ M, M' have at least 2 rows

$\Rightarrow$ garbled gate has at least 2λ bits

Just the beginning...

Theorem

A linear garbling scheme requires 2λ bits to garble a single AND *gate.*

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Purpose of lower bounds in a restricted model:

- ~~Give up, you can't do any better~~
- Try to do better by avoiding assumptions of the restricted model

Limitations of model

Limitation:
point-and-permute is sole source of non-linearity

Opportunity:
smaller GC using alternatives to point-permute [BallMalkinRosulek16]

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linear!	maybe non-linear (but still practical) techniques lead to smaller GC? e.g., compute $GF(2^\lambda)$-inverse of a wire label

Beyond garbled circuits

General model of “practical symmetric-key crypto” ?

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Similar models have been fruitful for lower bounds / impossibility results.

Generic group model [Shoup97]:

- Algorithm only uses prescribed group operations
- Generic ("structure preserving") signature schemes require 3 group elements, etc. [AbeGHO11,AbeGOT14]

[Black-box] Arithmetic cryptography

[IshaiPrabhakaranSahai09,ApplebaumAvronBrzuska15]

- Algorithm uses *arbitrary* field as black-box
- Asymptotic lower bounds & impossibility results

the end!