Epipolar Geometry -- Three Problems
• *Geometry* -- Given point pairs \((x_{1n}, x_{2n})\), find the transform \(F\) that holds between all other point pairs
Epipolar Geometry -- Three Problems

- **Geometry** -- Given point pairs \((x_{1n}, x_{2n})\), find the transform \(F\) that holds between all other point pairs.

- **Correspondence** -- Given \(F\) and \(x_1\), find \(x_2\).
Epipolar Geometry -- Three Problems

- **Geometry** -- Given point pairs \((x_{1n}, x_{2n})\), find the transform \(F\) that holds between all other point pairs

- **Correspondence** -- Given \(F\) and \(x_1\), find \(x_2\)

- **3D Reconstruction** -- Given \(x_1\) and \(x_2\), find \(X\)
Epipolar Geometry Problems
Epipolar Geometry Problems

- **Correspondence** -- Given $F$ and $x_1$, find $x_2$
Motivation: Epipolar Constraint

- Two views of the same object
- Can any point in image 2 correspond to the yellow point in image 1?
Motivation: Epipolar Constraint

Points \( x_2 \) corresponding to point \( x_1 \) must lie along the epipolar line \( l_2 \)

and

Points \( x_1 \) corresponding to point \( x_2 \) must lie along the epipolar line \( l_1 \)
Epipolar Constraint

Point $x_1$ can be a projection of multiple 3D points
corresponding points in image 2 must lie along the epipolar line
Stereo Matching

- Given $F$ and a set of points in image 1
- Find the corresponding points in image 2

$$l_2 = F x_1$$
Epipolar Constraint

\[ l_2 = e_2 \times x_2 = [e_2 \times] x_2 \]

a line passing through two points:

epipolar line
epipole
Epipolar Constraint

Geometry:
\[ x_2 = M_2 X \land X = M_1^{-1} x_1 \implies x_2 = M_2 M_1^{-1} x_1 \]
Epipolar constraint:
\[ x_2 = M_2 X \quad \land \quad X = M_1^{-1} x_1 \quad \Rightarrow \quad x_2 = M_2 M_1^{-1} x_1 \]

Epipolar constraint:
\[ l_2 = e_2 \times x_2 = [e_2 \times] x_2 \]
Epipolar constraint:

\[ l_2 = e_2 \times x_2 = [e_2 \times] x_2 \]

\[ \Rightarrow l_2 = [e_2 \times] M_2 M_1^{-1} x_1 = F x_1 \]

fundamental matrix
Finding Epipolar Lines

Given $F$ and $x_1$, find the epipolar line of $x_1$:

$$l_2 = F x_1$$

epipolar line in image 2
Finding Epipolar Lines -- Example

\[ l_{1n} = F^T x_{2n} \]

\[ l_{2n} = F x_{1n} \]

Epipolar lines for a few sample points
Improved Matching Formulation: Linear Assignment

\[
\min_Y \text{Trace}(A^\top Y)
\]

subject to:

\[
\forall i, j \quad y_{ij} \in [0, 1]
\]

\[
\forall i, \sum_j y_{ij} = 1 \quad \forall j, \sum_i y_{ij} = 1
\]
Improved Matching Formulation: Linear Assignment

\[ \min_{Y} \text{Trace}(A^T Y) \]

subject to:
\[ \forall i, j \ y_{ij} \in [0, 1] \]

\[ \forall i, \sum_{j} y_{ij} = 1 \]
\[ \forall j, \sum_{i} y_{ij} = 1 \]
Improved Matching Formulation: Linear Assignment

\[
\min \operatorname{Trace}(\mathbf{A}^\top \mathbf{Y})
\]
subject to:

\[
\forall i, j \ y_{ij} \in [0, 1]
\]

\[
\forall i, \sum_j y_{ij} = 1 \quad \forall j, \sum_i y_{ij} = 1
\]

distance matrix
Improved Matching Formulation: Linear Assignment

\[
\min_{Y} \text{Trace}(A^\top Y)
\]

subject to:

\[
\forall i, j \ y_{ij} \in [0, 1]
\]

\[
\forall i, \sum_{j} y_{ij} = 1 \quad \forall j, \sum_{i} y_{ij} = 1
\]

one-to-one matching
Improved Matching Formulation: Linear Assignment

\[
\min_{Y} \text{Trace}(A^\top Y)
\]

subject to:

\[\forall i, j \quad y_{ij} \in [0, 1]\]

\[\forall i, \sum_{j} y_{ij} = 1\]

\[\forall j, \sum_{i} y_{ij} = 1\]
Improved Matching Formulation: Linear Assignment

Hungarian Algorithm

\[
\begin{align*}
\min \ & \ \text{Trace}(A^T Y) \\
\text{subject to:} \ & \ \forall i, j \ y_{ij} \in [0, 1] \\
\ & \ \forall i, \ \sum_j y_{ij} = 1 \\
\ & \ \forall j, \ \sum_i y_{ij} = 1 \\
\end{align*}
\]

additional constraints:

\[
\forall j, \ x_j \in l_j = F x_i
\]

2D point in image 2 \hspace{2cm} \text{epipolar line in image 2}
Improved Matching Formulation: Linear Assignment

\[
\min_{Y} \text{Trace}(A^\top Y)
\]
subject to:
\[
\forall i, j \quad y_{ij} \in [0, 1]
\]
\[
\forall i, \quad \sum_{j} y_{ij} = 1 \\
\forall j, \quad \sum_{i} y_{ij} = 1
\]

additional constraints:
\[
\forall j : \quad x_j^\top F x_i = 0
\]

2D point in image 2

2D point in image 1
How to Incorporate the Additional Stereo Constraints?

\[ \forall j : \quad x_j^\top F x_i = 0 \]
How to Incorporate the Additional Stereo Constraints?

Additional constraints:

\[ \sum_{(i,j)} \left[ x_j^\top F x_i \right] \cdot y_{ij} = 0 \]

Indicator of matching pairs

\[ \forall j : \ x_j^\top F x_i = 0 \]

2D point in image 2

2D point in image 1
How to Incorporate the Additional Stereo Constraints?

**additional constraints:**

$$
\sum_{(i,j)} \left[ x_j^\top F x_i \right] \cdot y_{ij} = 0
$$

indicator of matching pairs

$$
\text{tr}(B^\top Y)
$$
Improved Matching Formulation: Linear Assignment

Hungarian Algorithm

\[
\min_{Y} \text{Trace}(A^\top Y)
\]

subject to:
\[
\forall i, j \ y_{ij} \in [0, 1]
\]
\[
\forall i, \sum_j y_{ij} = 1 \quad \forall j, \sum_i y_{ij} = 1
\]
Improved Matching Formulation: Linear Assignment

\[
\min_Y \left[ \text{Trace}(A^\top Y) + \text{Trace}(B^\top Y) \right]
\]

subject to:

\[\forall i, j \quad y_{ij} \in [0, 1]\]

\[\forall i, \sum_j y_{ij} = 1 \quad \forall j, \sum_i y_{ij} = 1\]
Improved Matching Formulation: Linear Assignment

Hungarian Algorithm

\[
\min_Y \text{Trace}((A + B)^\top Y)
\]

subject to:

\[
\forall i, j \quad y_{ij} \in [0, 1]
\]

\[
\forall i, \sum_j y_{ij} = 1 \quad \forall j, \sum_i y_{ij} = 1
\]
Epipolar Geometry Problems
3D Reconstruction -- Given $x_1$ and $x_2$, find $X$
3D Structure Reconstruction from Stereo Images

Goal:
Estimate $X$, given two observations of $X$ in stereo images and camera parameters $M_1 = K_1 [I \ 0]$ and $M_2 = K_1 [R \ t]$. 

}[Image of 3D structure reconstruction diagram]
3D Structure Reconstruction from Stereo Images

Goal:
Estimate $X$, given two observations of $X$ in stereo images and camera parameters $M_1 = K_1 [I \ 0]$ and $M_2 = K_1 [R \ t]$
Minimization of Geometric Error

\[ x_1 - M_1 X = 0 \]
\[ x_2 - M_2 X = 0 \]

\[ \| x_1 - M_1 X \|_2^2 = 0 \]
\[ \| x_2 - M_2 X \|_2^2 = 0 \]
Minimization of Geometric Error

\[ \min_X \left( \| x_1 - M_1 X \|^2 + \| x_2 - M_2 X \|^2 \right) \]
Minimization of Geometric Error

\[
\min_X \left( \| x_1 - M_1 X \|^2 + \| x_2 - M_2 X \|^2 \right)
\]

subject to:

\[
x_2^T F x_1 = 0
\]

epipolar constraint
Minimization of Geometric Error

\[ \min_X \left( \| x_1 - M_1 X \|^2 + \| x_2 - M_2 X \|^2 \right) \]

subject to: \[ X^T \begin{pmatrix} M_2^T & F & M_1 \end{pmatrix} X = 0 \] epipolar constraint
Minimization of Geometric Error -- Solution

\[
\min_X \left( \|x_1 - M_1 X\|^2 + \|x_2 - M_2 X\|^2 \right)
\]

subject to: \(X^T(M_2^T F M_1) X = 0\)

\[
\min_X \left( \|x_1 - M_1 X\|^2 + \|x_2 - M_2 X\|^2 + \lambda X^T(M_2^T F M_1) X \right)
\]
Minimization of Geometric Error -- Solution

$$\min_X \left( \|x_1 - M_1 X\|^2 + \|x_2 - M_2 X\|^2 \right)$$

subject to: $$X^T (M_2^T F M_1) X = 0$$

$$\min_X \left( \|x_1 - M_1 X\|^2 + \|x_2 - M_2 X\|^2 + \lambda X^T (M_2^T F M_1) X \right)$$

$$\min_X \left( \|x_1 - M_1 X\|^2 + \|x_2 - M_2 X\|^2 + \|AX\|_2^2 \right)$$
Minimization of Geometric Error -- Solution

\[
\min_X \left( \| x_1 - M_1 X \|^2 + \| x_2 - M_2 X \|^2 + \| AX \|^2 \right)
\]

\[\downarrow\]

\[
\min_X \sum_i \| a_i - A_i X \|^2
\]

Algorithm: Levenberg-Marquardt
Summary

epipolar line in image 2:

\[ l_2 = F x_1 \]

epipolar line in image 1:

\[ l_1 = F^T x_2 \]

fundamental equation:

\[ x_2^T F x_1 = 0 \]

No explicit relationship between \( x_1 \) and \( x_2 \), because \( F \) is singular
Example Problem: Finding Epipoles

Given the fundamental matrix, find the epipoles

\[
F = \begin{bmatrix}
    f_{11} & f_{12} & f_{13} \\
    f_{21} & f_{22} & f_{23} \\
    f_{31} & f_{32} & f_{33}
\end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}, \ e_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}
\]
Problem: Finding Epipoles

Since $e$ lies on the epipolar lines:

image 1: \[ e_1^T l_1 = 0 \]

image 2: \[ e_2^T l_2 = 0 \]
Problem: Finding Epipoles

image 1: \( e_1^T (F^T x_2) = 0 \)

image 2: \( e_2^T (Fx_1) = 0 \)
Problem: Finding Epipoles

homogeneous system of equations:

image 1: \[ F e_1 = 0 \]

image 2: \[ F^T e_2 = 0 \]
Problem: Finding Epipoles

image 1: \( F e_1 = 0 \) \( \iff \min_{e_1} F e_1 \) 
\[ \text{s.t. } \|e_1\|^2_2 = 1 \]

image 2: \( F^T e_2 = 0 \) \( \iff \min_{e_2} F^T e_2 \) 
\[ \text{s.t. } \|e_2\|^2_2 = 1 \]
Problem: Finding Epipoles

homogeneous system of equations:

image 1: \( e_1 \) is the eigenvector of \( F \)

image 2: \( e_2 \) is the eigenvector of \( F^T \)