CS 556: Computer Vision

Lecture 20

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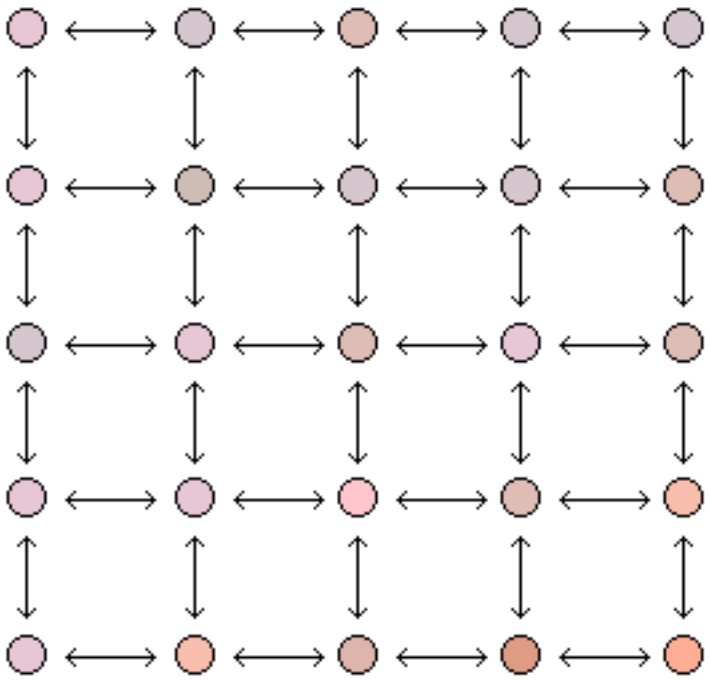
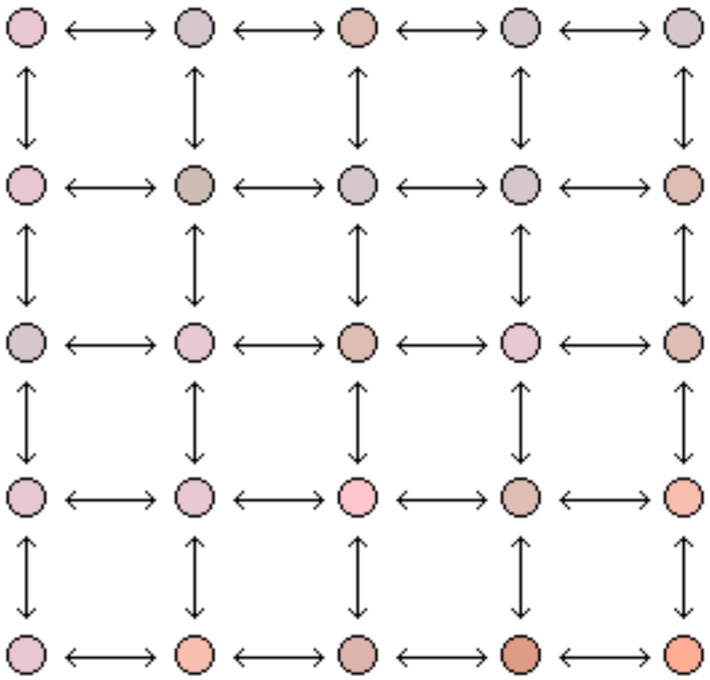
Perceptual Grouping as Graph-based Clustering

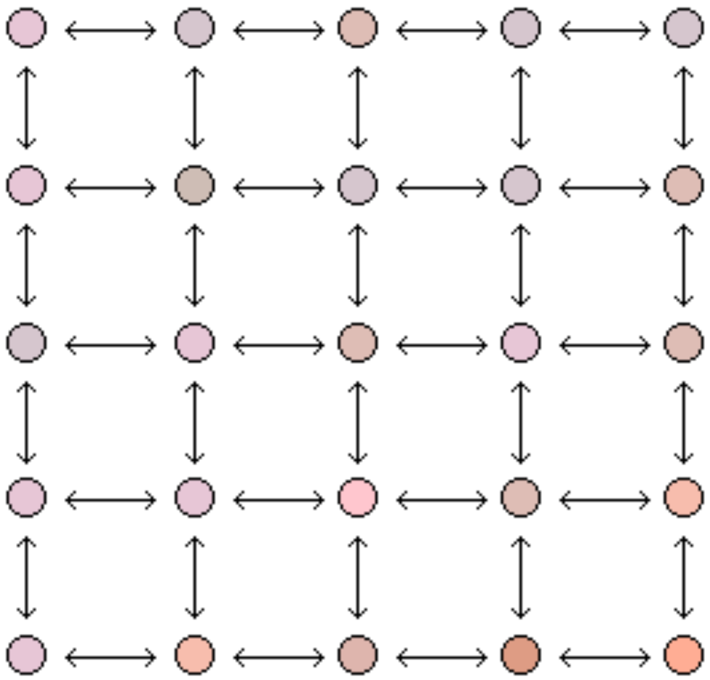


Perceptual Grouping as Graph-based Clustering



- Image elements are represented by a graph

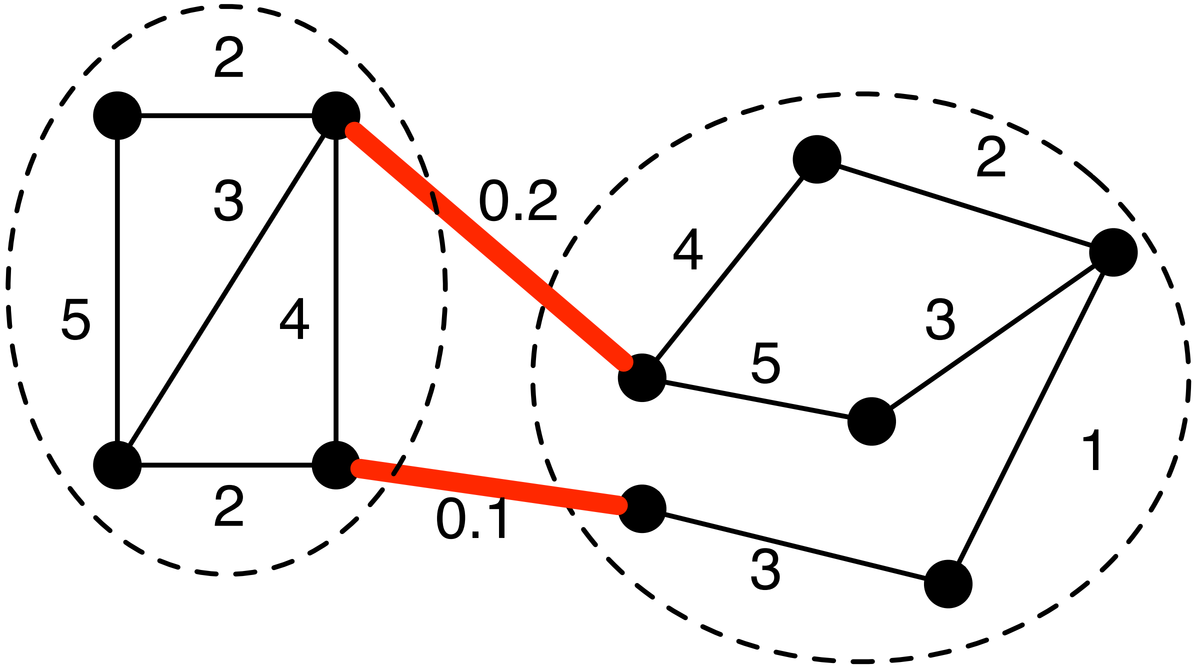
Perceptual Grouping as Graph-based Clustering



- Image elements are represented by a graph

- Grouping = Graph partitioning into subgraphs

Normalized-Cut Clustering

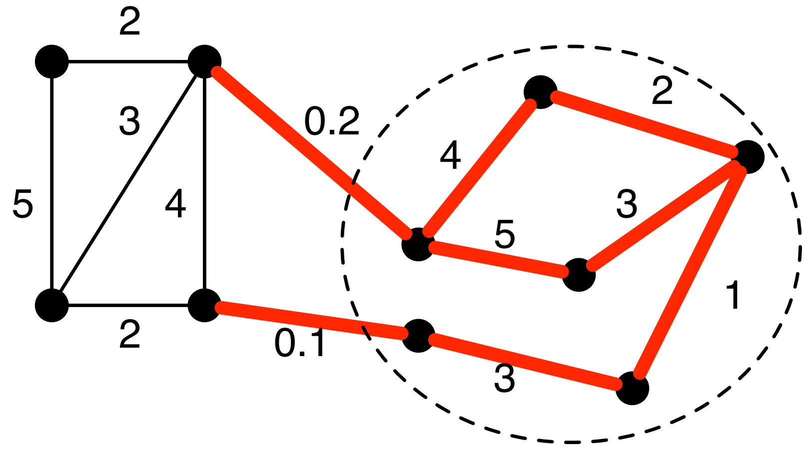


- Find two partitions G_1 and G_2 of graph G

- So that the following criterion is minimized:

$$\min_{G_1,G_2} \mathrm{Ncut}(G_1, G_2) = \frac{\mathrm{cut}(G_1, G_2)}{\mathrm{vol}(G_1)} + \frac{\mathrm{cut}(G_2, G_1)}{\mathrm{vol}(G_2)}$$

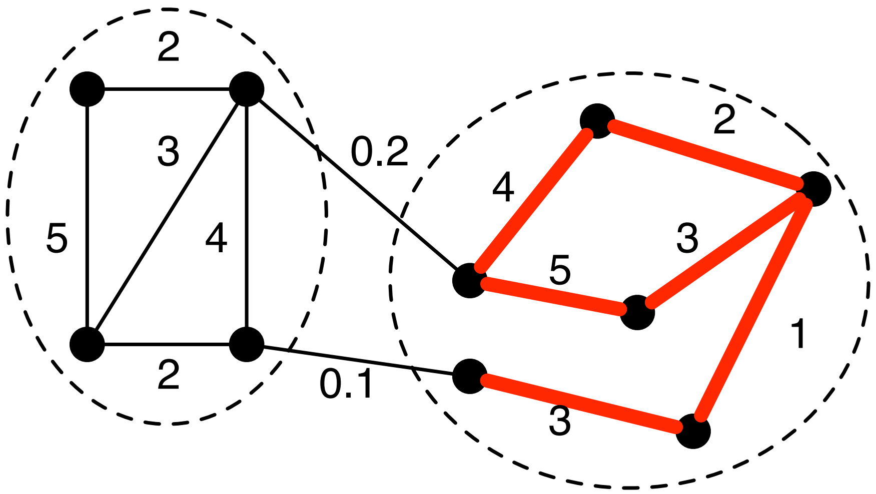
Volume of a Subgraph



$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots \\ 0 & d_2 & 0 & \\ \vdots & & & \\ \dots & 0 & 0 & d_N \end{bmatrix} \qquad \mathbf{1}_{G_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \; G_1$$

$$\Rightarrow \quad \mathrm{vol}(G_1) = \mathbf{1}_{G_1}^{\mathrm{T}} \; D \; \mathbf{1}_{G_1}$$

Association within a Subgraph

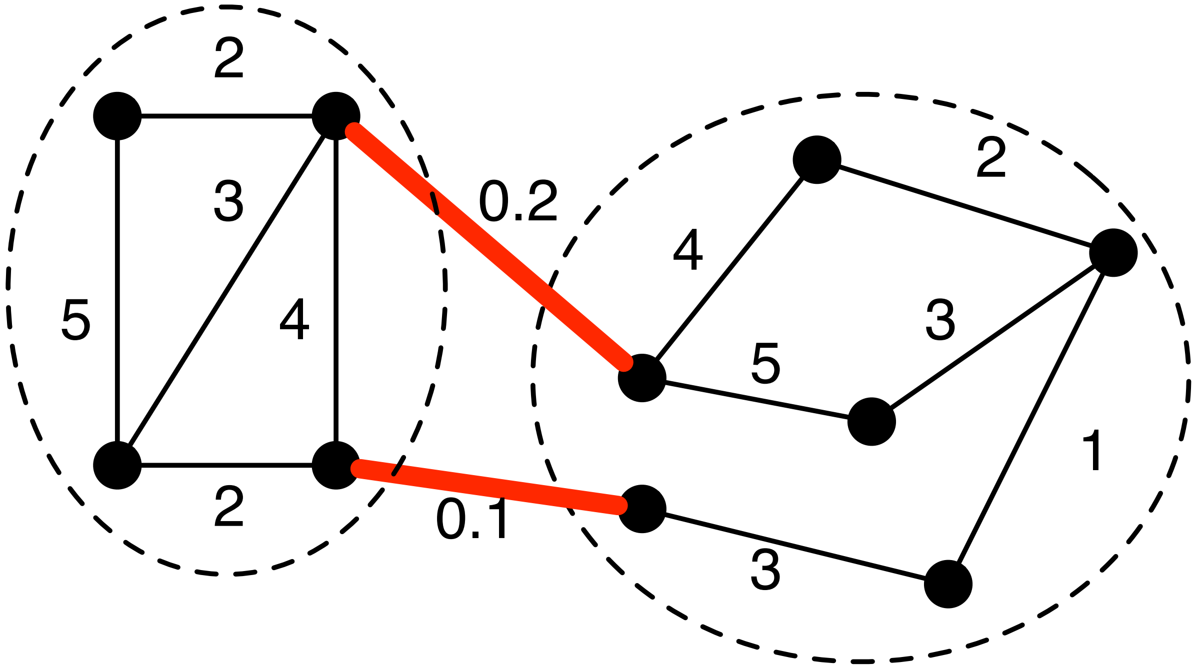


adjacency
matrix

$$A = \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \\ & & \end{bmatrix} \end{array}$$

$$\mathrm{assoc}(G_1) = \sum_{i,j \in G_1} a_{ij} = \mathbf{1}_{G_1}^{\mathrm{T}} \; A \; \mathbf{1}_{G_1}$$

Cut between Two Graph Partitions

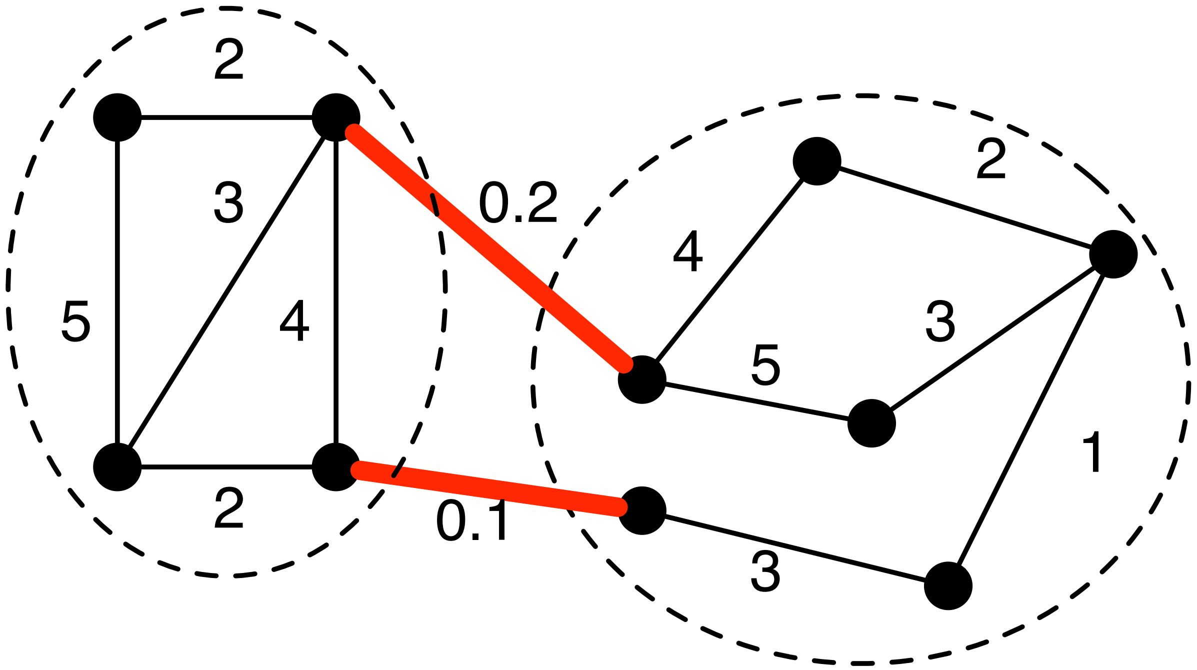


$$\mathrm{cut}(G_1, G_2) = \mathrm{vol}(G_1) - \mathrm{assoc}(G_1)$$

$$= \mathbf{1}_{G_1}^{\mathrm{T}}\ D\ \mathbf{1}_{G_1} - \mathbf{1}_{G_1}^{\mathrm{T}}\ A\ \mathbf{1}_{G_1}$$

$$= \mathbf{1}_{G_1}^{\mathrm{T}}\ (D - A)\ \mathbf{1}_{G_1}$$

Cut between Two Graph Partitions



$$\mathrm{cut}(G_1, G_2) = \mathbf{1}_{G_1}^{\mathrm{T}}\ (D - A)\ \mathbf{1}_{G_1}$$

$$\mathrm{cut}(G_2, G_1) = \mathbf{1}_{G_2}^{\mathrm{T}}\ (D - A)\ \mathbf{1}_{G_2}$$

Normalized-Cut Objective Function

$$\min_{G_1, G_2} \frac{\mathrm{cut}(G_1, G_2)}{\mathrm{vol}(G_1)} + \frac{\mathrm{cut}(G_2, G_1)}{\mathrm{vol}(G_2)}$$

Normalized-Cut Objective Function

$$\min_{G_1, G_2} \frac{\mathrm{cut}(G_1, G_2)}{\mathrm{vol}(G_1)} + \frac{\mathrm{cut}(G_2, G_1)}{\mathrm{vol}(G_2)}$$

$$\min_{G_1, G_2} \frac{\mathbf{1}_{G_1}^T (D - A) \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T D \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T (D - A) \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T D \mathbf{1}_{G_2}}$$

Normalized-Cut Objective Function

$$\min_{G_1,G_2} \frac{\mathrm{cut}(G_1, G_2)}{\mathrm{vol}(G_1)} + \frac{\mathrm{cut}(G_2, G_1)}{\mathrm{vol}(G_2)}$$

$$\min_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T(D - A)\mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T D\mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T(D - A)\mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T D\mathbf{1}_{G_2}}$$

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T\ A\ \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T\ D\ \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T\ A\ \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T\ D\ \mathbf{1}_{G_2}}$$

Normalized-Cut Objective Function

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \ A \ \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \ D \ \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \ A \ \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \ D \ \mathbf{1}_{G_2}}$$

$$\Downarrow$$

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \ D^{-1} A \ \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \ \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \ D^{-1} A \ \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \ \mathbf{1}_{G_2}}$$

Normalized-Cut Objective Function

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \; A \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; D \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; A \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; D \; \mathbf{1}_{G_2}}$$

$$\Downarrow$$

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \; \boxed{D^{-1} A} \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; \boxed{D^{-1} A} \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; \mathbf{1}_{G_2}}$$

normalized
Laplacian

Normalized-Cut Objective Function

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \; A \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; D \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; A \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; D \; \mathbf{1}_{G_2}}$$

$$\Downarrow$$

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \; \boxed{D^{-1} A} \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; \boxed{D^{-1} A} \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; \mathbf{1}_{G_2}}$$

normalized
Laplacian

$$\Downarrow$$

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \; L \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; L \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; \mathbf{1}_{G_2}}$$

Normalized-Cut Linearization

$$\max_{G_1,G_2} \frac{\mathbf{1}_{G_1}^T \; L \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; L \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; \mathbf{1}_{G_2}}$$

$$\Downarrow$$

$$X = [0 \; 0 \; 1 \; 0 \; 1 \ldots 0 \; 1 \; 1 \; 0]^T$$

indicator vector for nodes that
belong to a subgraph

Normalized-Cut Linearization

$$\max_{G_1, G_2} \frac{\boxed{\mathbf{1}_{G_1}^T} \; L \; \mathbf{1}_{G_1}}{\mathbf{1}_{G_1}^T \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}_{G_2}^T \; L \; \mathbf{1}_{G_2}}{\mathbf{1}_{G_2}^T \; \mathbf{1}_{G_2}}$$

$$\Downarrow$$

$$\max_{X_{G_1}, X_{G_2}} \frac{X_{G_1}^T \; L \; X_{G_1}}{X_{G_1}^T \; X_{G_1}} + \frac{X_{G_2}^T \; L \; X_{G_2}}{X_{G_2}^T \; X_{G_2}}$$

subject to: $X_{G_1} \in \{0,1\}^n$ $X_{G_2} \in \{0,1\}^n$

$\|X_{G_1}\| \neq 0$

$\|X_{G_2}\| \neq 0$

Normalized-Cut Linearization

$$\max_{G_1, G_2} \frac{\boxed{\mathbf{1}^T_{G_1}} \; L \; \mathbf{1}_{G_1}}{\mathbf{1}^T_{G_1} \; \mathbf{1}_{G_1}} + \frac{\mathbf{1}^T_{G_2} \; L \; \mathbf{1}_{G_2}}{\mathbf{1}^T_{G_2} \; \mathbf{1}_{G_2}}$$

$$\Downarrow$$

$$\max_{X_{G_1}, X_{G_2}} \frac{X^T_{G_1} \; L \; X_{G_1}}{X^T_{G_1} \; X_{G_1}} + \frac{X^T_{G_2} \; L \; X_{G_2}}{X^T_{G_2} \; X_{G_2}}$$

subject to: $X_{G_1} \in \{0,1\}^n \qquad X_{G_2} \in \{0,1\}^n$

$$\|X_{G_1}\| \neq 0$$

$$\|X_{G_2}\| \neq 0$$

$$\boxed{X^T_{G_1} X_{G_2} = 0}$$

orthogonality

Normalized-Cut Relaxation

$$\max_{X_{G_1}, X_{G_2}} \frac{X_{G_1}^T \ L \ X_{G_1}}{X_{G_1}^T \ X_{G_1}} + \frac{X_{G_2}^T \ L \ X_{G_2}}{X_{G_2}^T \ X_{G_2}}$$

subject to: $X_{G_1} \in [0,1]^n \quad X_{G_2} \in [0,1]^n$

$$\|X_{G_1}\| \neq 0 \qquad \|X_{G_2}\| \neq 0$$

$$X_{G_1}^T X_{G_2} = 0$$

How to Solve Normalized-Cut?

$$\max_{X_{G_1}, X_{G_2}} \frac{X_{G_1}^T \; L \; X_{G_1}}{X_{G_1}^T \; X_{G_1}} + \frac{X_{G_2}^T \; L \; X_{G_2}}{X_{G_2}^T \; X_{G_2}}$$

subject to: $X_{G_1} \in [0,1]^n \quad X_{G_2} \in [0,1]^n$

$$\|X_{G_1}\| \neq 0 \qquad \|X_{G_2}\| \neq 0$$

$$X_{G_1}^T X_{G_2} = 0$$

How to Solve Normalized-Cut?

Approximation 1

$$\max_{X_{G_1}} \quad \frac{X_{G_1}^T \ L \ X_{G_1}}{X_{G_1}^T \ X_{G_1}}$$

subject to: $X_{G_1} \in [0, 1]^n$

$$\|X_{G_1}\| \neq 0$$

How to Solve Normalized-Cut?

Approximation II

$$\max_{X_{G_1}} \quad \frac{X_{G_1}^T \; L \; X_{G_1}}{X_{G_1}^T \; X_{G_1}}$$

subject to: $\|X_{G_1}\| \neq 0$

How to Solve Normalized-Cut?

$$\max_{X_{G_1}} \quad \frac{X_{G_1}^T \; L \; X_{G_1}}{X_{G_1}^T \; X_{G_1}}$$

subject to: $$\|X_{G_1}\| \neq 0$$

How to Solve Normalized-Cut?

$$\max_{X_{G_1}} \quad \frac{X_{G_1}^T \ L \ X_{G_1}}{X_{G_1}^T \ X_{G_1}}$$

subject to: $\|X_{G_1}\| \neq 0$

Rayleigh-Ritz theorem

$X_{G_1} =$ eigenvector of the largest eigenvalue of *L*

How to Solve Normalized-Cut?

$$\max_{X_{G_1}, X_{G_2}} \frac{X_{G_1}^T \; L \; X_{G_1}}{X_{G_1}^T \; X_{G_1}} + \frac{X_{G_2}^T \; L \; X_{G_2}}{X_{G_2}^T \; X_{G_2}}$$

subject to: $X_{G_1} \in [0,1]^n \quad X_{G_2} \in [0,1]^n$

$$X_{G_1}^T X_{G_2} = 0$$

$X_{G_1} =$ eigenvector of the largest eigenvalue of *L*

$$X_{G_2} =?$$

How to Solve Normalized-Cut?

$$\max_{X_{G_1}, X_{G_2}} \frac{X_{G_1}^T \; L \; X_{G_1}}{X_{G_1}^T \; X_{G_1}} + \frac{X_{G_2}^T \; L \; X_{G_2}}{X_{G_2}^T \; X_{G_2}}$$

subject to: $X_{G_1} \in [0,1]^n \quad X_{G_2} \in [0,1]^n$

$$X_{G_1}^T X_{G_2} = 0$$

$X_{G_1} =$ eigenvector of the largest eigenvalue of *L*

$X_{G_2} =$ eigenvector of the second largest eigenvalue of *L*

Ncuts: General Case

cut

$$\min_{G_1, G_2, \ldots, G_K} \quad \sum_{k=1}^{K} \frac{\mathbf{1}_{G_k}^{\mathrm{T}} \ (D - A) \ \mathbf{1}_{G_k}}{\mathbf{1}_{G_k}^{\mathrm{T}} \ D \ \mathbf{1}_{G_k}}$$

volume

Ncuts: General Case

cut

$$\min_{G_1,G_2,\ldots,G_K} \sum_{k=1}^{K} \frac{\mathbf{1}_{G_k}^{\mathrm{T}} \, (D-A) \, \mathbf{1}_{G_k}}{\mathbf{1}_{G_k}^{\mathrm{T}} \, D \, \mathbf{1}_{G_k}}$$

volume

$\Downarrow$ relaxation

$$\max_{X_1,\ldots,X_K} \sum_{k=1}^{K} \frac{X_k^{\mathrm{T}} \, D^{-1} A \, X_k}{X_k^{\mathrm{T}} \, X_k}$$

s.t.

$$X_k \in [0,1]^n \qquad \forall k, l, \quad X_k^T X_l = 0$$

Ncuts: General Case

Solution:

K eigenvectors of $D^{-1}A$

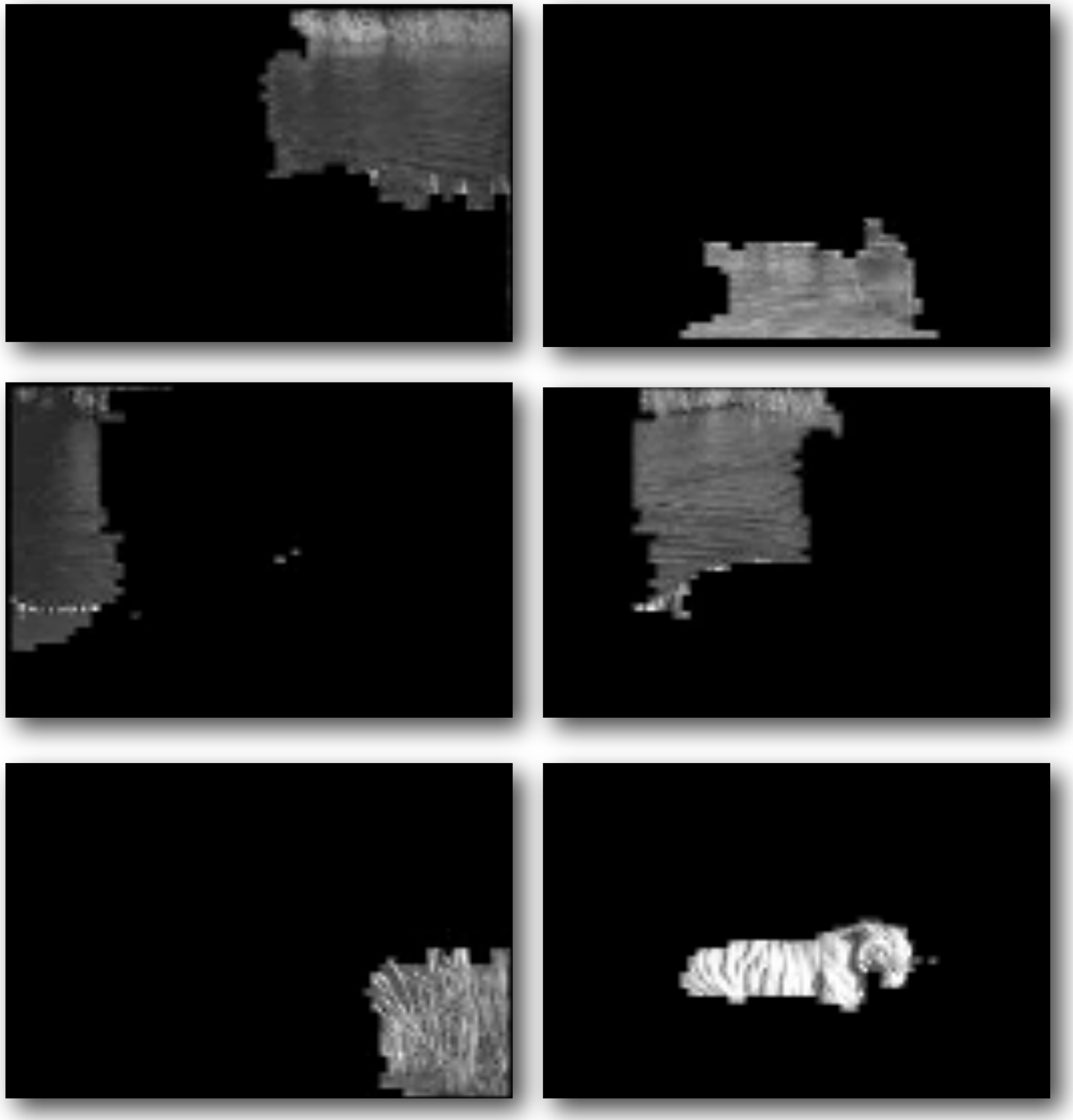
corresponding to

the K largest eigenvalues of $D^{-1}A$

Experiments -- Ncuts as Image Segmentation

input image

segments



Shi&Malik 1998

Experiments -- Ncuts as Image Segmentation

input
image

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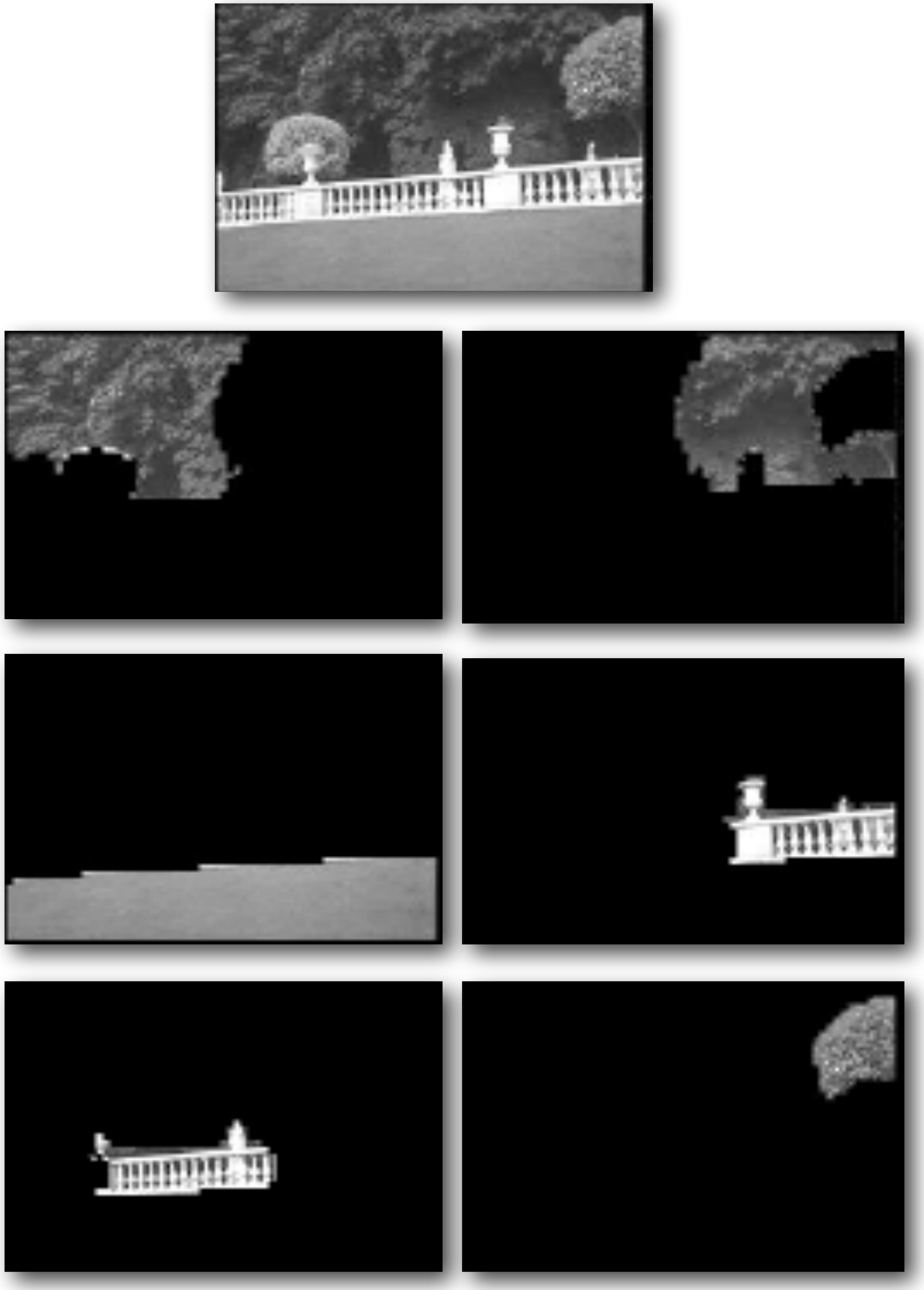

Shi&Malik 1998

Experiments -- Ncuts as Image Segmentation

source: Jianbo Shi

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