Markov Random Fields and Stochastic Image Models

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   (c) Simulation
   (d) Parameter estimation
4. Application of MRF's to Segmentation
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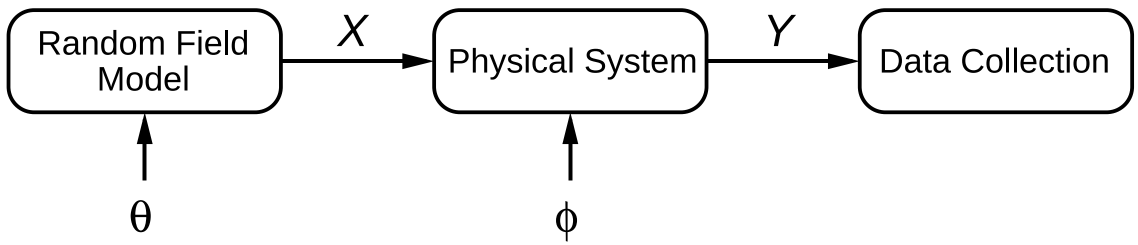
The Bayesian Approach

θ - Random field model parameters

X - Unknown image

ϕ - Physical system model parameters

Y - Observed data

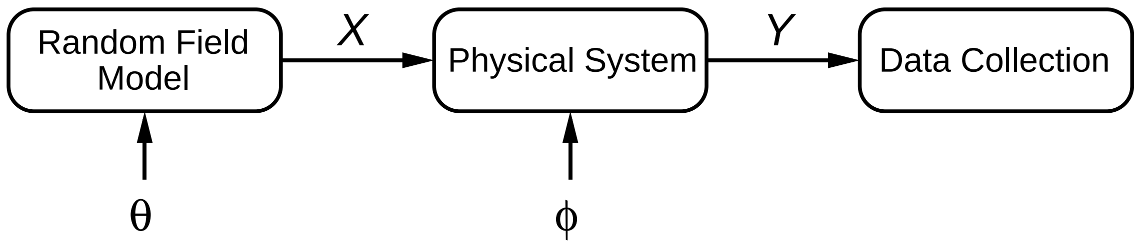


- Random field may model:
  - Achromatic/color/multispectral image
  - Image of discrete pixel classifications
  - Model of object cross-section
- Physical system may model:
  - Optics of image scanner
  - Spectral reflectivity of ground covers (remote sensing)
  - Tomographic data collection

Bayesian Versus Frequentist?

- How does the Bayesian approach differ?
  - Bayesian makes assumptions about prior behavior.
  - Bayesian requires that you choose a model.
  - A **good** prior model can improve accuracy.
  - But model mismatch can impair accuracy
- When should you use the frequentist approach?
  - When (# of data samples)>>(# of unknowns).
  - When an accurate prior model does not exist.
  - When prior model is not needed.
- When should you use the Bayesian approach?
  - When (# of data samples)≈(# of unknowns).
  - When model mismatch is tolerable.
  - When accuracy without prior is poor.

Examples of Bayesian Versus Frequentist?

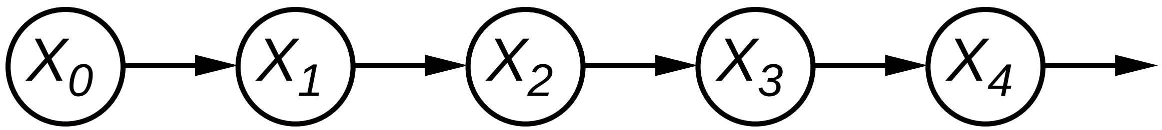


- Bayesian model of image X
  - (# of image points)$\approx$(# of data points.)
  - Images have unique behaviors which may be modeled.
  - Maximum likelihood estimation works poorly.
  - Reduce model mismatch by estimating parameter θ.
- Frequentist model for θ and ϕ
  - (# of model parameters)$<<$(# of data points.)
  - Parameters are difficult to model.
  - Maximum likelihood estimation works well.

Markov Chains

- Topics to be covered:
  - 1-D properties
  - Parameter estimation
  - 2-D Markov Chains
- **Notation:** Upper case $\Rightarrow$ Random variable

Markov Chains



- Definition of (homogeneous) Markov chains

$$p(x_n|x_i \; i < n) = p(x_n|x_{n-1})$$

- Therefore, we may show that the probability of a sequence is given by

$$p(x) = p(x_0) \prod_{n=1}^{N} p(x_n|x_{n-1})$$

- Notice: X_n is **not** independent of X_{n+1}

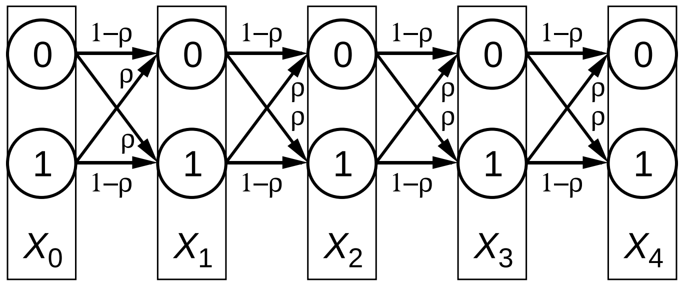
$$p(x_n|x_i \; i \neq n) = p(x_n|x_{n-1}, x_{n+1})$$

Parameters of Markov Chain

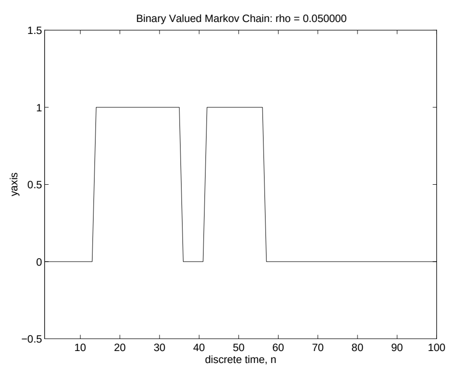
- Transition parameters are:

$$\theta_{j,i} = p(x_n = i | x_{n-1} = j)$$

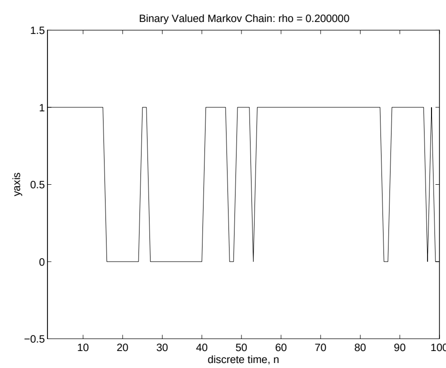
- Example: $\theta = \begin{bmatrix} 1-\rho & \rho \\ \rho & 1-\rho \end{bmatrix}$



- ρ is the probability of changing state.



$\rho = 0.05$



$\rho = 0.2$

Parameter Estimation for Markov Chains

- Maximum likelihood (ML) parameter estimation

$$\hat{\theta} = \arg\max_{\theta} p(x|\theta)$$

- For Markov chain

$$\hat{\theta}_{j,i} = \frac{h_{j,i}}{\sum_k h_{j,k}}$$

where $h_{j,i}$ is the histogram of transitions

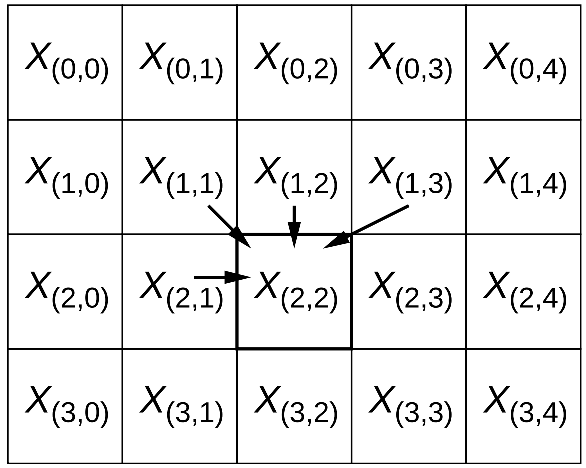
$$h_{j,i} = \sum_n \delta(x_n = i \ \& \ x_{n-1} = j)$$

- Example

$$x_n = 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1$$

$$\theta = \begin{bmatrix} h_{0,0} & h_{0,1} \\ h_{1,0} & h_{1,1} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 6 \end{bmatrix}$$

2-D Markov Chains



- Advantages:
  - Simple expressions for probability
  - Simple parameter estimation
- Disadvantages:
  - No natural ordering of pixels in image
  - Anisotropic model behavior

Discrete State Markov Random Fields

- Topics to be covered:
  - Definitions and theorems
  - 1-D MRF's
  - Ising model
  - M-Level model
  - Line process model

Markov Random Fields

- Noncausal model
- Advantages of MRF's
  - Isotropic behavior
  - Only local dependencies
- Disadvantages of MRF's
  - Computing probability is difficult
  - Parameter estimation is difficult
- Key theoretical result: Hammersley-Clifford theorem

Definition of Neighborhood System and Clique

- Define

  S - set of lattice points

  s - a lattice point, $s \in S$

  X_s - the value of X at s

  ∂s - the neighboring points of s

- A neighborhood system ∂s must be symmetric

$$r \in \partial s \Rightarrow s \in \partial r \ \ \text{also } s \notin \partial s$$

- A clique is a set of points, c, which are all neighbors of each other

$$\forall s, r \in c,\ r \in \partial s$$

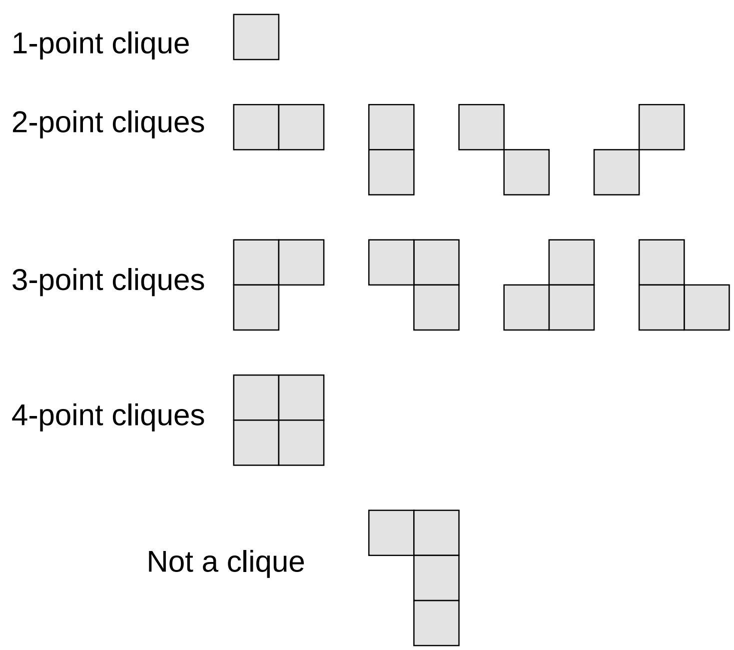
Example of Neighborhood System and Clique

- Example of 8 point neighborhood

$X_{(0,0)}$	$X_{(0,1)}$	$X_{(0,2)}$	$X_{(0,3)}$	$X_{(0,4)}$
$X_{(1,0)}$	$X_{(1,1)}$	$X_{(1,2)}$	$X_{(1,3)}$	$X_{(1,4)}$
$X_{(2,0)}$	$X_{(2,1)}$	$X_{(2,2)}$	$X_{(2,3)}$	$X_{(2,4)}$
$X_{(3,0)}$	$X_{(3,1)}$	$X_{(3,2)}$	$X_{(3,3)}$	$X_{(3,4)}$
$X_{(4,0)}$	$X_{(4,1)}$	$X_{(4,2)}$	$X_{(4,3)}$	$X_{(4,4)}$

Neighbors of $X_{(2,2)}$

- Example of cliques for 8 point neighborhood



Gibbs Distribution

x_c - The value of X at the points in clique c.

$V_c(x_c)$ - A potential function is any function of x_c.

- A (discrete) density is a Gibbs distribution if

$$p(x) = \frac{1}{Z} \exp\left\{ -\sum_{c \in \mathcal{C}} V_c(x_c) \right\}$$

  $\mathcal{C}$ is the set of all cliques

  Z is the normalizing constant for the density.

- Z is known as the **partition function**.
- $U(x) = \sum_{c \in \mathcal{C}} V_c(x_c)$ is known as the **energy function**.

Markov Random Field

- Definition: A random object X on the lattice S with neighborhood system ∂s is said to be a Markov random field if for all $s \in S$

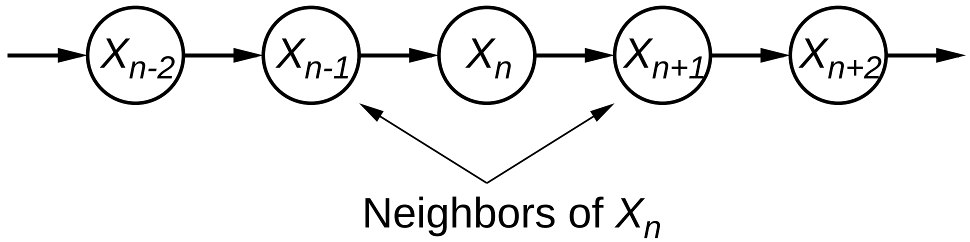
$$p(x_s | x_r \text{ for } r \neq s) = p(x_s | x_{\partial r})$$

Hammersley-Clifford Theorem[14]

$$\left(\begin{array}{c} X \text{ is a Markov random field} \\ \& \\ \forall x,\ P\{X = x\} > 0 \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} P\{X = x\} \text{ has the form} \\ \text{of a Gibbs distribution} \end{array} \right)$$

- Gives you a method for writing the density for a MRF
- Does not give the value of Z, the partition function.
- Positivity, $P\{X = x\} > 0$, is a technical condition which we will generally assume.

Markov Chains are MRF's



- Neighbors of n are $\partial n = \{n-1, n+1\}$
- Cliques have the form $c = \{n-1, n\}$
- Density has the form

$$\begin{aligned} p(x) &= p(x_0) \prod_{n=1}^{N} p(x_n | x_{n-1}) \\ &= p(x_0) \exp\left\{ \sum_{n=1}^{N} \log p(x_n | x_{n-1}) \right\} \end{aligned}$$

- The potential functions have the form

$$V(x_n, x_{n-1}) = \log p(x_n | x_{n-1})$$

1-D MRF's are Markov Chains

- Let X_n be a 1-D MRF with $\partial n = \{n-1, n+1\}$
- The discrete density has the form of a Gibbs distribution

$$p(x) = p(x_0) \exp\left\{ \sum_{n=1}^{N} V(x_n, x_{n-1}) \right\}$$

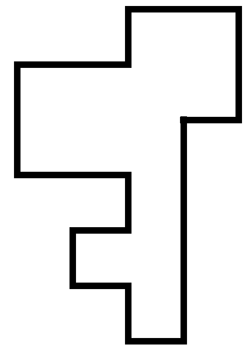
- It may be shown that this is a Markov Chain.
- Transition probabilities may be difficult to compute.

The Ising Model: A 2-D MRF[100]

0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0
0	0	0	1	1	1	1	0
0	0	0	1	1	1	0	0
0	0	0	0	0	1	0	0
0	0	0	0	1	1	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0

Cliques: $\boxed{X_r \mid X_s}$ $\quad \begin{array}{|c|}\hline X_r \\ \hline X_s \\ \hline\end{array}$

Boundary:



- Potential functions are given by

$$V(x_r, x_s) = \beta\delta(x_r \neq x_s)$$

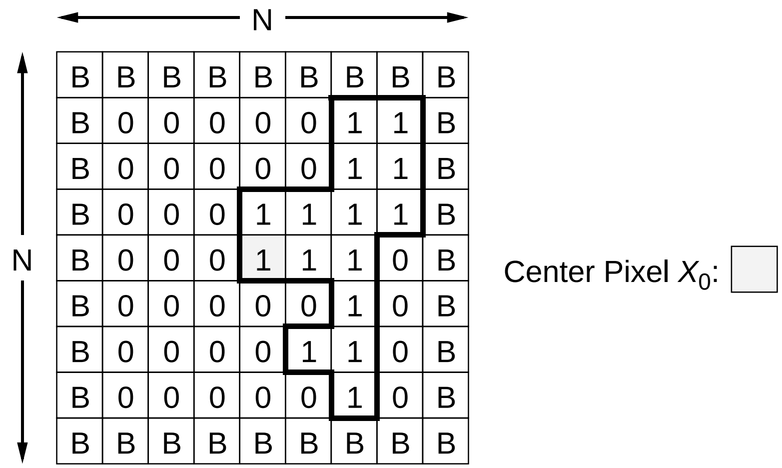
where β is a model parameter.

- Energy function is given by

$$\sum_{c \in \mathcal{C}} V_c(x_c) = \beta(\text{Boundary length})$$

- Longer boundaries $\Rightarrow$ less probable

Critical Temperature Behavior[127, 126, 100]



- $\frac{1}{\beta}$ is analogous to temperature.
- Peierls showed that for $\beta > \beta_c$

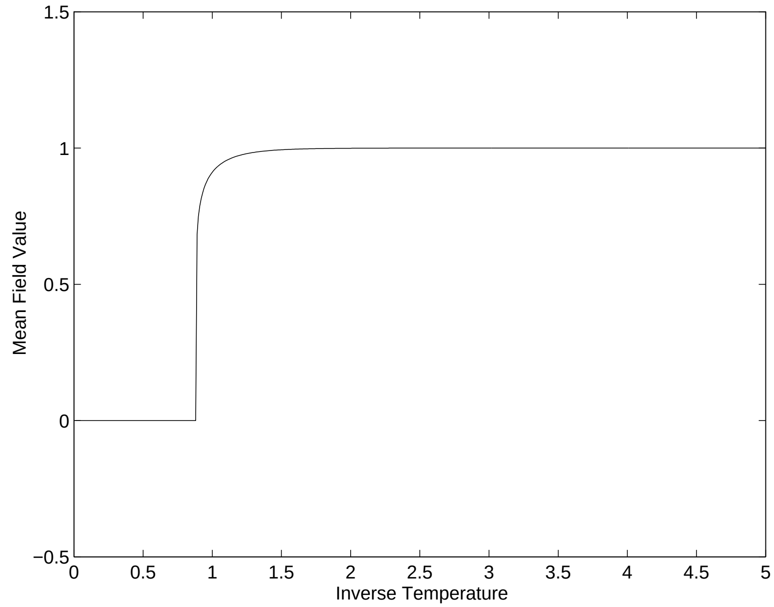
$$\lim_{N \to \infty} P(X_0 = 0 | B = 0) \neq \lim_{N \to \infty} P(X_0 = 0 | B = 1)$$

- The effect of the boundary does not diminish as $N \to \infty$!
- $\beta_c \approx .88$ is known as the critical temperature.

Critical Temperature Analysis[122]

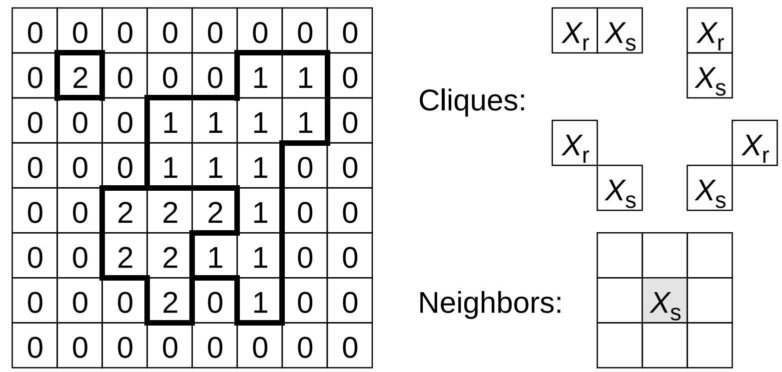
- Amazingly, Onsager was able to compute

$$E[X_0|B=1] = \begin{cases} \left(1 - \frac{1}{(\sinh(\beta))^4}\right)^{1/8} & \text{if } \beta > \beta_c \\ 0 & \text{if } \beta < \beta_c \end{cases}$$



- Onsager also computed an analytic expression for $Z(T)$!

M-Level MRF[16]



- Define $\mathcal{C}_1 \triangleq$ (hor./vert. cliques) and $\mathcal{C}_2 \triangleq$ (diag. cliques)
- Then

$$V(x_r, x_s) = \begin{cases} \beta_1 \delta(x_r \neq x_s) & \text{for } \{x_r, x_s\} \in \mathcal{C}_1 \\ \beta_2 \delta(x_r \neq x_s) & \text{for } \{x_r, x_s\} \in \mathcal{C}_2 \end{cases}$$

- Define

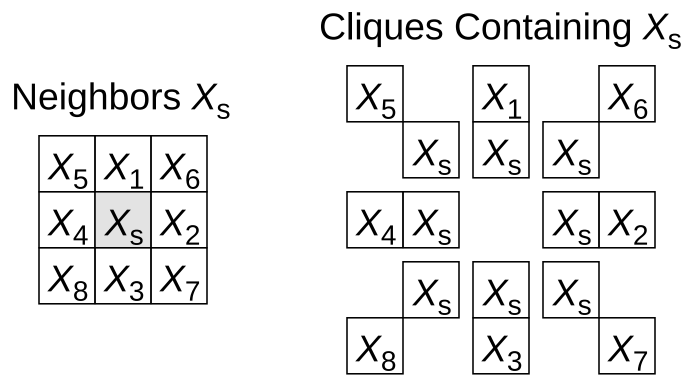
$$t_1(x) \triangleq \sum_{\{s,r\} \in \mathcal{C}_1} \delta(x_r \neq x_s)$$

$$t_2(x) \triangleq \sum_{\{s,r\} \in \mathcal{C}_2} \delta(x_r \neq x_s)$$

- Then the probability is given by

$$p(x) = \frac{1}{Z} \exp\left\{-(\beta_1 t_1(x) + \beta_2 t_2(x))\right\}$$

Conditional Probability of a Pixel



- The probability of a pixel given all other pixels is

$$p(x_s|x_{i\neq s}) = \frac{\frac{1}{Z}\exp\left\{-\Sigma_{c\in\mathcal{C}} V_c(x_c)\right\}}{\Sigma_{x_s=0}^{M-1}\frac{1}{Z}\exp\left\{-\Sigma_{c\in\mathcal{C}} V_c(x_c)\right\}}$$

- Notice: Any term $V_c(x_c)$ which does not include x_s cancels.

$$p(x_s|x_{i\neq s}) = \frac{\exp\left\{-\beta_1\Sigma_{i=1}^{4}\delta(x_s\neq x_i)-\beta_2\Sigma_{i=5}^{8}\delta(x_s\neq x_i)\right\}}{\Sigma_{x_s=0}^{M-1}\exp\left\{-\beta_1\Sigma_{i=1}^{4}\delta(x_s\neq x_i)-\beta_2\Sigma_{i=5}^{8}\delta(x_s\neq x_i)\right\}}$$

Conditional Probability of a Pixel (Continued)

Neighbors X_s

1	1	0
1	X_s	0
0	0	0

$V_1(0, x_{\partial s}) = 2 \quad V_2(0, x_{\partial s}) = 1$

$V_1(1, x_{\partial s}) = 2 \quad V_2(1, x_{\partial s}) = 3$

- Define

$$\begin{aligned} v_1(x_s, \partial x_s) &\triangleq \# \text{ of horz./vert. neighbors} \neq x_s \\ v_2(x_s, \partial x_s) &\triangleq \# \text{ of diag. neighbors} \neq x_s \end{aligned}$$

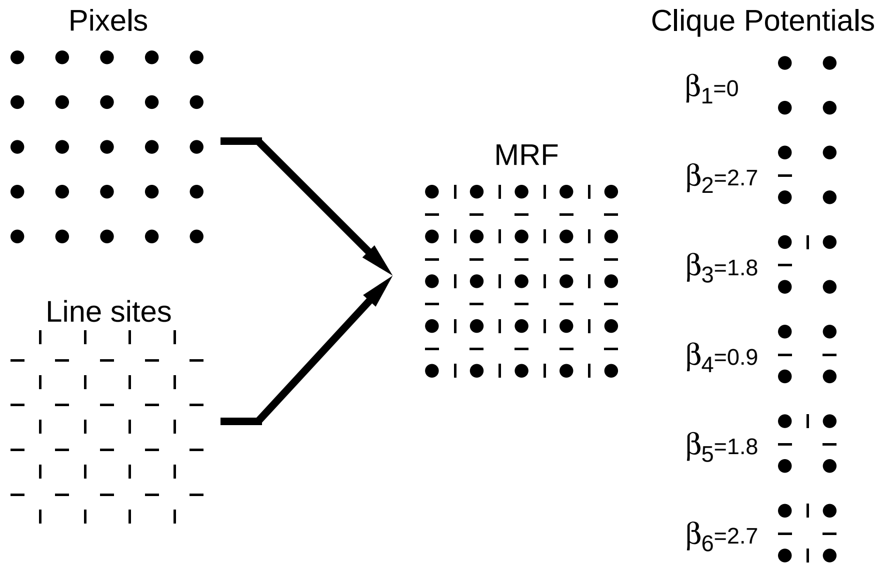
- Then

$$p(x_s | x_{i \neq s}) = \frac{1}{Z'} \exp\left\{ -\beta_1 v_1(x_s, \partial x_s) - \beta_2 v_2(x_s, \partial x_s) \right\}$$

where Z' is an easily computed normalizing constant

- When $\beta_1, \beta_2 > 0$, X_s is most likely to be the majority neighboring class.

Line Process MRF [68]



- Line sites fall between pixels
- The values $\beta_1, \cdots, \beta_2$ determine the potential of line sites
- The potential of pixel values is

$$V(x_s, x_r, l_{r,s}) = \begin{cases} (x_s - x_r)^2 & \text{if } l_{r,s} = 0 \\ 0 & \text{if } l_{r,s} = 1 \end{cases}$$

- The field is
  - Smooth between line sites
  - Discontinuous at line sites

Simulation

- Topics to be covered:
  - Metropolis sampler
  - Gibbs sampler
  - Generalized Metropolis sampler

Generating Samples from a Gibbs Distribution

- How do we generate a random variable X with a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\{-U(x)\}$$

- Generally, this problem is difficult.
- Markov Chains can be generated sequentially
- Non-causal structure of MRF's makes simulation difficult.

The Metropolis Sampler[118, 100]

- How do we generate a sample from a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\{-U(x)\}$$

- Start with the sample x^k, and generate a new sample W with probability $q(w|x^k)$.

  Note: $q(w|x^k)$ must be symmetric.

$$q(w|x^k) = q(x^k|w)$$

- Compute $\Delta E(W) = U(W) - U(x^k)$, then do the following:

  If $\Delta E(W) < 0$

  – Accept: $X^{k+1} = W$

  If $\Delta E(W) \geq 0$

  – Accept: $X^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
  – Reject: $X^{k+1} = x^k$ with probability $1 - \exp\{-\Delta E(W)\}$

Ergodic Behavior of Metropolis Sampler

- The sequence of random fields, X^k, form a Markov chain.
- Let $p(x^{k+1}|x^k)$ be the transition probabilities of the Markov chain.
- Then X^k is reversible

$$p(x^{k+1}|x^k)\exp\{-U(x^k)\} = \exp\{-U(x^{k+1})\}p(x^k|x^{k+1})$$

- Therefore, if the Markov chain is irreducible, then

$$\lim_{k\to\infty} P\{X^k = x\} = \frac{1}{Z}\exp\{-U(x)\}$$

- If every state can be reached, then as $k \to \infty$, X^k will be a sample from the Gibbs distribution.

Example Metropolis Sampler for Ising Model

	0	
1	x_s	0
	0	

- Assume $x_s^k = 0$.
- Generate a binary R.V., W, such that $P\{W = 0\} = 0.5$.

$$\begin{aligned} \Delta E(W) &= U(W) - U(x_s^k) \\ &= \begin{cases} 0 & \text{if } W = 0 \\ 2\beta & \text{if } W = 1 \end{cases} \end{aligned}$$

If $\Delta E(W) < 0$

- Accept $X_s^{k+1} = W$

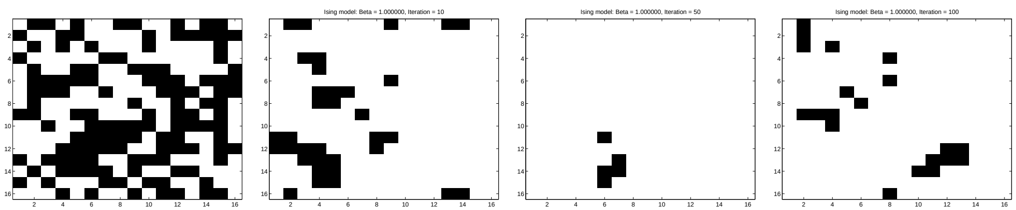
If $\Delta E(W) \geq 0$

- Accept: $X_s^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
- Reject: $X_s^{k+1} = x_s^k$ with probability $1 - \exp\{-\Delta E(W)\}$

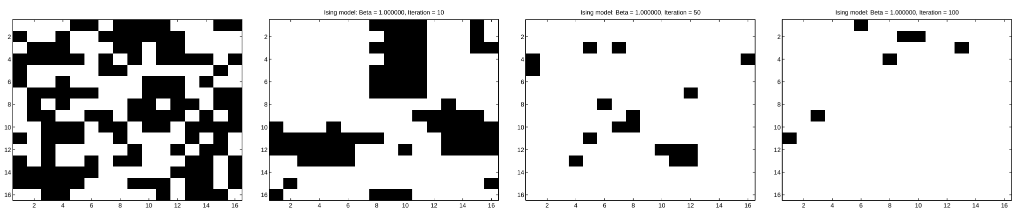
- Repeat this procedure for each pixel.
- **Warning:** for $\beta > \beta_c$ convergence can be extremely slow!

Example Simulation for Ising Model($\beta = 1.0$)

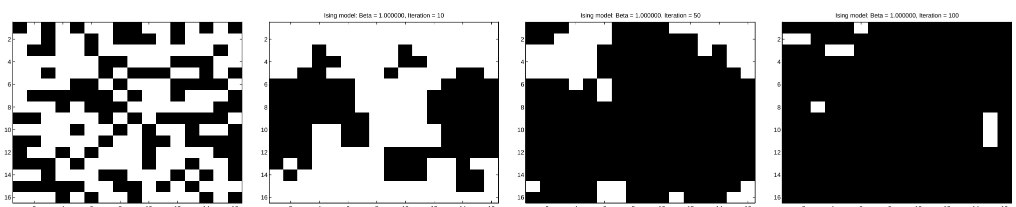
- Test 1



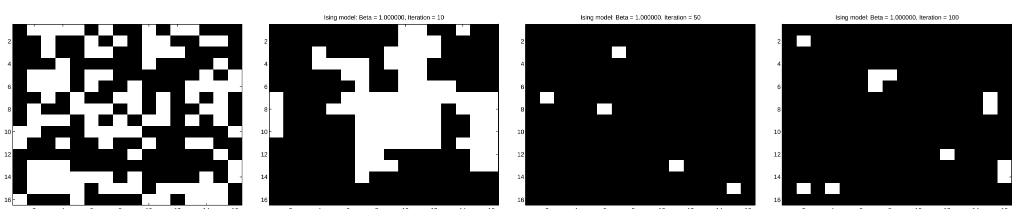
- Test 2



- Test 3



- Test 3



Advantages and Disadvantages of Metropolis Sampler

- Advantages
  - Can be implemented whenever ΔE is easy to compute.
  - Has guaranteed geometric convergence.
- Disadvantages
  - Can be slow if there are many rejections.
  - Is constrained to use a symmetric transition function $q(x^{k+1}|x^k)$.

Gibbs Sampler[68]

- Replace each point with a sample from its conditional distribution

$$p(x_s|x_i^k \; i \neq s) = p(x_s|x_{\partial s})$$

- Scan through all the points in the image.
- Advantage
  - Eliminates need for rejections $\Rightarrow$ faster convergence
- Disadvantage
  - Generating samples from $p(x_s|x_{\partial s})$ can be difficult.

Generalized Metropolis Sampler[80, 129]

- Hastings and Peskun generalized the Metropolis sampler for transition functions $q(w|x^k)$ which are not symmetric.
- The acceptance probability is then

$$\alpha(x_s^k, w) = \min\left\{1, \frac{q(x^k|w)}{q(w|x^k)} \exp\{-\Delta E(w)\}\right\}$$

- Special cases

$$q(w|x^k) = q(x^k|z) \Rightarrow \text{conventional Metropolis}$$
$$q(w_s|x^k) = p(x_s^k|x_{\partial s}^k)\big|_{x_s^k = w_s} \Rightarrow \text{Gibbs sampler}$$

- Advantage
  – Transition function may be chosen to minimize rejections[76]

Parameter Estimation for Discrete State MRF's

- Topics to be covered:
  - Why is it difficult?
  - Coding/maximum pseudolikehood
  - Least squares

Why is Parameter Estimation Difficult?

- Consider the ML estimate of β for an Ising model.
- Remember that

$$t_1(x) = (\#\text{ horz. and vert. neighbors of different value.})$$

- Then the ML estimate of β is

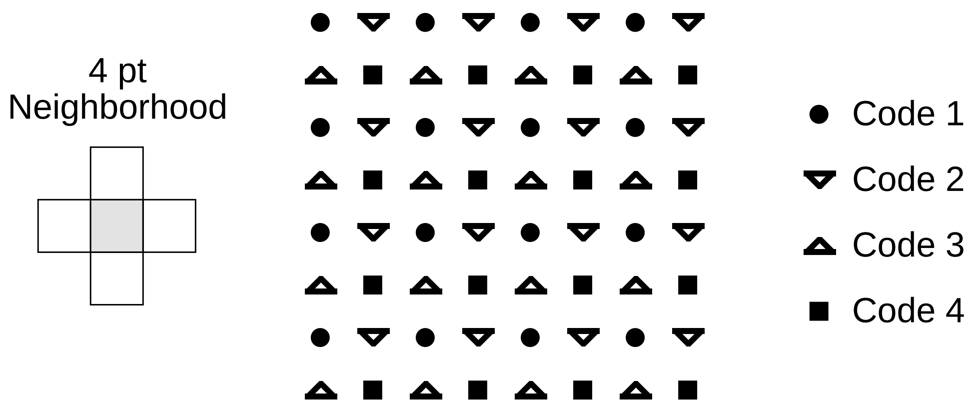
$$\begin{aligned}\hat{\beta} &= \arg\max_{\beta} \left\{ \frac{1}{Z(\beta)} \exp\left\{-\beta t_1(x)\right\} \right\} \\ &= \arg\max_{\beta} \left\{-\beta t_1(x) - \log Z(\beta)\right\}\end{aligned}$$

- However, $\log Z(\beta)$ has an intractable form

$$\log Z(\beta) = \log \sum_x \exp\left\{-\beta t_1(x)\right\}$$

- Partition function can not be computed.

Coding Method/Maximum Pseudolikelihood[15, 16]



- Assume a 4 point neighborhood
- Separate points into four groups or codes.
- Group (code) contains points which are conditionally independent given the other groups (codes).

$$\hat{\beta} = \arg\max_{\beta} \prod_{s \in \mathrm{Code}_k} p(x_s | x_{\partial s})$$

- This is tractable (but not necessarily easy) to compute

Least Squares Parameter Estimation[49]

- It can be shown that for an Ising model
$$\log \frac{P\{X_s = 1 | x_{\partial s}\}}{P\{X_s = 0 | x_{\partial s}\}} = -\beta \left(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}) \right)$$
- For each unique set of neighboring pixel values, $x_{\partial s}$, we may compute
  - The observed rate of $\log \frac{P\{X_s=1|x_{\partial s}\}}{P\{X_s=0|x_{\partial s}\}}$
  - The value of $(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}))$
  - This produces a set of over-determined linear equations which can be solved for β.
- This least squares method is easily implemented.

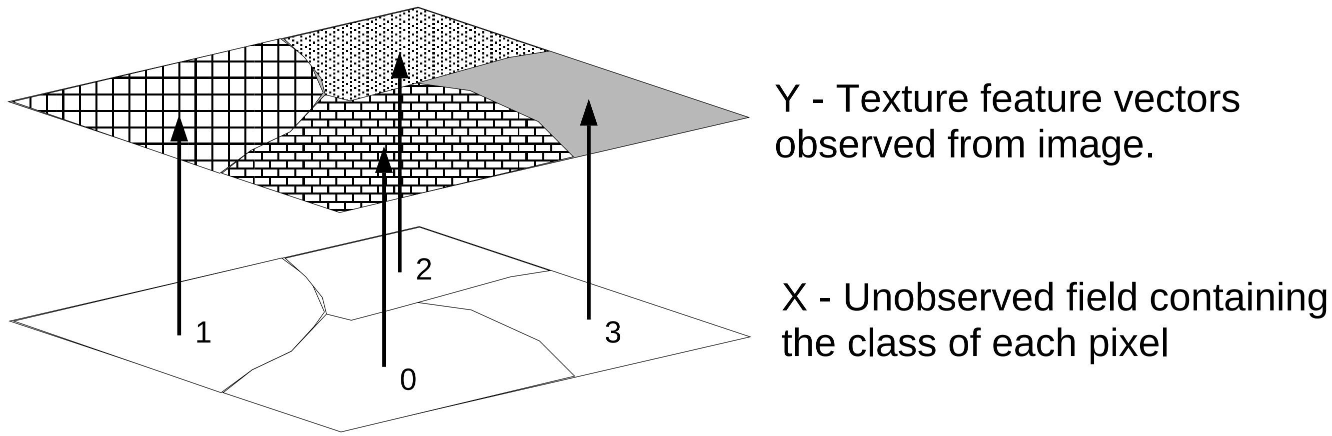
Theoretical Results in Parameter Estimation for MRF's

- Inconsistency of ML estimate for Ising model[130, 131]
  - Caused by critical temperature behavior.
  - Single sample of Ising model cannot distinguish between high β with mean $1/2$, and low β with large mean.
  - Not identifiable
- Consistency of maximum pseudolikelihood estimate[69]
  - Requires an identifiable parameterization.

Application of MRF's to Segmentation

- Topics to be covered:
  - The Model
  - Bayesian Estimation
  - MAP Optimization
  - Parameter Estimation
  - Other Approaches

Bayesian Segmentation Model



- Discrete MRF is used to model the segmentation field.
- Each class is represented by a value $X_s \in \{0, \cdots, M-1\}$
- The joint probability of the data and segmentation is

$$P\{Y \in dy, X = x\} = p(y|x)p(x)$$

where

- $p(y|x)$ is the data model
- $p(x)$ is the segmentation model

Bayes Estimation

- $C(x, X)$ is the cost of guessing x when X is the correct answer.
- $\hat{X}$ is the estimated value of X.
- $E[C(\hat{X}, X)]$ is the expected cost (risk).

- Objective: Choose the estimator $\hat{X}$ which minimizes $E[C(\hat{X}, X)]$.

Maximum *A Posteriori* (MAP) Estimation

- Let $C(x, X) = \delta(x \neq X)$
- Then the optimum estimator is given by

$$
\begin{aligned}
\hat{X}_{MAP} &= \arg\max_x p_{x|y}(x|Y) \\
&= \arg\max_x \log \frac{p_{y,x}(Y, x)}{p_y(Y)} \\
&= \arg\max_x \{\log p(Y|x) + \log p(x)\}
\end{aligned}
$$

- Advantage:
  - Can be computed through direct optimization
- Disadvantage:
  - Cost function is unreasonable for many applications

Maximizer of the Posterior Marginals (MPM) Estimation[116]

- Let $C(x, X) = \sum_{s \in S} \delta(x_s \neq X_s)$
- Then the optimum estimator is given by

$$\hat{X}_{MPM} = \arg\max_{x_s} p_{x_s|Y}(x_s|Y)$$

- Compute the most likely class for each pixel
- Method:
  - Use simulation method to generate samples from $p_{x|y}(x|y)$.
  - For each pixel, choose the most frequent class.
- Advantage:
  - Minimizes number of misclassified pixels
- Disadvantage:
  - Difficult to compute

MAP Optimization for Segmentation

- Assume the data model

$$p_{y|x}(y|x) = \prod_{s \in S} p(y_s|x_s)$$

- And the prior model (Ising model)

$$p_x(x) = \frac{1}{Z'} \exp\{-\beta t_1(x)\}$$

- Then the MAP estimate has the form

$$\hat{x} = \arg\min_x \left\{-\log p_{y|x}(y|x) + \beta t_1(x)\right\}$$

- This optimization problem is very difficult

Iterated Conditional Modes [16]

- The problem:

$$\hat{x}_{MAP} = \arg\min_x \left\{ - \sum_{s \in S} \log p_{y_s|x_s}(y_s|x_s) + \beta t_1(x) \right\}$$

- Iteratively minimize the function with respect to each pixel, x_s.

$$\hat{x}_s = \arg\min_{x_s} \left\{ - \log p_{y_s|x_s}(y_s|x_s) + \beta v_1(x_s|x_{\partial s}) \right\}$$

- This converges to a local minimum in the cost function

Simulated Annealing [68]

- Consider the Gibbs distribution

$$\frac{1}{Z}\exp\left\{-\frac{1}{T}U(x)\right\}$$

where

$$U(x) = \sum_{s\in S} \log p_{y_s|x_s}(y_s|x_s) + \beta t_1(x)$$

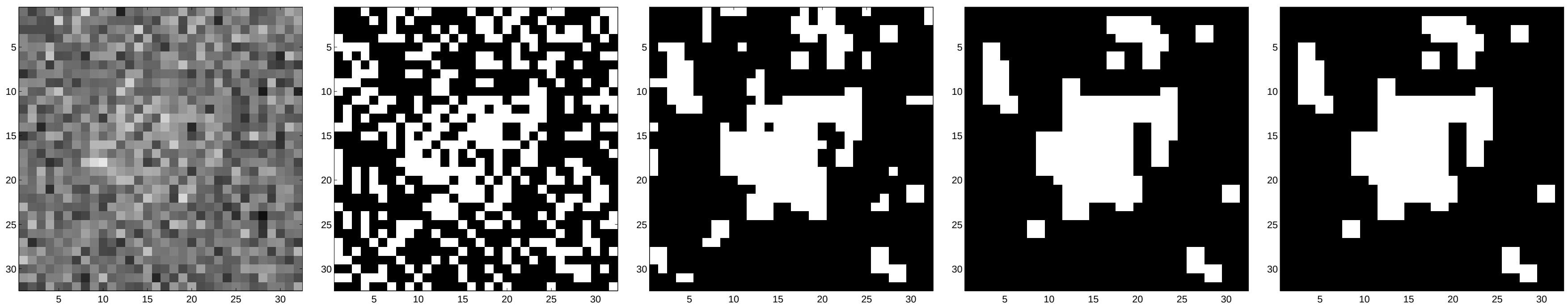
- As $T \rightarrow 0$, the distribution becomes clustered about $\hat{x}_{MAP}$.
- Use simulation method to generate samples from distribution.
- Slowly let $T \rightarrow 0$.
- If $T_k = \frac{T_1}{1+\log k}$ for iteration k, the the simulation converges to $\hat{x}_{MAP}$ almost surely.
- Problem: This is very slow!

Multiscale MAP Segmentation

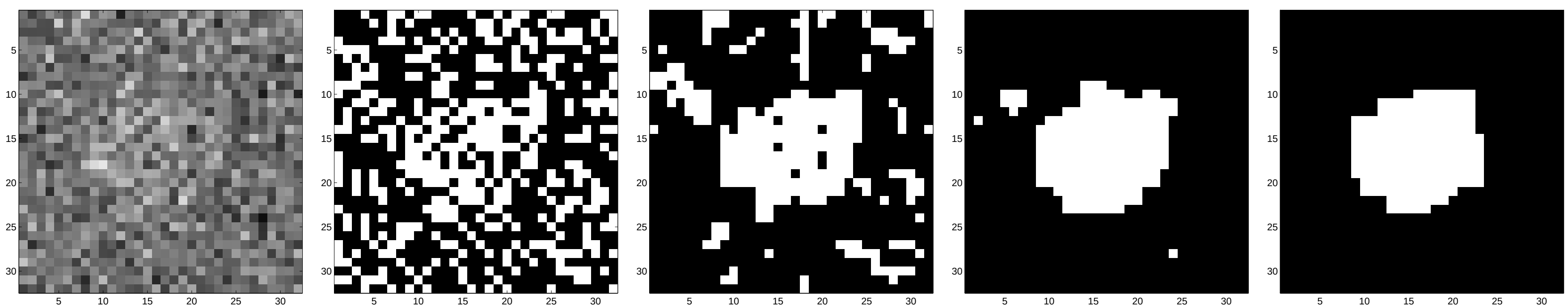
- Renormalization theory[72]
  - Theoretically results in the exact MAP segmentation
  - Requires the computation of intractable functions
  - Can be implemented with approximation
- Multiscale resolution segmentation[23]
  - Performs ICM segmentation in a coarse-to-fine sequence
  - Each MAP optimization is initialized with the solution from the previous coarser resolution
  - Used the fact that a discrete MRF constrained to be block constant is still a MRF.
- Multiscale Markov random fields[97]
  - Extended MRF to the third dimension of scale
  - Formulated a parallel computational approach

Segmentation Example

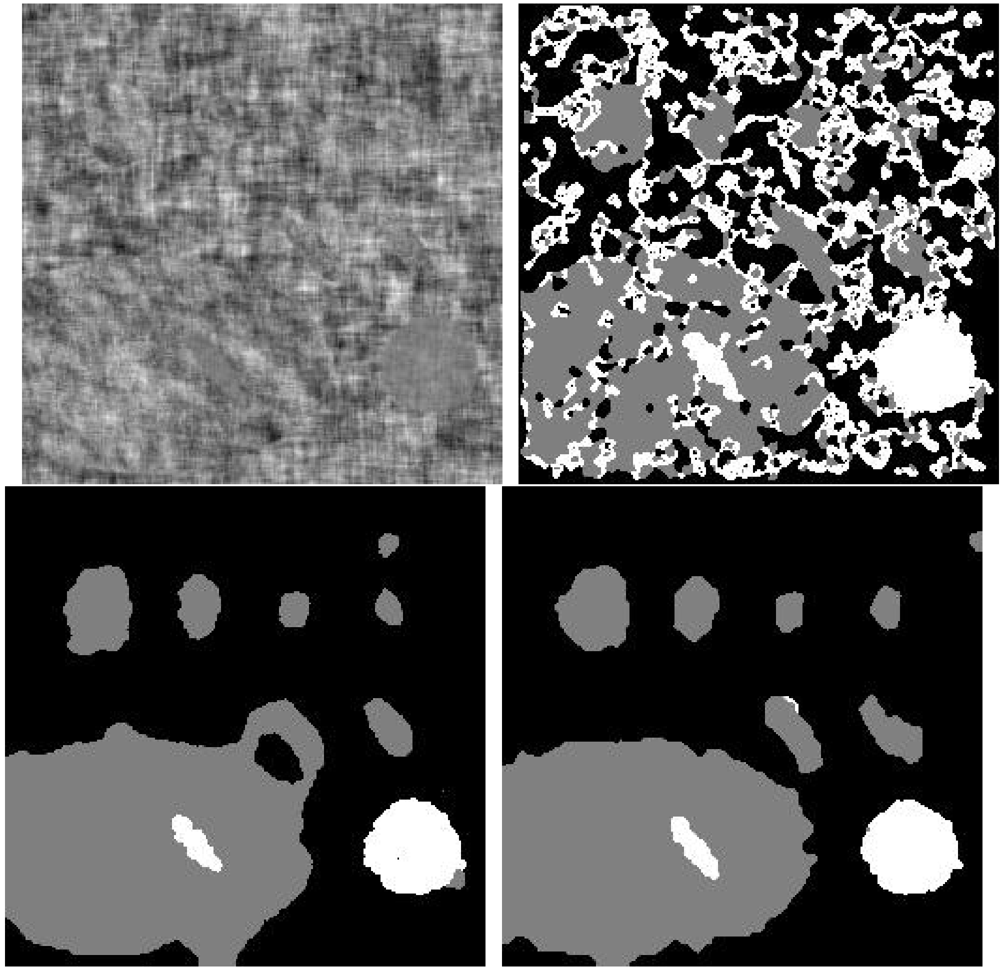
- Iterated Conditional Modes (ICM): ML ; ICM 1; ICM 5; ICM 10



- Simulated Annealing (SA): ML ; SA 1; SA 5; SA 10



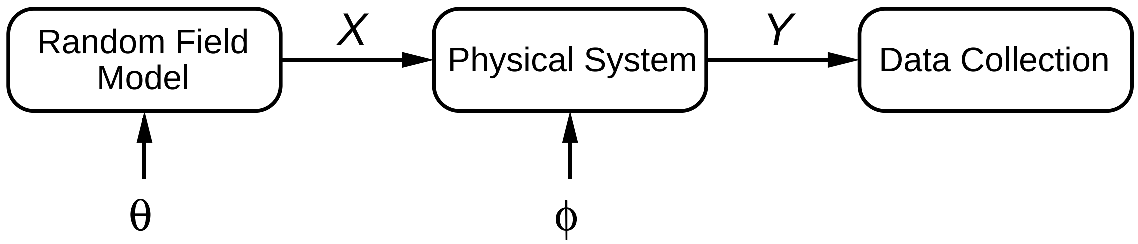
Texture Segmentation Example



a	b
c	d

a) Synthetic image with 3 textures b) ICM - 29 iterations c) Simulated Annealing - 100 iterations d) Multiresolution - 7.8 iterations

Parameter Estimation



- Question: How do we estimate θ from Y?
- Problem: We don't know X!
- Solution 1: Joint MAP estimation [104]

$$(\hat{\theta}, \hat{x}) = \arg\max_{\theta, x} p(y, x|\theta)$$

  – Problem: The solution is biased.

- Solution 2: Expectation maximization algorithm [9, 70]

$$\hat{\theta}^{k+1} = \arg\max_{\theta} E[\log p(Y, X|\theta)|Y = y, \theta^k]$$

  – Expectation may be computed using simulation techniques or mean field theory.

Other Approaches to using Discrete MRFs

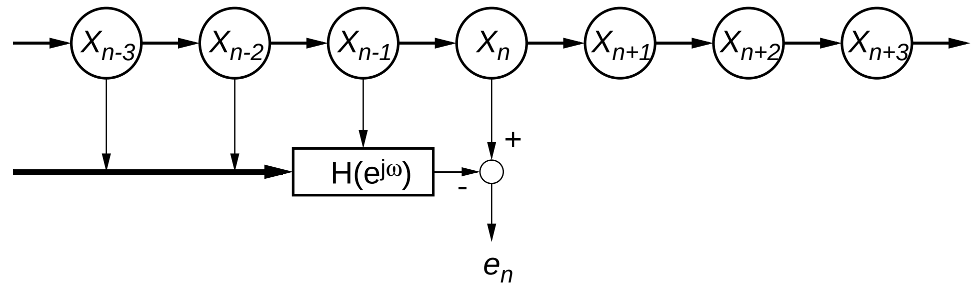
- Dynamic programming does not work in 2-D, but a number of researchers have formulated approximate recursive solutions to MAP estimation[48, 169].
- Mean field theory has also been studied as a method for computing the MPM estimate[176].

Gaussian Random Process Models

- Topics to be covered:
  - Autoregressive (AR) models
  - Simultaneous Autoregressive (SAR) models
  - Gaussian MRF's
  - Generalization to 2-D

Autoregressive (AR) Models

$$e_n = x_n - \sum_{k=1}^{\infty} x_{n-k} h_k$$



- $H(e^{j\omega})$ is an optimal predictor $\Rightarrow e(n)$ is white noise.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{2} x^t \mathbf{A}^t \mathbf{A} x \right\}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & & -h_{m-n} \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

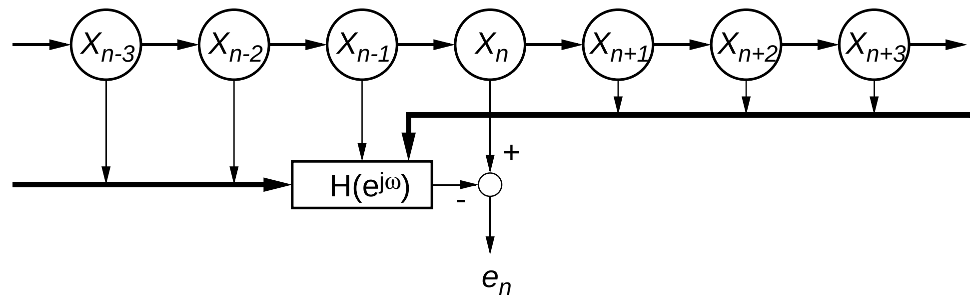
$$Z = (2\pi)^{N/2} |\mathbf{A}|^{-1} = (2\pi)^{N/2}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{|1 - H(e^{j\omega})|^2}$$

Simultaneous Autoregressive (SAR) Models[95, 94]

$$e_n = x_n - \sum_{k=1}^{\infty} (x_{n-k} - x_{n+k}) h_k$$



- $e(n)$ is white noise $\Rightarrow H(e^{j\omega})$ is **not** an optimal non-causal predictor.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{2} x^t \mathbf{A}^t \mathbf{A} x \right\}$$

where

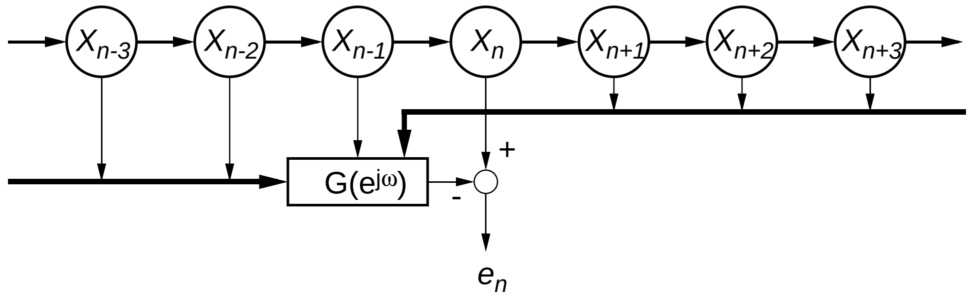
$$\mathbf{A} = \begin{bmatrix} 1 & & -h_{m-n} \\ & \ddots & \\ -h_{n-m} & & 1 \end{bmatrix}$$

$$Z = (2\pi)^{N/2} |\mathbf{A}|^{-1} \approx (2\pi)^{N/2} \exp\left\{ -\frac{N}{2\pi} \int_{-\pi}^{\pi} \log |1 - H(e^{j\omega})| d\omega \right\}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{|1 - H(e^{j\omega})|^2}$$

Conditional Markov (CM) Models (i.e. MRF's)[95, 94]



$$e_n = x_n - \sum_{k=1}^{\infty}(x_{n-k} - x_{n+k})g_k$$

- $G(e^{j\omega})$ is an optimal non-causal predictor $\Rightarrow$ $e(n)$ is **not** white noise.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z}\exp\left\{-\frac{1}{2}x^t\mathbf{B}x\right\}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & & -g_{m-n} \\ & \ddots & \\ -g_{n-m} & & 1 \end{bmatrix}$$

$$Z = (2\pi)^{N/2}|\mathbf{B}|^{-1/2} \approx (2\pi)^{N/2}\exp\left\{-\frac{N}{4\pi}\int_{-\pi}^{\pi}\log(1 - G(e^{j\omega}))d\omega\right\}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{1 - G(e^{j\omega})}$$

Generalization to 2-D

- Same basic properties hold.
- Circulant matrices become circulant block circulant.
- Toeplitz matrices become Toeplitz block Toeplitz.
- SAR and MRF models are more important in 2-D.

Non-Gaussian Continuous State MRF's

- Topics to be covered:
  - Quadratic functions
  - Non-Convex functions
  - Continuous MAP estimation
  - Convex functions

Why use Non-Gaussian MRF's?

- Gaussian MRF's do not model edges well.
- In applications such as image restoration and tomography, Gaussian MRF's either
  - Blur edges
  - Leave excessive amounts of noise

Gaussian MRF's

- Gaussian MRF's have density functions with the form

$$p(x) = \frac{1}{Z} \exp\left\{ -\sum_{s\in S} a_s x_s^2 - \sum_{\{s,r\}\in C} b_{sr} |x_s - x_r|^2 \right\}$$

- We will assume $a_s = 0$.
- The terms $|x_s - x_r|^2$ penalize rapid changes in gray level.
- MAP estimate has the form

$$\hat{x} = \arg\min_x \left\{ -\log p(y|x) + \sum_{\{s,r\}\in C} b_{sr} |x_s - x_r|^2 \right\}$$

- **Problem:** Quadratic function, $|\cdot|^2$, excessively penalizes image edges.

Non-Gaussian MRF's Based on Pair-Wise Cliques

- We will consider MRF's with pair-wise cliques

$$p(x) = \frac{1}{Z} \exp\left\{ - \sum_{\{s,r\} \in C} b_{sr} \rho\left(\frac{x_s - x_r}{\sigma} \right) \right\}$$

$|x_s - x_r|$ - is the change in gray level.

σ - controls the gray level variation or scale.

$\rho(\Delta)$:

– Known as the potential function.

– Determines the cost of abrupt changes in gray level.

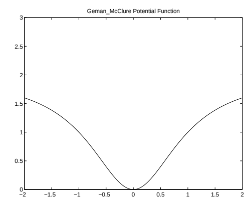
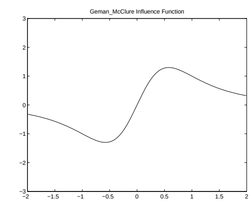
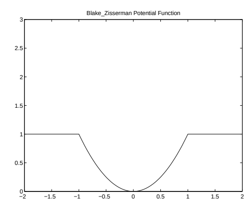
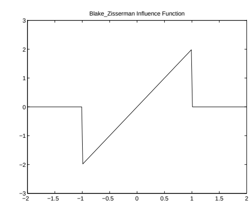
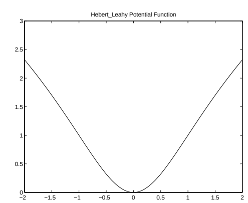
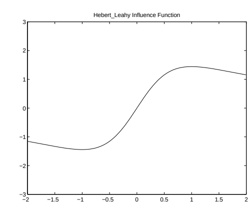
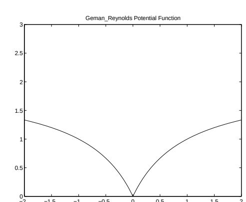
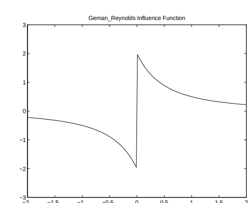
– $\rho(\Delta) = |\Delta|^2$ is the Gaussian model.

$\rho'(\Delta) = \frac{d\rho(\Delta)}{d\Delta}$:

– Known as the influence function from "M-estimation"[139, 85].

– Determines the attraction of a pixel to neighboring gray levels.

Non-Convex Potential Functions

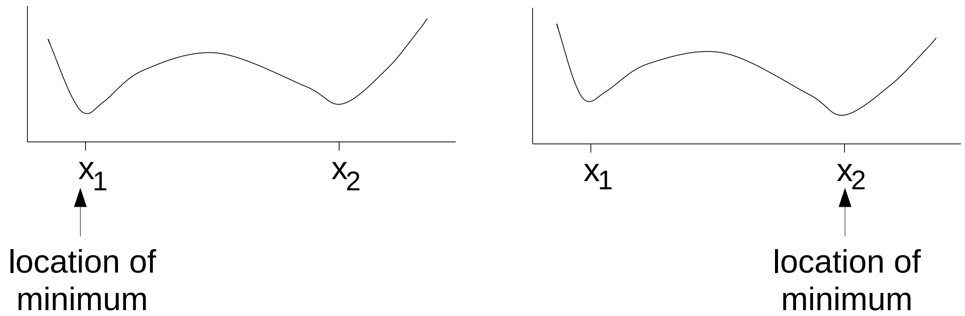
Authors	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Geman and McClure	$\frac{\Delta^2}{1+\Delta^2}$	[70, 71]		
Blake and Zisserman	$\min\left\{\Delta^2, 1\right\}$	[20, 19]		
Hebert and Leahy	$\log\left(1+\Delta^2\right)$	[81]		
Geman and Reynolds	$\frac{\|\Delta\|}{1+\|\Delta\|}$	[66]		

Properties of Non-Convex Potential Functions

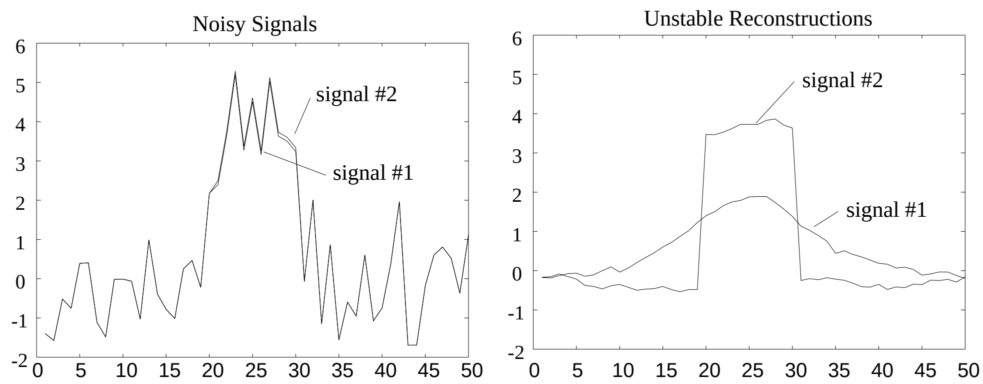
- Advantages
  - Very sharp edges
  - Very general class of potential functions
- Disadvantages
  - Difficult (impossible) to compute MAP estimate
  - Usually requires the choice of an edge threshold
  - **MAP estimate is a discontinuous function of the data**

Continuous (Stable) MAP Estimation[25]

- Minimum of non-convex function can change abruptly.

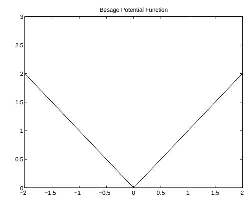
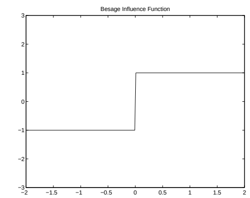
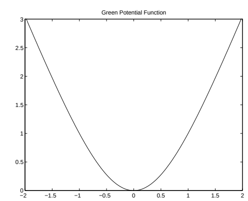
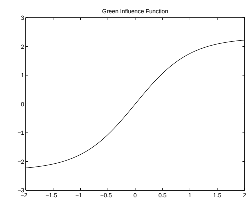
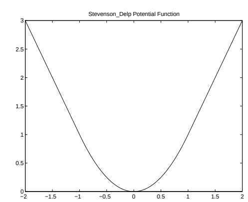
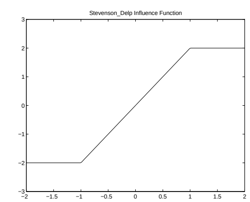
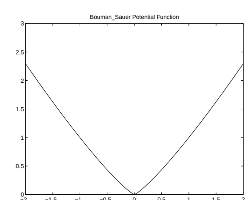
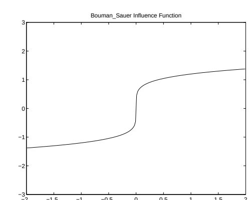


- Discontinuous MAP estimate for Blake and Zisserman potential.



- Theorem:[25] - If the log of the posterior density is **strictly convex**, then the MAP estimate is a continuous function of the data.

Convex Potential Functions

Authors(Name)	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Besag	$\lvert\Delta\rvert$	[17]		
Green	$\log\cosh\Delta$	[75]		
Stevenson and Delp (Huber function)	$\min\{\lvert\Delta\rvert^2, 2\lvert\Delta\rvert - 1\}$	[155]		
Bouman and Sauer (Generalized Gaussian MRF)	$\lvert\Delta\rvert^p$	[25]		

Properties of Convex Potential Functions

- Both $\log\cosh(\Delta)$ and Huber functions
  - Quadratic for $|\Delta| << 1$
  - Linear for $|\Delta| >> 1$
  - Transition from quadratic to linear determines edge threshold.
- Generalized Gaussian MRF (GGMRF) functions
  - Include $|\Delta|$ function
  - Do not require an edge threshold parameter.
  - Convex and differentable for $p > 1$.

Parameter Estimation for Continuous MRF's

- Topics to be covered:
  - Estimation of scale parameter, σ
  - Estimation of temperature, T, and shape, p

ML Estimation of Scale Parameter, σ, for Continuous MRF's [26]

- For any continuous state Gibbs distribution

$$p(x) = \frac{1}{Z(\sigma)} \exp\{-U(x/\sigma)\}$$

  the partition function has the form

$$Z(\sigma) = \sigma^N Z(1)$$

- Using this result the ML estimate of σ is given by

$$\left. \frac{\sigma}{N} \frac{d}{d\sigma} U(x/\sigma) \right|_{\sigma=\hat{\sigma}} - 1 = 0$$

- This equation can be solved numerically using any root finding method.

ML Estimation of σ for GGMRF's [108, 26]

- For a Generalized Gaussian MRF (GGMRF)

$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ -\frac{1}{p\sigma^p} U(x) \right\}$$

where the energy function has the property that for all $\alpha > 0$

$$U(\alpha x) = \alpha^p U(x)$$

- Then the ML estimate of σ is

$$\hat{\sigma} = \left(\frac{1}{N} U(x) \right)^{(1/p)}$$

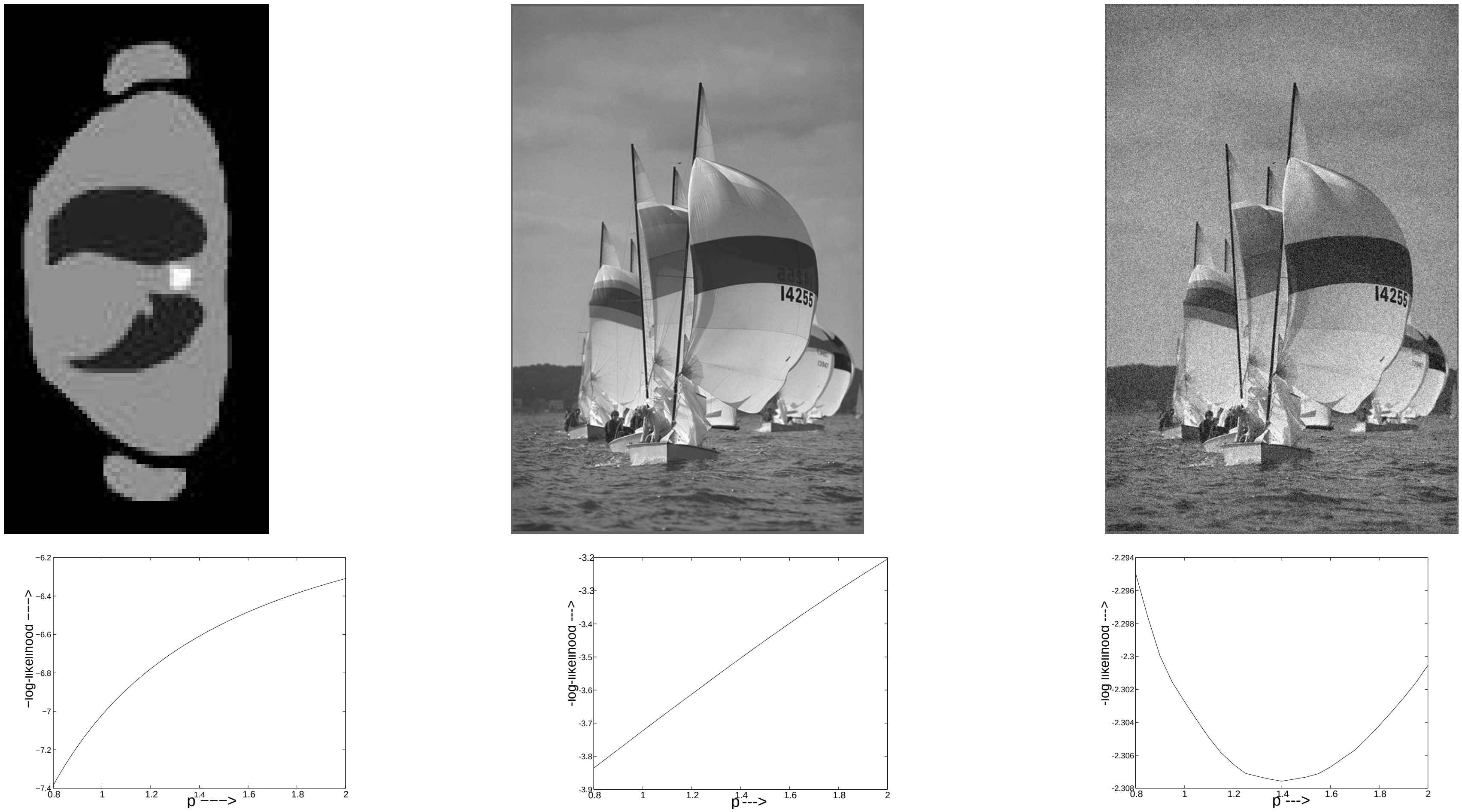
- Notice for that for the i.i.d. Gaussian case, this is

$$\hat{\sigma} = \sqrt{\frac{1}{N} \sum_s |x_s|^2}$$

Estimation of Temperature, T, and Shape, p, Parameters

- ML estimation of T[71]
  - Used to estimate T for any distribution.
  - Based on "off line" computation of log partition function.
- Adaptive method [133]
  - Used to estimate p parameter of GGMRF.
  - Based on measurement of kurtosis.
- ML estimation of p[145, 144]
  - Used to estimate p parameter of GGMRF.
  - Based on "off line" computation of log partition function.

Example Estimation of p Parameter



(a) (b) (c)

- ML estimation of p for (a) transmission phantom (b) natural image (c) image corrupted with Gaussian noise. The plot below each image shows the corresponding negative log-likelihood as a function of p. The ML estimate is the value of p that minimizes the plotted function.

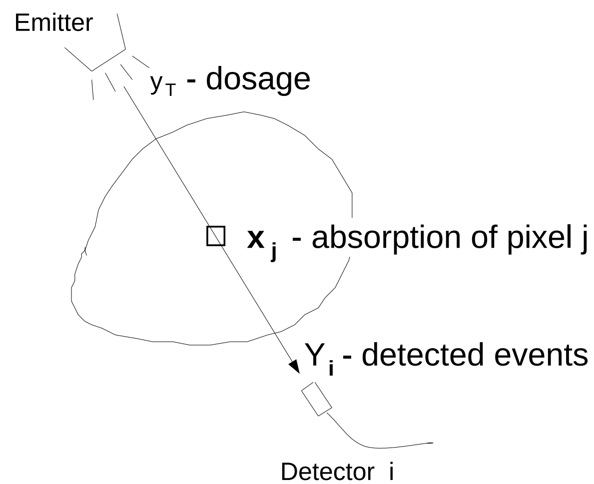
Application to Tomography

- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation

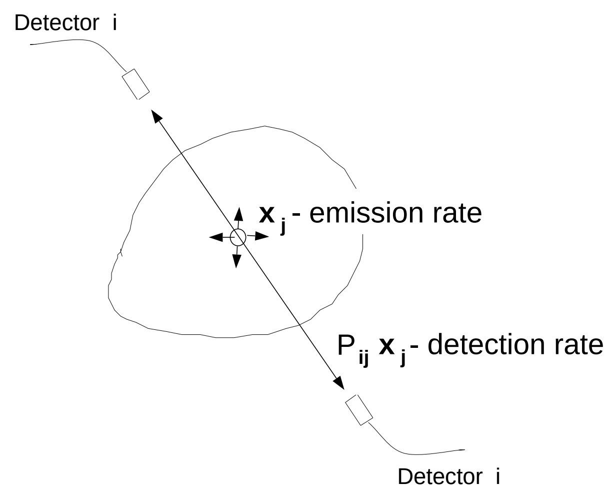
The Tomography Problem

- Recover image cross-section from integral projections

- Transmission problem



- Emission problem



Statistical Data Model[27]

- Notation
  - y - vector of photon counts
  - x - vector of image pixels
  - P - projection matrix
  - $P_{j,*}$ - j^{th} row of projection matrix
- Emission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-P_{i*}x \ + y_i \log\{P_{i*}x\} - \log(y_i!)\right)$$

- Transmission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-y_T e^{-P_{i*}x} + y_i(\log y_T - P_{i*}x) - \log(y_i!)\right)$$

- Common form

$$\boxed{\log p(y|x) = -\Sigma_{i=1}^{M} f_i(P_{i*}x)}$$

  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!

Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

$$
\begin{aligned}
\hat{x} &= \arg\max_{x} p(x|y) \\
&= \arg\max_{x>0} \left\{ -\sum_{i=1}^{M} f_i(P_{i*}x) - \sum_{k<j} b_{k,j}\, \rho(x_k - x_j) \right\}
\end{aligned}
$$

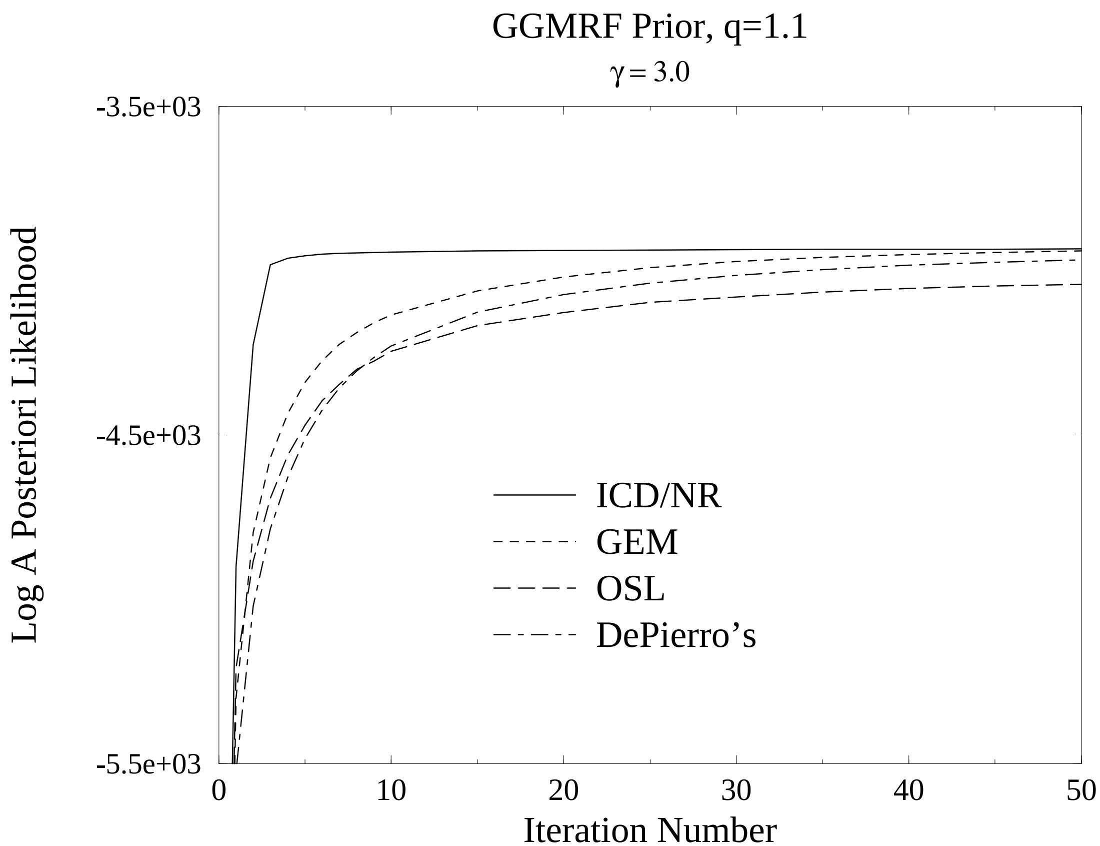
- Can be solved using direct optimization
- Incorporates positivity constraint

MAP Optimization Strategies

- Expectation maximization (EM) based optimization strategies
  - ML reconstruction[151, 107]
  - MAP reconstruction[81, 75, 84]
  - Slow convergence; Similar to gradient search.
  - Accelerated EM approach[59]
- Direct optimization
  - Preconditioned gradient descent with soft positivity constraint[45]
  - ICM iterations (also known as ICD and Gauss-Seidel)[27]

Convergence of ICM Iterations: MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel



- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green's (OSL), and Hebert/Leahy's GEM, and De Pierro's method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$.

Estimation of σ from Tomographic Data

- Assume a GGMRF prior distribution of the form

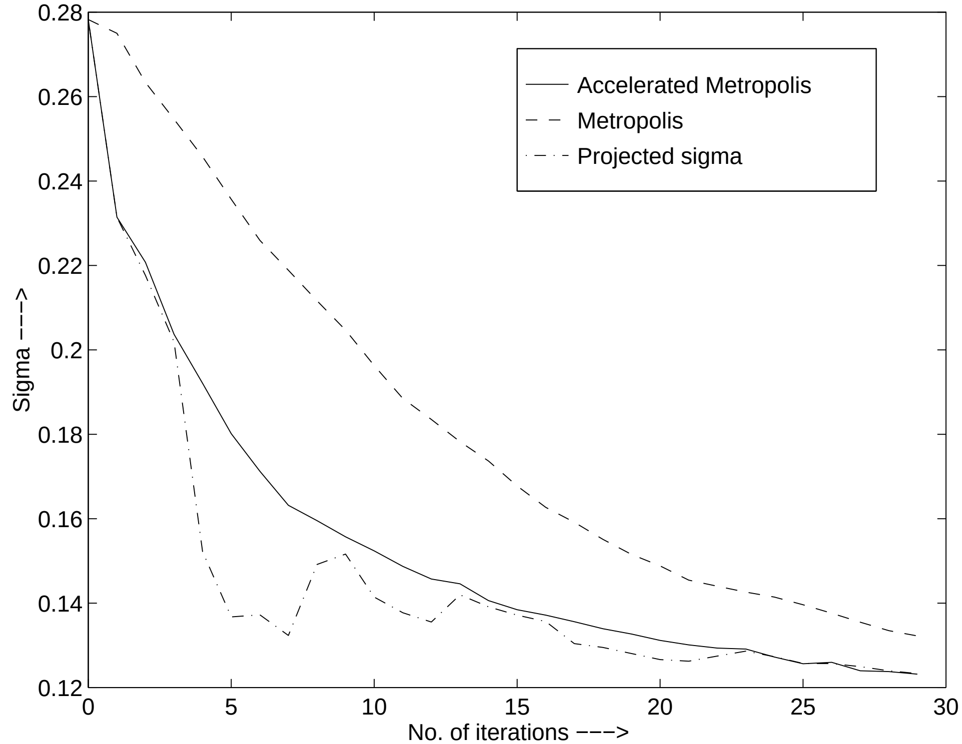
$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ \frac{1}{p\sigma^p} U(x) \right\}$$

- Problem: We don't know X!
- EM formulation for incomplete data problem

$$\begin{aligned} \sigma^{(k+1)} &= \arg\max_{\sigma} E\left\{ \log p(X|\sigma) | Y = y, \sigma^{(k)} \right\} \\ &= \left(E\left\{ \frac{1}{N} U(X) | Y = y, \sigma^{(k)} \right\} \right)^{1/p} \end{aligned}$$

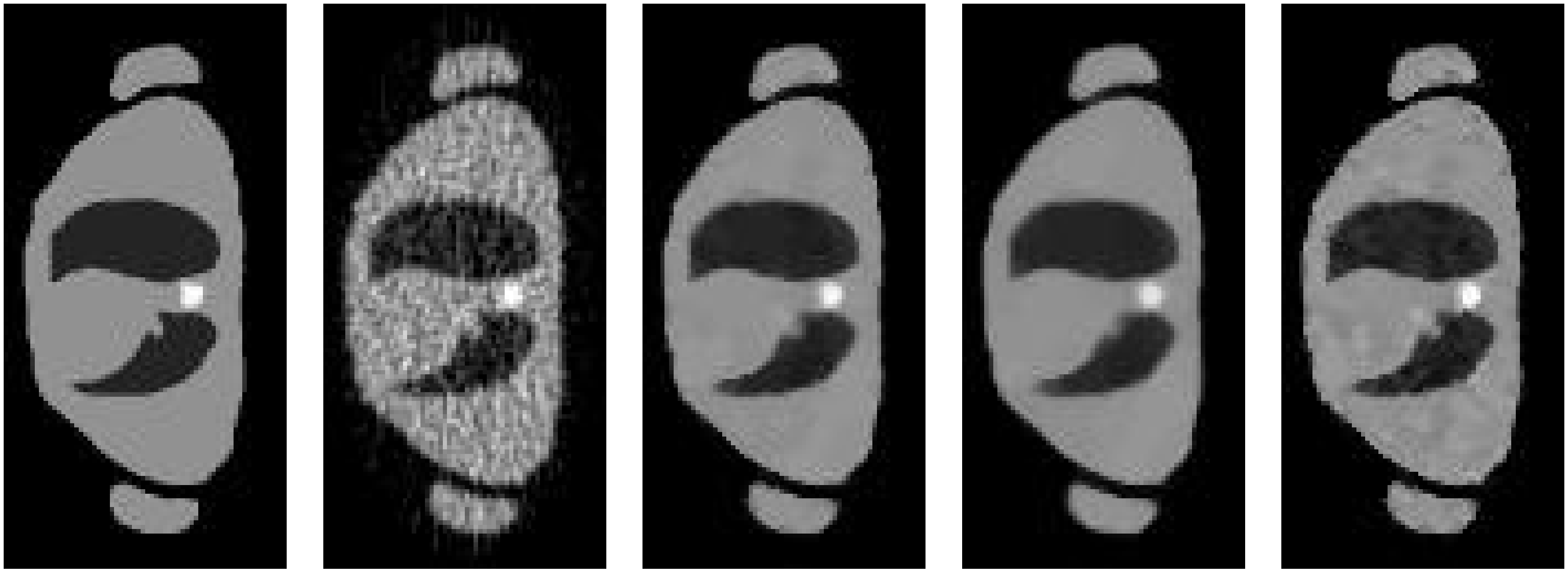
- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.

Example of Estimation of σ from Tomographic Data



- The above plot shows the EM updates for σ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter s denotes the standard deviation of the symmetric transition distribution for the CM method.

Example of Tomographic Reconstructions

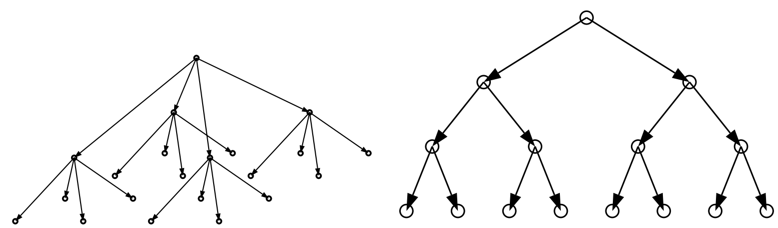


a	b	c	d	e

- (a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter σ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$
- Phantom courtesy of J. Fessler, University of Michigan

Multiscale Stochastic Models

- Generate a Markov chain in scale



- Some references
  - Continuous models[12, 5, 111]
  - Discrete models[29, 111]
- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM's.

Multiscale Stochastic Models for Continuous State Estimation

- Theory of 1-D systems can be extended to multiscale trees[6, 7].
- Can be used to efficiently estimate optical flow[111].
- These models can approximate MRF's[112].
- The structure of the model allows exact calculation of log likelihoods for texture segmentation[113].

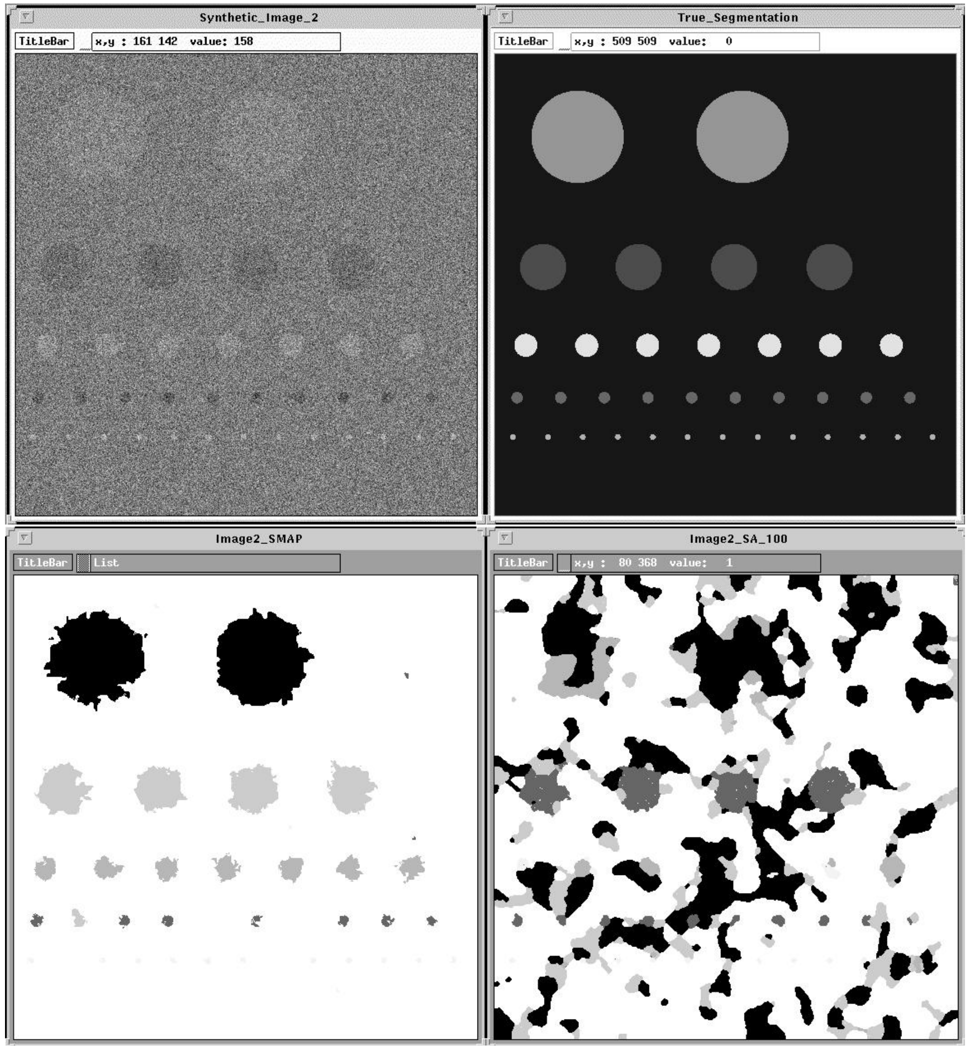
Multiscale Stochastic Models for Segmentation[29]

- Multiscale model results in non-iterative segmentation
- Sequential MAP (SMAP) criteria minimizes size of largest misclassification.
- Computational comparison

	Replacements per pixel				
	SMAP	SMAP + par. est.	SA 500	SA 100	ICM
image1	1.33	3.13	504	105	28
image2	1.33	3.55	506	108	28
image3	1.33	3.14	505	104	10

Segmentation of Synthetic Test Image

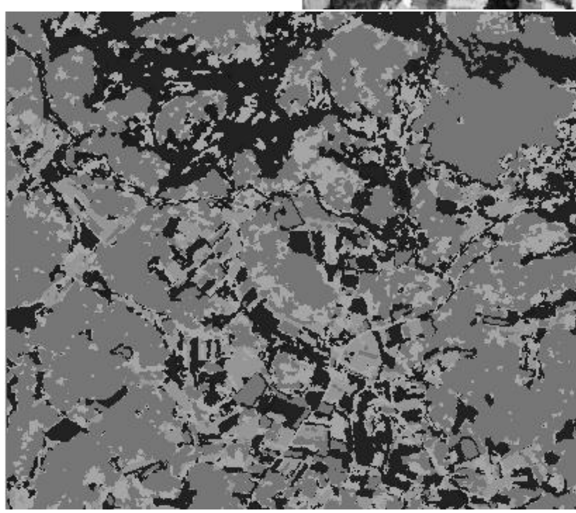
Synthetic Image — Correct Segmentation

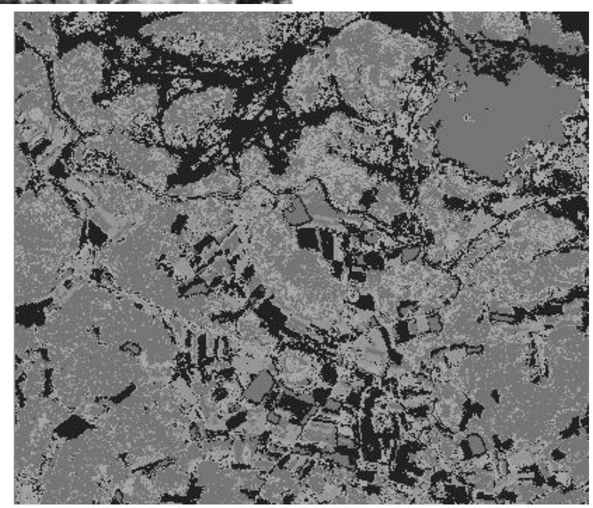


SMAP — 100 Iterations of SA

Multispectral Spot Image Segmentation

SPOT image





SMAP

Maximum Likelihood

High Level Image Models

- MRF's have been used to
  - model the relative location of objects in a scene[119].
  - model relational constraints for object matching problems[109].
- Multiscale stochastic models
  - have been used to model complex assemblies for automated inspection[166].
  - have been used to model 2-D patterns for application in image search[154].

References

[1] M. Aitkin and D. B. Rubin. Estimation and hypothesis testing in finite mixture models. *Journal of the Royal Statistical Society B*, 47(1):67–75, 1985.

[2] H. Akaike. A new look at the statistical model identification. *IEEE Trans. Automat. Contr.*, AC-19(6):716–723, December 1974.

[3] S. Alliney and S. A. Ruzinsky. An algorithm for the minimization of mixed l_1 and l_2 norms with application to Bayesian estimation. *IEEE Trans. on Signal Processing*, 42(3):618–627, March 1994.

[4] P. Barone, A. Frigessi, and M. Piccioni, editors. *Stochastic models, statistical methods, and algorithms in image analysis.* Springer-Verlag, Berlin, 1992.

[5] M. Basseville, A. Benveniste, K. Chou, S. Golden, and R. Nikoukhah. Modeling and estimation of multiresolution stochastic processes. *IEEE Trans. on Information Theory*, 38(2):766–784, 1992.

[6] M. Basseville, A. Benveniste, and A. Willsky. Multiscale autoregressive processes, part i: Schur-Levinson parametrizations. *IEEE Trans. on Signal Processing*, 40(8):1915–1934, 1992.

[7] M. Basseville, A. Benveniste, and A. Willsky. Multiscale autoregressive processes, part ii: Lattice structures for whitening and modeling. *IEEE Trans. on Signal Processing*, 40(8):1915–1934, 1992.

[8] L. Baum, T. Petrie, G. Soules, and N. Weiss. A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains. *Ann. Math. Statistics*, 41(1):164–171, 1970.

[9] L. E. Baum and T. Petrie. Statistical inference for probabilistic functions of finite state Markov chains. *Ann. Math. Statistics*, 37:1554–1563, 1966.

[10] F. Beckman. The solution of linear equations by the conjugate gradient method. In A. Ralston, H. Wilf, and K. Enslein, editors, *Mathematical Methods for Digital Computers*. Wiley, 1960.

[11] M. G. Bello. A combined Markov random field and wave-packet transform-based approach for image segmentation. *IEEE Trans. on Image Processing*, 3(6):834–846, November 1994.

[12] A. Benveniste, R. Nikoukhah, and A. Willsky. Multiscale system theory. In *Proceedings of the 29th Conference on Decision and Control*, volume 4, pages 2484–2489, Honolulu, Hawaii, December 5-7 1990.

[13] J. Besag. Nearest-neighbour systems and the auto-logistic model for binary data. *Journal of the Royal Statistical Society B*, 34(1):75–83, 1972.

[14] J. Besag. Spatial interaction and the statistical analysis of lattice systems. *Journal of the Royal Statistical Society B*, 36(2):192–236, 1974.

[15] J. Besag. Efficiency of pseudolikelihood estimation for simple Gaussian fields. *Biometrica*, 64(3):616–618, 1977.

[16] J. Besag. On the statistical analysis of dirty pictures. *Journal of the Royal Statistical Society B*, 48(3):259–302, 1986.

[17] J. Besag. Towards Bayesian image analysis. *Journal of Applied Statistics*, 16(3):395–407, 1989.

[18] J. E. Besag and P. A. P. Moran. On the estimation and testing of spatial interaction in Gaussian lattice processes. *Biometrika*, 62(3):555–562, 1975.

[19] A. Blake. Comparison of the efficiency of deterministic and stochastic algorithms for visual reconstruction. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 11(1):2–30, January 1989.

[20] A. Blake and A. Zisserman. *Visual Reconstruction*. MIT Press, Cambridge, Massachusetts, 1987.

[21] C. A. Bouman and B. Liu. A multiple resolution approach to regularization. In *Proc. of SPIE Conf. on Visual Communications and Image Processing*, pages 512–520, Cambridge, MA, Nov. 9-11 1988.

[22] C. A. Bouman and B. Liu. Segmentation of textured images using a multiple resolution approach. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, pages 1124–1127, New York, NY, April 11-14 1988.

[23] C. A. Bouman and B. Liu. Multiple resolution segmentation of textured images. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 13:99–113, Feb. 1991.

[24] C. A. Bouman and K. Sauer. An edge-preserving method for image reconstruction from integral projections. In *Proc. of the Conference on Information Sciences and Systems*, pages 382–387, The Johns Hopkins University, Baltimore, MD, March 20-22 1991.

[25] C. A. Bouman and K. Sauer. A generalized Gaussian image model for edge-preserving map estimation. *IEEE Trans. on Image Processing*, 2:296–310, July 1993.

[26] C. A. Bouman and K. Sauer. Maximum likelihood scale estimation for a class of Markov random fields. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, volume 5, pages 537–540, Adelaide, South Australia, April 19-22 1994.

[27] C. A. Bouman and K. Sauer. A unified approach to statistical tomography using coordinate descent optimization. *IEEE Trans. on Image Processing*, 5(3):480–492, March 1996.

[28] C. A. Bouman and M. Shapiro. Multispectral image segmentation using a multiscale image model. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, pages III–565 – III–568, San Francisco, California, March 23-26 1992.

[29] C. A. Bouman and M. Shapiro. A multiscale random field model for Bayesian image segmentation. *IEEE Trans. on Image Processing*, 3(2):162–177, March 1994.

[30] A. Brandt. *Multigrid Techniques: 1984 Guide with Applications to Fluid Dynamics*, volume Nr. 85. GMD-Studien, 1984. ISBN:3-88457-081-1.

[31] W. Briggs. *A Multigrid Tutorial*. Society for Industrial and Applied Mathematics, Philadelphia, 1987.

[32] P. Charbonnier, L. Blanc-Feraud, G. Aubert, and M. Barlaud. Two deterministic half-quadratic reqularization algorithms for computed imaging. In *Proc. of IEEE Int'l Conf. on Image Proc.*, volume 2, pages 168–176, Austin, TX, November, 13-16 1994.

[33] R. Chellappa and S. Chatterjee. Classification of textures using Gaussian Markov random fields. *IEEE Trans. on Acoust. Speech and Signal Proc.*, ASSP-33(4):959–963, August 1985.

[34] K. Chou, S. Golden, and A. Willsky. Modeling and estimation of multiscale stochastic processes. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, pages 1709–1712, Toronto, Canada, May 14-17 1991.

[35] B. Claus and G. Chartier. Multiscale signal processing: Isotropic random fields on homogeneous trees. *IEEE Trans. on Circ. and Sys.:Analog and Dig. Signal Proc.*, 41(8):506–517, August 1994.

[36] F. Cohen and D. Cooper. Simple parallel hierarchical and relaxation algorithms for segmenting noncausal Markovian random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-9(2):195–219, March 1987.

[37] F. S. Cohen and Z. Fan. Maximum likelihood unsupervised texture image segmentation. *CVGIP:Graphical Models and Image Proc.*, 54(3):239–251, May 1992.

[38] F. S. Cohen, Z. Fan, and M. A. Patel. Classification of rotated and scaled textured images using Gaussian Markov random field models. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 13(2):192–202, February 1991.

[39] M. L. Comer and E. J. Delp. Parameter estimation and segmentation of noisy or textured images using the EM algorithm and MPM estimation. In *Proc. of IEEE Int'l Conf. on Image Proc.*, volume II, pages 650–654, Austin, Texas, November 1994.

[40] M. L. Comer and E. J. Delp. Multiresolution image segmentation. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, pages 2415–2418, Detroit, Michigan, May 1995.

[41] D. B. Cooper. Maximum likelihood estimation of Markov process blob boundaries in noisy images. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-1:372–384, 1979.

[42] D. B. Cooper, H. Elliott, F. Cohen, L. Reiss, and P. Symosek. Stochastic boundary estimation and object recognition. *Comput. Vision Graphics and Image Process.*, 12:326–356, April 1980.

[43] R. Cristi. Markov and recursive least squares methods for the estimation of data with discontinuities. *IEEE Trans. on Acoust. Speech and Signal Proc.*, 38(11):1972–1980, March 1990.

[44] G. R. Cross and A. K. Jain. Markov random field texture models. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-5(1):25–39, January 1983.

[45] E. *Ü*. Mumcuoğlu, R. Leahy, S. R. Cherry, and Z. Zhou. Fast gradient-based methods for Bayesian reconstruction of transmission and emission pet images. *IEEE Trans. on Medical Imaging*, 13(4):687–701, December 1994.

[46] A. Dempster, N. Laird, and D. Rubin. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society B*, 39(1):1–38, 1977.

[47] H. Derin and W. Cole. Segmentation of textured images using Gibbs random fields. *Comput. Vision Graphics and Image Process.*, 35:72–98, 1986.

[48] H. Derin, H. Elliot, R. Cristi, and D. Geman. Bayes smoothing algorithms for segmentation of binary images modeled by Markov random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-6(6):707–719, November 1984.

[49] H. Derin and H. Elliott. Modeling and segmentation of noisy and textured images using Gibbs random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-9(1):39–55, January 1987.

[50] H. Derin and P. A. Kelly. Discrete-index Markov-type random processes. *Proc. of the IEEE*, 77(10):1485–1510, October 1989.

[51] H. Derin and C. Won. A parallel image segmentation algorithm using relaxation with varying neighborhoods and its mapping to array processors. *Comput. Vision Graphics and Image Process.*, 40:54–78, October 1987.

[52] R. W. Dijkerman and R. R. Mazumdar. Wavelet representations of stochastic processes and multiresolution stochastic models. *IEEE Trans. on Signal Processing*, 42(7):1640–1652, July 1994.

[53] P. C. Doerschuk. Bayesian signal reconstruction, Markov random fields, and X-ray crystallography. *J. Opt. Soc. Am. A*, 8(8):1207–1221, 1991.

[54] R. Dubes and A. Jain. Random field models in image analysis. *Journal of Applied Statistics*, 16(2):131–164, 1989.

[55] R. Dubes, A. Jain, S. Nadabar, and C. Chen. MRF model-based algorithms for image segmentation. In *Proc. of the $10^t h$ Internat. Conf. on Pattern Recognition*, pages 808–814, Atlantic City, NJ, June 1990.

[56] I. M. Elfadel and R. W. Picard. Gibbs random fields, cooccurrences and texture modeling. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 16(1):24–37, January 1994.

[57] H. Elliott, D. B. Cooper, F. S. Cohen, and P. F. Symosek. Implementation, interpretation, and analysis of a suboptimal boundary finding algorithm. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-4(2):167–182, March 1982.

[58] W. Enkelmann. Investigations of multigrid algorithms. *Comput. Vision Graphics and Image Process.*, 43:150–177, 1988.

[59] J. Fessler and A. Hero. Space-alternating generalized expectation-maximization algorithms. *IEEE Trans. on Acoust. Speech and Signal Proc.*, 42(10):2664–2677, October 1994.

[60] N. P. Galatsanos and A. K. Katsaggelos. Methods for choosing the regularization parameter and estimating the noise variance in image restoration and their relation. *IEEE Trans. on Image Processing*, 1(3):322–336, July 1992.

[61] S. B. Gelfand and S. K. Mitter. Recursive stochastic algorithms for global optimization in r^d. *SIAM Journal on Control and Optimization*, 29(5):999–1018, September 1991.

[62] S. B. Gelfand and S. K. Mitter. Metropolis-type annealing algorithms for global optimization in r^d. *SIAM Journal on Control and Optimization*, 31(1):111–131, January 1993.

[63] S. B. Gelfand and S. K. Mitter. On sampling methods and annealing algorithms. In R. Chellappa and A. Jain, editors, *Markov Random Fields: Theory and Applications*, pages 499–515. Academic Press, Inc., Boston, 1993.

[64] D. Geman. Bayesian image analysis by adaptive annealing. In *Digest 1985 Int. Geoscience and remote sensing symp.*, Amherst, MA, October 7-9 1985.

[65] D. Geman, S. Geman, C. Graffigne, and P. Dong. Boundary detection by constrained optimization. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 12(7):609–628, July 1990.

[66] D. Geman and G. Reynolds. Constrained restoration and the recovery of discontinuities. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 14(3):367–383, 1992.

[67] D. Geman, G. Reynolds, and C. Yang. Stochastic algorithms for restricted image spaces and experiments in deblurring. In R. Chellappa and A. Jain, editors, *Markov Random Fields: Theory and Applications*, pages 39–68. Academic Press, Inc., Boston, 1993.

[68] S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-6:721–741, Nov. 1984.

[69] S. Geman and C. Graffigne. Markov random field image models and their applications to computer vision. In *Proc. of the Intl Congress of Mathematicians*, pages 1496–1517, Berkeley, California, 1986.

[70] S. Geman and D. McClure. Bayesian images analysis: An application to single photon emission tomography. In *Proc. Statist. Comput. sect. Amer. Stat. Assoc.*, pages 12–18, Washington, DC, 1985.

[71] S. Geman and D. McClure. Statistical methods for tomographic image reconstruction. *Bull. Int. Stat. Inst.*, LII-4:5–21, 1987.

[72] B. Gidas. A renormalization group approach to image processing problems. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 11(2):164–180, February 89.

[73] J. Goutsias. Unilateral approximation of Gibbs random field images. *CVGIP:Graphical Models and Image Proc.*, 53(3):240–257, May 1991.

[74] A. Gray, J. Kay, and D. Titterington. An empirical study of the simulation of various models used for images. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 16(5):507–513, May 1994.

[75] P. J. Green. Bayesian reconstruction from emission tomography data using a modified EM algorithm. *IEEE Trans. on Medical Imaging*, 9(1):84–93, March 1990.

[76] P. J. Green and X. liang Han. Metropolis methods, Gaussian proposals and antithetic variables. In P. Barone, A. Frigessi, and M. Piccioni, editors, *Stochastic Models, Statistical methods, and Algorithms in Image Analysis*, pages 142–164. Springer-Verlag, Berlin, 1992.

[77] M. I. Gurelli and L. Onural. On a parameter estimation method for Gibbs-Markov random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 16(4):424–430, April 1994.

[78] W. Hackbusch. *Multi-Grid Methods and Applications.* Sprinter-Verlag, Berlin, 1980.

[79] K. Hanson and G. Wecksung. Bayesian approach to limited-angle reconstruction computed tomography. *J. Opt. Soc. Am.*, 73(11):1501–1509, November 1983.

[80] W. K. Hastings. Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57(1):97–109, 1970.

[81] T. Hebert and R. Leahy. A generalized EM algorithm for 3-d Bayesian reconstruction from Poisson data using Gibbs priors. *IEEE Trans. on Medical Imaging*, 8(2):194–202, June 1989.

[82] T. J. Hebert and K. Lu. Expectation-maximization algorithms, null spaces, and MAP image restoration. *IEEE Trans. on Image Processing*, 4(8):1084–1095, August 1995.

[83] F. Heitz and P. Bouthemy. Multimodal estimation of discontinuous optical flow using Markov random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 15(12):1217–1232, December 1993.

[84] G. T. Herman, A. R. De Pierro, and N. Gai. On methods for maximum a posteriori image reconstruction with normal prior. *J. Visual Comm. Image Rep.*, 3(4):316–324, December 1992.

[85] P. Huber. *Robust Statistics*. John Wiley & Sons, New York, 1981.

[86] B. Hunt. The application of constrained least squares estimation to image restoration by digital computer. *IEEE Trans. on Comput.*, c-22(9):805–812, September 1973.

[87] B. Hunt. Bayesian methods in nonlinear digital image restoration. *IEEE Trans. on Comput.*, c-26(3):219–229, 1977.

[88] J. Hutchinson, C. Koch, J. Luo, and C. Mead. Computing motion using analog and binary resistive networks. *Computer*, 21:53–63, March 1988.

[89] A. Jain. Advances in mathematical models for image processing. *Proc. of the IEEE*, 69:502–528, May 1981.

[90] B. D. Jeffs and M. Gunsay. Restoration of blurred star field images by maximally sparse optimization. *IEEE Trans. on Image Processing*, 2(2):202–211, April 1993.

[91] F.-C. Jeng and J. W. Woods. Compound Gauss-Markov random fields for image estimation. *IEEE Trans. on Signal Processing*, 39(3):683–697, March 1991.

[92] B. Jeon and D. Landgrebe. Classification with spatio-temporal interpixel class dependency contexts. *IEEE Trans. on Geoscience and Remote Sensing*, 30(4), JULY 1992.

[93] S. R. Kadaba, S. B. Gelfand, and R. L. Kashyap. Bayesian decision feedback for segmentation of binary images. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, pages 2543–2546, Detroit, Michigan, May 8-12 1995.

[94] R. Kashyap and R. Chellappa. Estimation and choice of neighbors in spatial-interaction models of images. *IEEE Trans. on Information Theory*, IT-29(1):60–72, January 1983.

[95] R. Kashyap, R. Chellappa, and A. Khotanzad. Texture classification using features derived from random field models. *Pattern Recogn. Let.*, 1(1):43–50, October 1982.

[96] R. L. Kashyap and K.-B. Eom. Robust image modeling techniques with an image restoration application. *IEEE Trans. on Acoust. Speech and Signal Proc.*, 36(8):1313–1325, August 1988.

[97] Z. Kato, M. Berthod, and J. Zerubia. Parallel image classification using multiscale Markov random fields. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, volume 5, pages 137–140, Minneapolis, MN, April 27-30 1993.

[98] A. Katsaggelos. Image identification and restoration based on the expectation-maximization algorithm. *Optical Engineering*, 29(5):436–445, May 1990.

[99] P. Kelly, H. Derin, and K. Hartt. Adaptive segmentation of speckled images using a hierarchical random field model. *IEEE Trans. on Acoust. Speech and Signal Proc.*, 36(10):1628–1641, October 1988.

[100] R. Kindermann and J. Snell. *Markov Random Fields and their Applications.* American Mathematical Society, Providence, 1980.

[101] J. Konrad and E. Dubois. Bayesian estimation of motion vector fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 14(9):910–927, September 1992.

[102] R. Lagendijk, A. Tekalp, and J. Biemond. Maximum likelihood image and blur identification: A unifying approach. *Optical Engineering*, 29(5):422–435, May 1990.

[103] R. L. Lagendijk, J. Biemond, and D. E. Boekee. Identification and restoration of noisy blurred images using the expectation-maximization algorithm. *IEEE Trans. on Acoust. Speech and Signal Proc.*, 38(7):1180–1191, July 1990.

[104] S. Lakshmanan and H. Derin. Simultaneous parameter estimation and segmentation of Gibbs random fields using simulated annealing. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 11(8):799–813, August 1989.

[105] S. Lakshmanan and H. Derin. Gaussian Markov random fields at multiple resolutions. In R. Chellappa and A. Jain, editors, *Markov Random Fields: Theory and Applications*, pages 131–157. Academic Press, Inc., Boston, 1993.

[106] S. Lakshmanan and H. Derin. Valid parameter space for 2-d Gaussian Markov random fields. *IEEE Trans. on Information Theory*, 39(2):703–709, March 1993.

[107] K. Lange. Convergence of EM image reconstruction algorithms with Gibbs smoothing. *IEEE Trans. on Medical Imaging*, 9(4):439–446, December 1990.

[108] K. Lange. An overview of Bayesian methods in image reconstruction. In *Proc. of the SPIE Conference on Digital Image Synthesis and Inverse Optics*, volume SPIE-1351, pages 270–287, San Diego, CA, 1990.

[109] S. Z. Li. *Markov Random Field Modeling in Computer Vision*. to be published, 1996.

[110] J. Liu and Y.-H. Yang. Multiresolution color image segmentation. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 16(7):689–700, July 1994.

[111] M. R. Luettgen, W. C. Karl, and A. S. Willsky. Efficient multiscale regularization with applications to the computation of optical flow. *IEEE Trans. on Image Processing*, 3(1), January 1994.

[112] M. R. Luettgen, W. C. Karl, A. S. Willsky, and R. R. Tenney. Multiscale representations of Markov random fields. *IEEE Trans. on Signal Processing*, 41(12), December 1993. special issue on Wavelets and Signal Processing.

[113] M. R. Luettgen and A. S. Willsky. Likelihood calculation for a class of multiscale stochastic models, with application to texture discrimination. *IEEE Trans. on Image Processing*, 4(2):194–207, February 1995.

[114] B. Manjunath, T. Simchony, and R. Chellappa. Stochastic and deterministic networks for texture segmentation. *IEEE Trans. on Acoust. Speech and Signal Proc.*, 38(6):1039–1049, June 1990.

[115] J. Mao and A. K. Jain. Texture classification and segmentation using multiresolution simultaneous autoregressive models. *Pattern Recognition*, 25(2):173–188, 1992.

[116] J. Marroquin, S. Mitter, and T. Poggio. Probabilistic solution of ill-posed problems in computational vision. *Journal of the American Statistical Association*, 82:76–89, March 1987.

[117] S. McCormick, editor. *Multigrid Methods*. Society for Industrial and Applied Mathematics, Philadelphia, 1987.

[118] N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller. Equations of state calculations by fast computing machines. *J. Chem. Phys.*, 21:1087–1091, 1953.

[119] J. W. Modestino and J. Zhang. A Markov random field model-based approach to image interpretation. In R. Chellappa and A. Jain, editors, *Markov Random Fields: Theory and Applications*, pages 369–408. Academic Press, Inc., Boston, 1993.

[120] H. H. Nguyen and P. Cohen. Gibbs random fields, fuzzy clustering, and the unsupervised segmentation of textured images. *CVGIP:Graphical Models and Image Proc.*, 55(1):1–19, January 1993.

[121] Y. Ogata. A Monte Carlo method for an objective Bayesian procedure. *Ann. Inst. of Statist. Math.*, 42(3):403–433, 1990.

[122] L. Onsager. Crystal statistics i. a two-dimensional model. *Physical Review*, 65:117–149, 1944.

[123] L. Onural. Generating connected textured fractal patterns using Markov random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 13(8):819–825, August 1991.

[124] L. Onural, M. B. Alp, and M. I. Gurelli. Gibbs random field model based weight selection for the 2-d adaptive weighted median filter. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 16(8):831–837, August 1994.

[125] T. Pappas. An adaptive clustering algorithm for image segmentation. *IEEE Trans. on Signal Processing*, 4:901–914, April 1992.

[126] R. E. Peierls. On Ising's model of ferromagnetism. *Proc. Camb. Phil. Soc.*, 32:477–481, 1936.

[127] R. E. Peierls. Statistical theory of adsorption with interaction between the adsorbed atoms. *Proc. Camb. Phil. Soc.*, 32:471–476, 1936.

[128] P. Perez and F. Heitz. Multiscale Markov random fields and constrained relaxation in low level image analysis. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, volume 3, pages 61–64, San Francisco, CA, March 23-26 1992.

[129] P. H. Peskun. Optimum Monte-Carlo sampling using Markov chains. *Biometrika*, 60(3):607–612, 1973.

[130] D. Pickard. Asymptotic inference for an ising lattice iii. non-zero field and ferromagnetic states. *J. Appl. Prob.*, 16:12–24, 1979.

[131] D. Pickard. Inference for discrete Markov fields: The simplest nontrivial case. *Journal of the American Statistical Association*, 82:90–96, March 1987.

[132] D. N. Politis. Markov chains in many dimensions. *Adv. Appl. Prob.*, 26:756–774, 1994.

[133] W. Pun and B. Jeffs. Shape parameter estimation for generalized Gaussian Markov random field models used in MAP image restoration. In *29th Asilomar Conference on Signals, Systems, and Computers*, Oct. 29 - Nov. 1 1995.

[134] L. R. Rabiner. A tutorial on hidden Markov models and selected applications in speech recognition. *Proc. of the IEEE*, 77(2):257–286, February 1989.

[135] L. R. Rabiner and B. H. Juang. An introduction to hidden Markov models. *IEEE Signal Proc. Magazine*, 3(1):4–16, January 1986.

[136] E. Redner and H. Walker. Mixture densities, maximum likelihood and the EM algorithm. *SIAM Review*, 26(2), April 1984.

[137] S. Reeves and R. Mersereau. Optimal estimation of the regularization parameter and stabilizing functional for regularized image restoration. *Optical Engineering*, 29(5):446–454, May 1990.

[138] S. J. Reeves and R. M. Mersereau. Blur identification by the method of generalized cross-validation. *IEEE Trans. on Image Processing*, 1(3):301–321, July 1992.

[139] W. Rey. *Introduction to Robust and Quasi-Robust Statistical Methods*. Springer-Verlag, Berlin, 1980.

[140] E. Rignot and R. Chellappa. Segmentation of polarimetric synthetic aperature radar data. *IEEE Trans. on Image Processing*, 1(3):281–300, July 1992.

[141] B. D. Ripley. *Stochastic Simulation*. John Wiley & Sons, New York, 1987.

[142] J. Rissanen. A universal prior for integers and estimation by minimum description length. *Annals of Statistics*, 11(2):417–431, 1983.

[143] B. Rouchouze, P. Mathieu, T. Gaidon, and M. Barlaud. Motion estimation based on Markov random fields. In *Proc. of IEEE Int'l Conf. on Image Proc.*, volume 3, pages 270–274, Austin, TX, November, 13-16 1994.

[144] S. S. Saquib, C. A. Bouman, and K. Sauer. ML parameter estimation for Markov random fields, with applications to Bayesian tomography. Technical Report TR-ECE 95-24, School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, October 1995.

[145] S. S. Saquib, C. A. Bouman, and K. Sauer. Efficient ML estimation of the shape parameter for generalized Gaussian MRF. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, volume 4, pages 2229–2232, Atlanta, GA, May 7-10 1996.

[146] K. Sauer and C. Bouman. Bayesian estimation of transmission tomograms using segmentation based optimization. *IEEE Trans. on Nuclear Science*, 39:1144–1152, 1992.

[147] K. Sauer and C. A. Bouman. A local update strategy for iterative reconstruction from projections. *IEEE Trans. on Signal Processing*, 41(2), February 1993.

[148] R. Schultz, R. Stevenson, and A. Lumsdaine. Maximum likelihood parameter estimation for non-Gaussian prior signal models. In *Proc. of IEEE Int'l Conf. on Image Proc.*, volume 2, pages 700–704, Austin, TX, November 1994.

[149] R. R. Schultz and R. L. Stevenson. A Bayesian approach to image expansion for improved definition. *IEEE Trans. on Image Processing*, 3(3):233–242, May 1994.

[150] R. R. Schultz and R. L. Stevenson. Stochastic modeling and estimation of multispectral image data. *IEEE Trans. on Image Processing*, 4(8):1109–1119, August 1995.

[151] L. Shepp and Y. Vardi. Maximum likelihood reconstruction for emission tomography. *IEEE Trans. on Medical Imaging*, MI-1(2):113–122, October 1982.

[152] J. Silverman and D. Cooper. Bayesian clustering for unsupervised estimation of surface and texture models. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 10(4):482–495, July 1988.

[153] T. Simchony, R. Chellappa, and Z. Lichtenstein. Relaxation algorithms for MAP estimation of gray-level images with multiplicative noise. *IEEE Trans. on Information Theory*, 36(3):608–613, May 1990.

[154] S. Sista, C. A. Bouman, and J. P. Allebach. Fast image search using a multiscale stochastic model. In *Proc. of IEEE Int'l Conf. on Image Proc.*, pages 225–228, Washington, DC, October 23-26 1995.

[155] R. Stevenson and E. Delp. Fitting curves with discontinuities. *Proc. of the first international workshop on robust computer vision*, pages 127–136, October 1-3 1990.

[156] H. Tan, S. Gelfand, and E. Delp. A comparative cost function approach to edge detection. *IEEE Trans. on Systems Man and Cybernetics*, 19(6):1337–1349, December 1989.

[157] T. Taxt, P. Flynn, and A. Jain. Segmentation of document images. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 11(12):1322–1329, December 1989.

[158] D. Terzopoulos. Image analysis using multigrid relaxation methods. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-8(2):129–139, March 1986.

[159] D. Terzopoulos. The computation of visible-surface representations. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 10(4):417–438, July 1988.

[160] C. Therrien. An estimation-theoretic approach to terrain image segmentation. *Comput. Vision Graphics and Image Process.*, 22:313–326, 1983.

[161] C. Therrien, T. Quatieri, and D. Dudgeon. Statistical model-based algorithm for image analysis. *Proc. of the IEEE*, 74(4):532–551, April 1986.

[162] C. W. Therrien, editor. *Decision Estimation and Classification: An Introduction to Pattern Recognition and Related Topics*. John Wiley & Sons, New York, 1989.

[163] A. Thompson, J. Kay, and D. Titterington. A cautionary note about the cross validatory choice. *J. statist. Comput. Simul.*, 33:199–216, 1989.

[164] A. M. Thompson, J. C. Brown, J. W. Kay, and D. M. Titterington. A study of methods of choosing the smoothing parameter in image restoration. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 13(4):326–339, 1991.

[165] A. Tikhonov and V. Arsenin. *Solutions of Ill-Posed Problems*. Winston and Sons, New York, 1977.

[166] D. Tretter, C. A. Bouman, K. Khawaja, and A. Maciejewski. A multiscale stochastic image model for automated inspection. *IEEE Trans. on Image Processing*, 4(12):507–517, December 1995.

[167] C. Won and H. Derin. Segmentation of noisy textured images using simulated annealing. In *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, pages 14.4.1–14.4.4, Dallas, TX, 1987.

[168] C. S. Won and H. Derin. Unsupervised segmentation of noisy and textured images using Markov random fields. *CVGIP:Graphical Models and Image Proc.*, 54(4):308–328, July 1992.

[169] J. Woods, S. Dravida, and R. Mediavilla. Image estimation using double stochastic Gaussian random field models. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, PAMI-9(2):245–253, March 1987.

[170] C. Wu. On the convergence properties of the EM algorithm. *Annals of Statistics*, 11(1):95–103, 1983.

[171] C.-h. Wu and P. C. Doerschuk. Cluster expansions for the deterministic computation of Bayesian estimators based on Markov random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 17(3):275–293, March 1995.

[172] C.-h. Wu and P. C. Doerschuk. Texture-based segmentation using Markov random field models and approximate Bayesian estimators based on trees. *J. Math. Imaging and Vision*, 5(4), December 1995.

[173] C.-h. Wu and P. C. Doerschuk. Tree approximations to Markov random fields. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 17(4):391–402, April 1995.

[174] Z. Wu and R. Leahy. An approximate method of evaluating the joint likelihood for first-order GMRFs. *IEEE Trans. on Image Processing*, 2(4):520–523, October 1993.

[175] J. Zerubia and R. Chellappa. Mean field annealing using compound Gauss-Markov random fields for edge detection and image estimation. *IEEE Trans. on Neural Networks*, 4(4):703–709, July 1993.

[176] J. Zhang. The mean field theory in EM procedures for Markov random fields. *IEEE Trans. on Signal Processing*, 40(10):2570–2583, October 1992.

[177] J. Zhang. The mean field theory in EM procedures for blind Markov random field image restoration. *IEEE Trans. on Image Processing*, 2(1):27–40, January 1993.

[178] J. Zhang and G. G. Hanauer. The application of mean field theory to image motion estimation. *IEEE Trans. on Image Processing*, 4(1):19–33, January 1995.

[179] J. Zhang and J. W. Modestino. A model-fitting approach to cluster validation with application to stochastic model-based image segmentation. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 12(10):1009–1017, October 1990.

[180] J. Zhang, J. W. Modestino, and D. A. Langan. Maximum-likelihood parameter estimation for unsupervised stochastic model-based image segmentation. *IEEE Trans. on Image Processing*, 3(4):404–420, July 1994.

[181] M. Zhang, R. Haralick, and J. Campbell. Multispectral image context classification using stochastic relaxation. *IEEE Trans. on Systems Man and Cybernetics*, vol. 20(1):128–140, February 1990.