ECE 468: Digital Image Processing

Lecture 23

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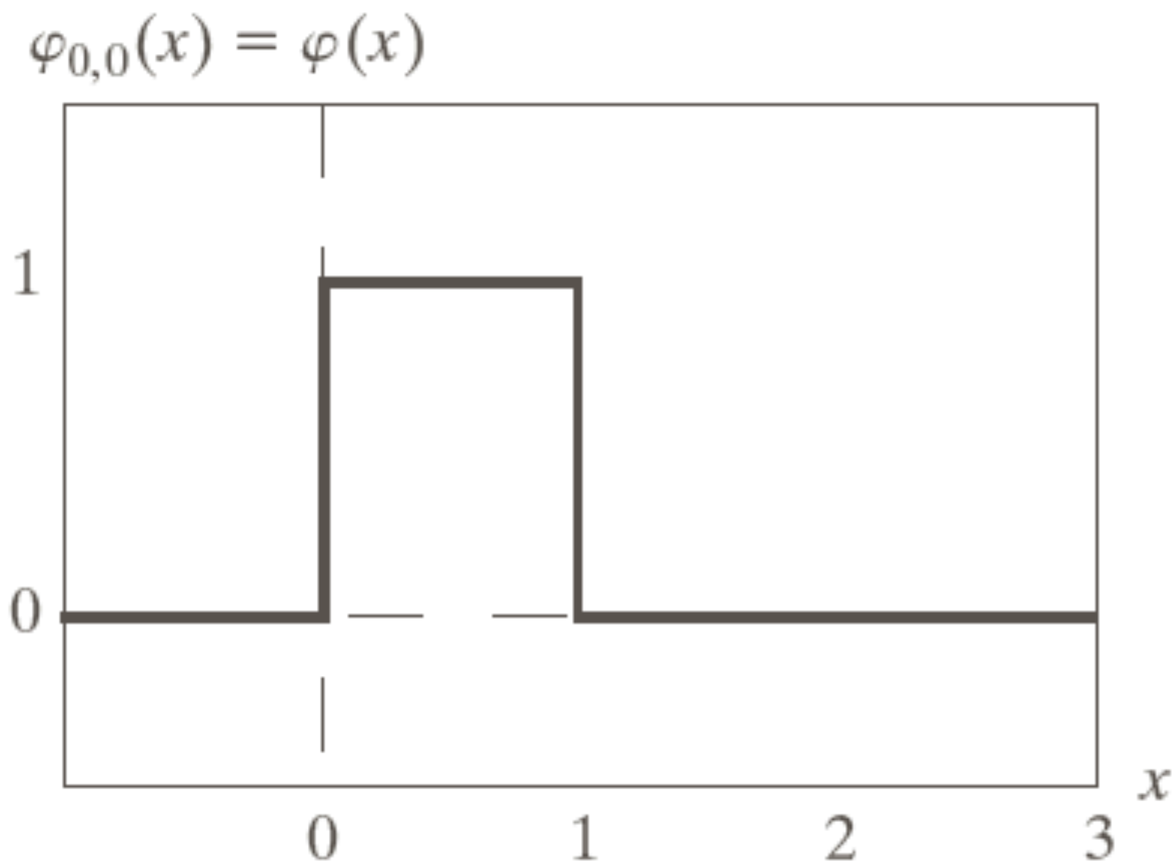
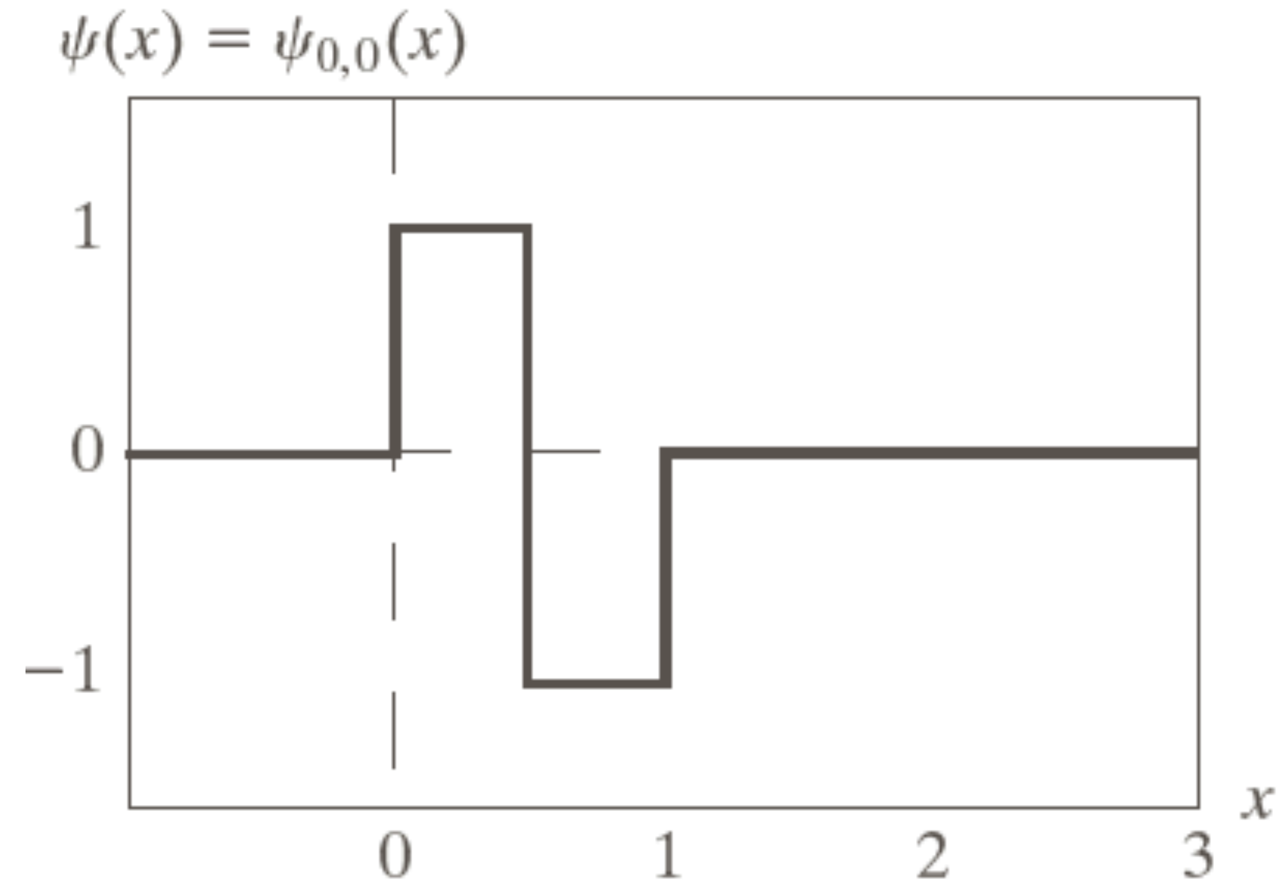
Outline

- Discrete Wavelet Transform (Textbook 7.3.2)
- Fast Wavelet Transform (Textbook 7.4)

Scaling and Wavelet Haar Functions





Scaling Functions

Given: $\varphi(x)$

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

Relations Between the Scaling Functions

$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n)$$

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$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n)$$

$$\varphi(2^j x - k) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2(2^j x - k) - n)$$

Relations Between the Scaling Functions

$$\varphi(2^j x - k) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2(2^j x - k) - n)$$

$$= \sum_n h_\varphi(n)\sqrt{2}\varphi(2^{j+1}x - (2k + n))$$

$$m = 2k + n$$

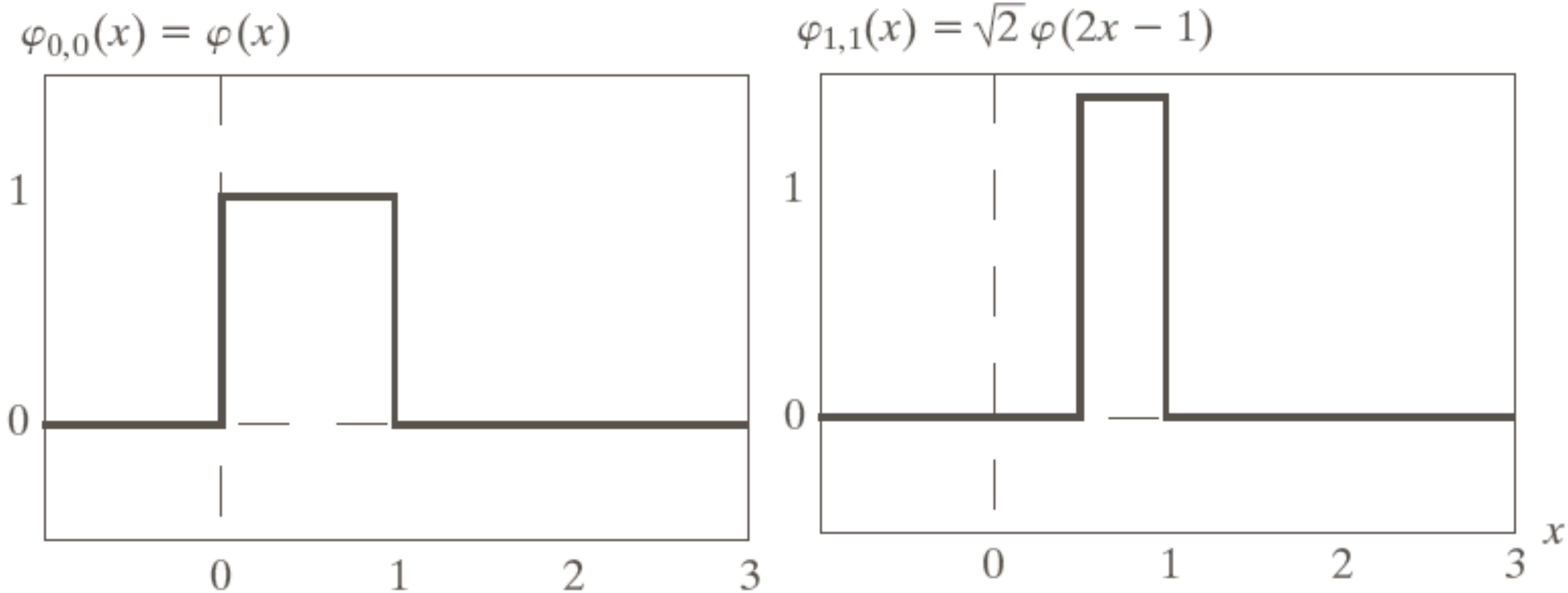
$$= \sum_m h_\varphi(m - 2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

Relations Between the Scaling Functions

$$\varphi(2^j x - k) = \sum_m h_\varphi(m - 2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

$$h_\varphi(m - 2k) = \langle \varphi(2^j x - k), \sqrt{2}\varphi(2^{j+1}x - m) \rangle$$

example: $$h_\varphi(1 - 2 \cdot 0) = h_\varphi(1) = \frac{\sqrt{2}}{2}$$



Relations Between the Scaling and Wavelet Functions

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x - n)$$

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$$\psi(2^j x - k) = \sum_n h_\psi(n)\sqrt{2}\varphi(2(2^j x - k) - n)$$

$$m = 2k + n$$

$$= \sum_m h_\psi(m - 2k)\sqrt{2}\varphi(2^{j+1} x - m)$$

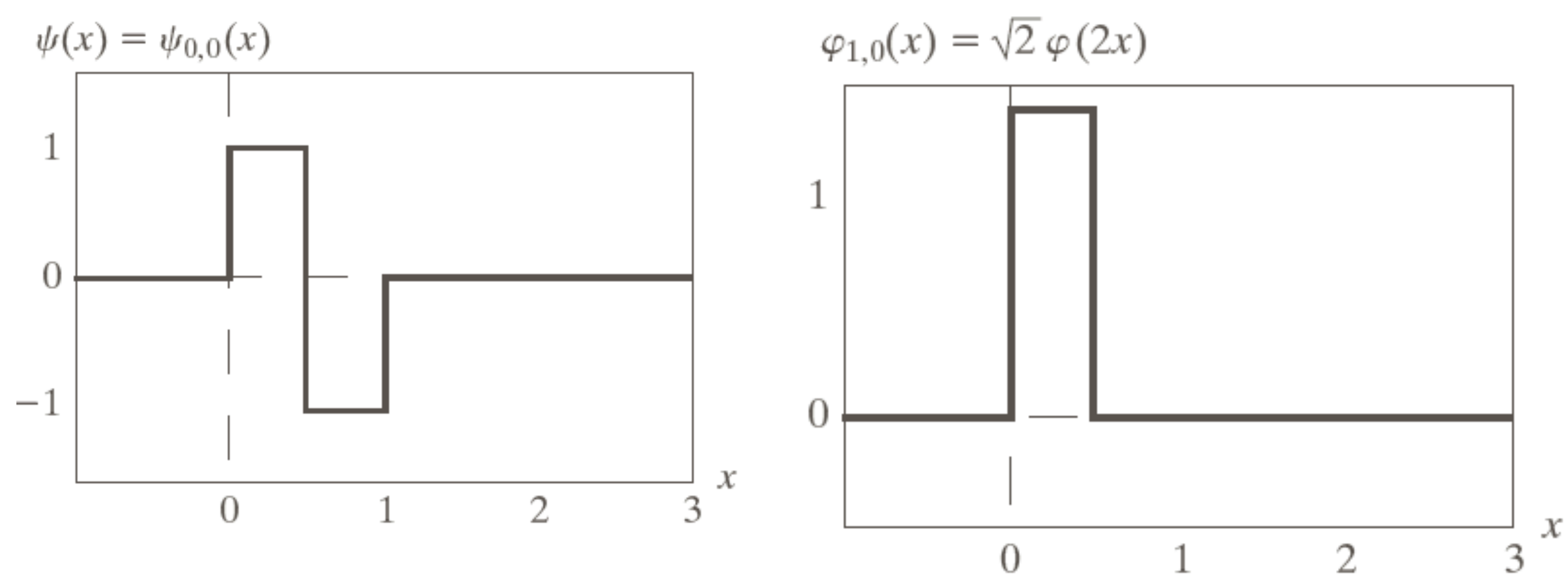
Relations Between the Scaling and Wavelet Functions

$$\psi(2^j x - k) = \sum_m h_\psi(m - 2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

$$h_\psi(m - 2k) = \langle \psi(2^j x - k), \sqrt{2}\varphi(2^{j+1}x - m) \rangle$$

example:

$$h_\psi(0 - 2 \cdot 0) = h_\psi(0) = \frac{\sqrt{2}}{2}$$

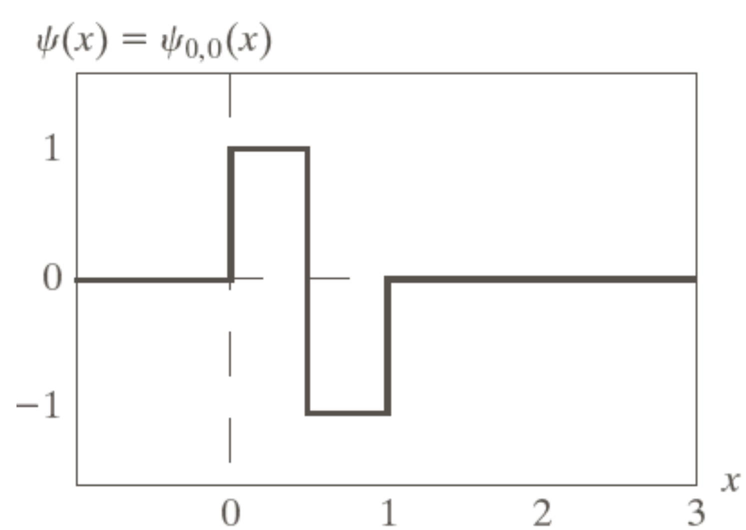


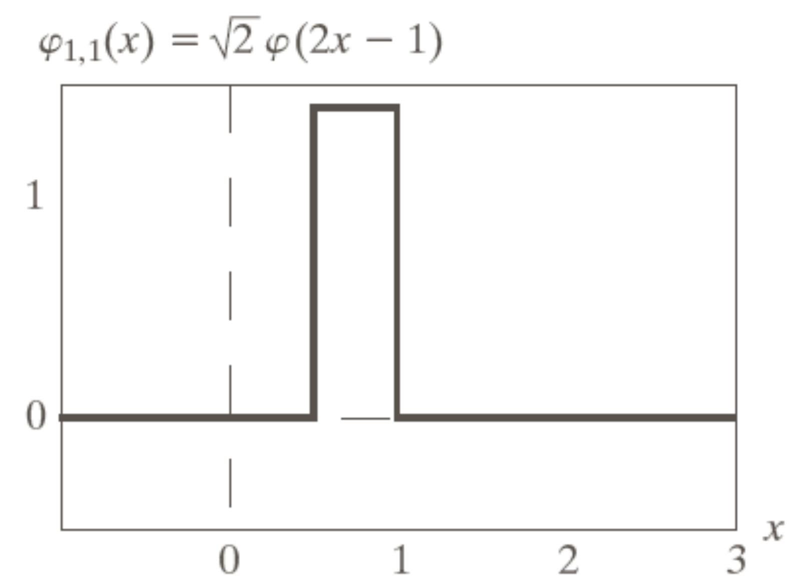
Relations Between the Scaling and Wavelet Functions

$$\psi(2^j x - k) = \sum_m h_\psi(m - 2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

$$h_\psi(m - 2k) = \langle \psi(2^j x - k), \sqrt{2}\varphi(2^{j+1}x - m) \rangle$$

example: $h_\psi(1 - 2 \cdot 0) = h_\psi(1) = -\frac{\sqrt{2}}{2}$





Discrete Wavelet Transform

$$f(x) = \frac{1}{\sqrt{M}} \sum_{k} W^{\varphi}_{j_0,k} \varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0=0}^{J-1} \sum_{k} W^{\psi}_{j,k} \psi_{j,k}(x)$$

$$j_0 = 0$$

$$M = 2^J$$

$$x = 0, 1, 2, \ldots, M-1$$

$$k = 0, 1, 2, \ldots, 2^j - 1$$

Discrete Wavelet Transform

$$W^{\varphi}_{j_0,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\varphi_{j_0,k}(x)$$

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j,k}(x), \; j \geq j_0$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x)\varphi_{j,k}(x)$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\varphi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \varphi(2^j x - k)$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\varphi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \varphi(2^j x - k)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \sum_m h_{\varphi}(m - 2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x) \varphi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \varphi(2^j x - k)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \sum_m h_{\varphi}(m - 2k) \sqrt{2} \varphi(2^{j+1} x - m)$$

$$= \sum_m h_{\varphi}(m - 2k) \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j+1}{2}} \varphi(2^{j+1} x - m)$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\varphi_{j,k}(x)$$

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$$= \sum_m h_{\varphi}(m-2k) \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j+1}{2}} \varphi(2^{j+1}x - m)$$

$$= \sum_m h_{\varphi}(m-2k) W^{\varphi}_{j+1,m}$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\varphi_{j,k}(x)$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x)\varphi_{j,k}(x)$$

$$= \sum_{m} h_{\varphi}(m-2k) W^{\varphi}_{j+1,m}$$

Fast Wavelet Transform

$$W^{\varphi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x) \varphi_{j,k}(x)$$

$$= \sum_{m} h_{\varphi}(m - 2k) W^{\varphi}_{j+1,m}$$

$$= h_{\varphi}(-2k) * W^{\varphi}_{j+1,k}$$

Fast Wavelet Transform

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x) \psi_{j,k}(x)$$

Fast Wavelet Transform

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x)\psi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_{x} f(x) 2^{\frac{j}{2}} \psi(2^j x - k)$$

Fast Wavelet Transform

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \psi(2^j x - k)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \sum_m h_{\psi}(m - 2k)\sqrt{2}\varphi(2^{j+1} x - m)$$

Fast Wavelet Transform

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_x f(x) \psi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \psi(2^j x - k)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \sum_m h_{\psi}(m - 2k) \sqrt{2} \varphi(2^{j+1} x - m)$$

$$= \sum_m h_{\psi}(m - 2k) \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j+1}{2}} \varphi(2^{j+1} x - m)$$

Fast Wavelet Transform

$$W_{j,k}^{\psi} = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j,k}(x)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \psi(2^j x - k)$$

$$= \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j}{2}} \sum_m h_\psi(m-2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

$$= \sum_m h_\psi(m-2k) \frac{1}{\sqrt{M}} \sum_x f(x) 2^{\frac{j+1}{2}} \varphi(2^{j+1}x - m)$$

$$= \sum_m h_\psi(m-2k) W_{j+1,m}^{\varphi}$$

Fast Wavelet Transform

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x)\psi_{j,k}(x)$$

Fast Wavelet Transform

$$W_{j,k}^{\psi} = \frac{1}{\sqrt{M}} \sum_{x} f(x) \psi_{j,k}(x)$$

$$= \sum_{m} h_{\psi}(m - 2k) W_{j+1,m}^{\varphi}$$

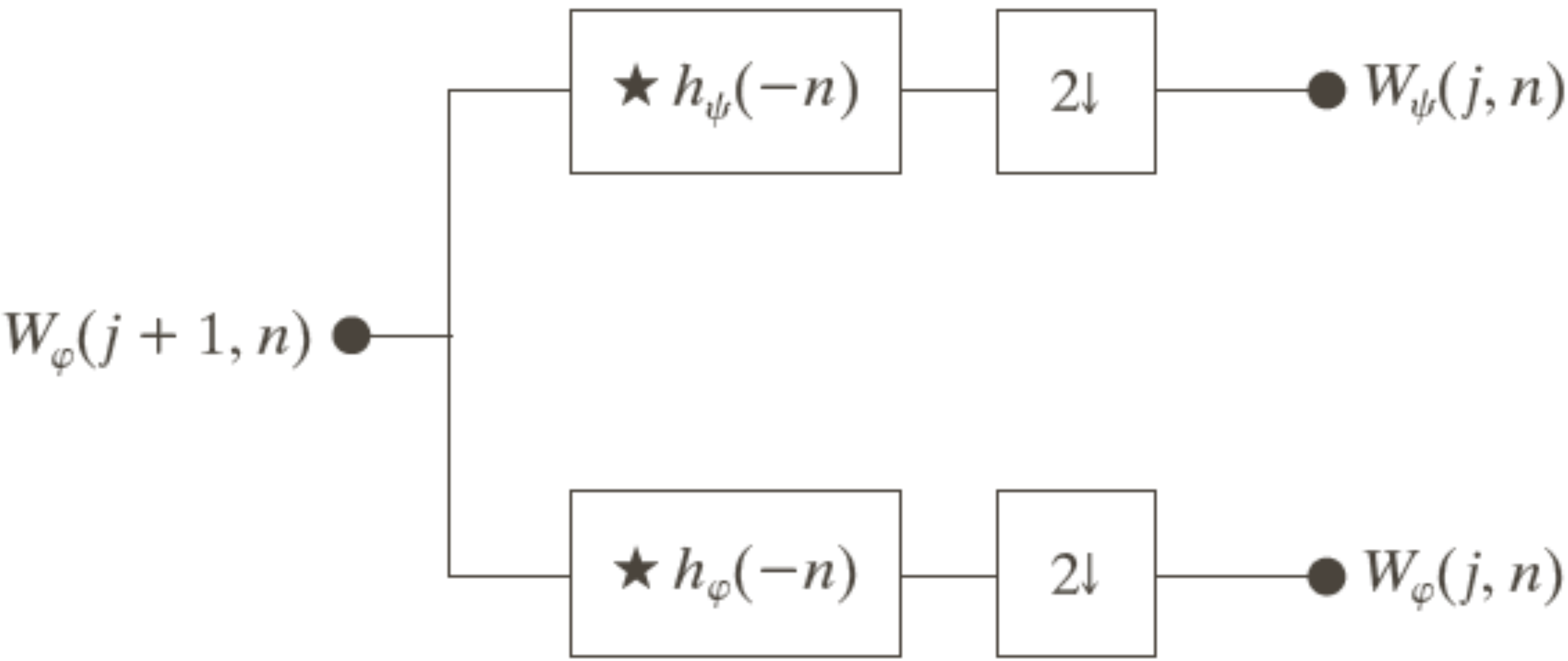
Fast Wavelet Transform

$$W^{\psi}_{j,k} = \frac{1}{\sqrt{M}} \sum_{x} f(x) \psi_{j,k}(x)$$

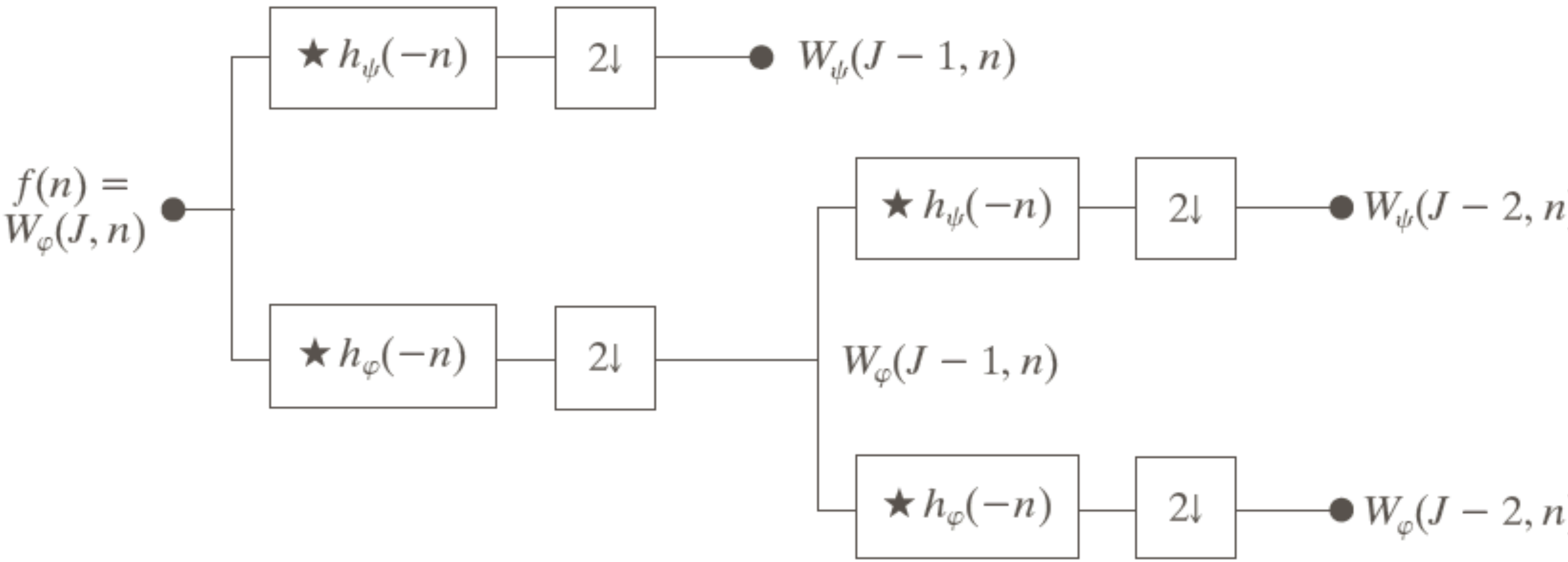
$$= \sum_{m} h_{\psi}(m - 2k) W^{\varphi}_{j+1,m}$$

$$= h_{\psi}(-2k) * W^{\varphi}_{j+1,k}$$

Discrete Wavelet Transform



Discrete Wavelet Transform

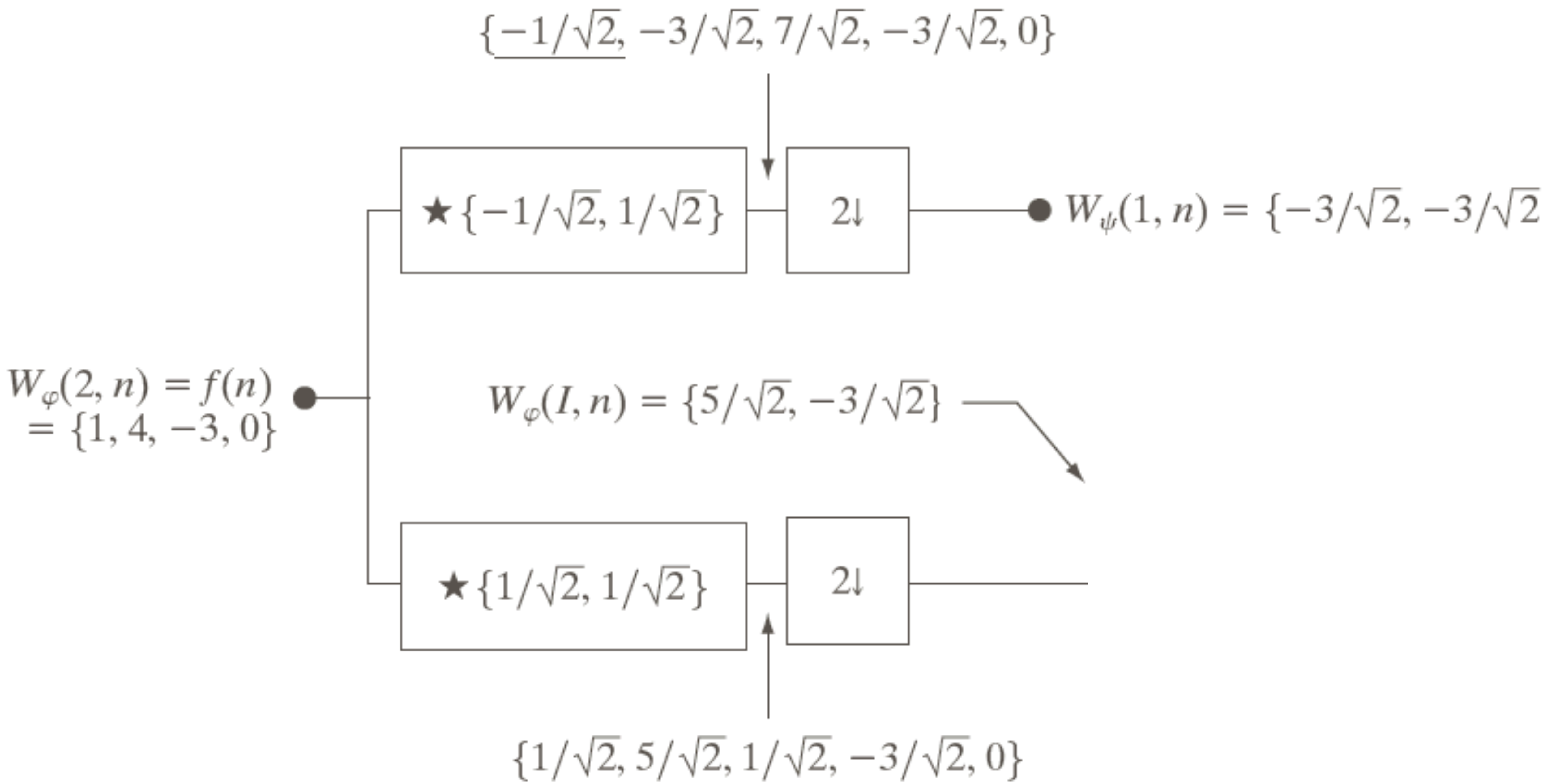


Example: Discrete Wavelet Transform

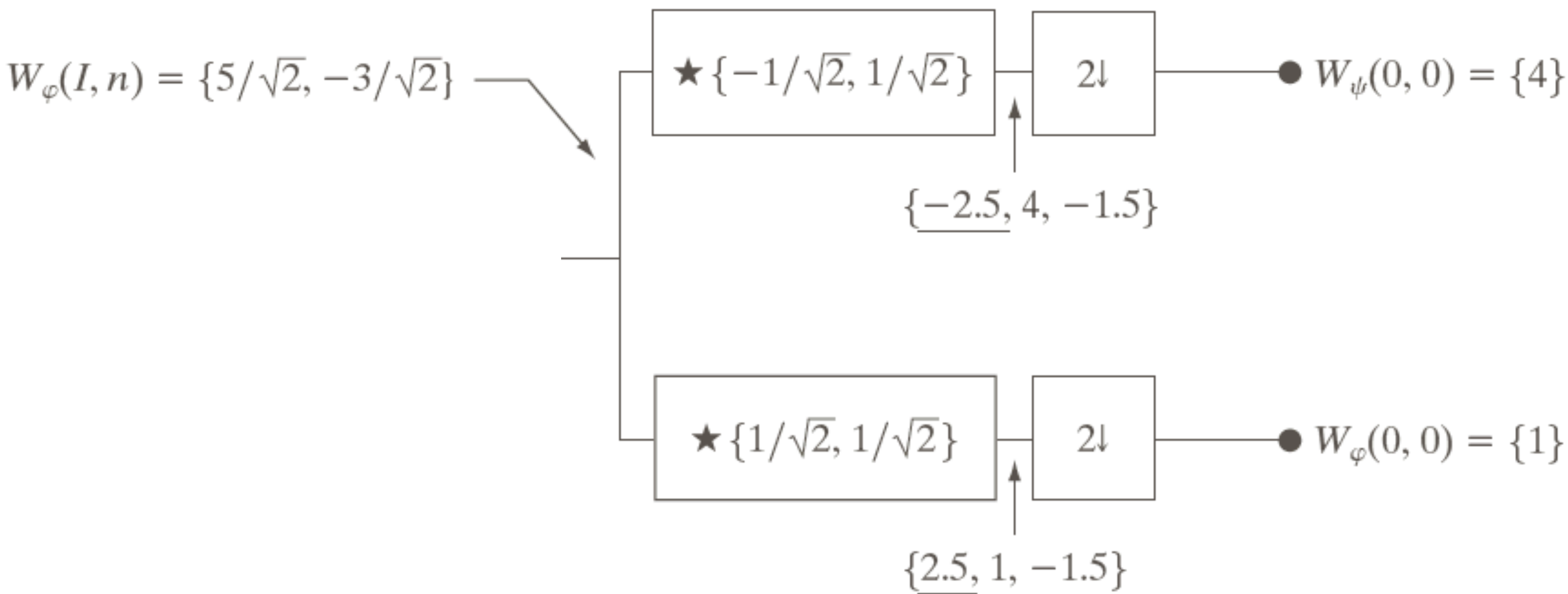
n	$h_\varphi(n)$
0	$1/\sqrt{2}$
1	$1/\sqrt{2}$

TABLE 7.2 Orthonormal Haar filter coefficients for $h_\varphi(n)$.

Example: Discrete Wavelet Transform



Example: Discrete Wavelet Transform



Example: Discrete Wavelet Transform

$$f(n) = \{1, 4, -3, 0\} \quad \Longrightarrow \quad j = 2: \quad W^{\varphi}_{2,k} = \{1, 4, -3, 0\}$$

$$f(x) = \frac{1}{\sqrt{M}} \sum_k W^{\varphi}_{0,k} \varphi_{0,k}(x) + \frac{1}{\sqrt{M}} \sum_{j=0}^{1} \sum_k W^{\psi}_{j,k} \psi_{j,k}(x)$$

$$j = 1: \quad W^{\varphi}_{1,k} = \{\frac{5}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\} \quad W^{\psi}_{1,k} = \{\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\}$$

$$j = 0: \quad W^{\varphi}_{0,0} = \{1\} \quad W^{\psi}_{0,0} = \{4\}$$

Inverse Discrete Wavelet Transform

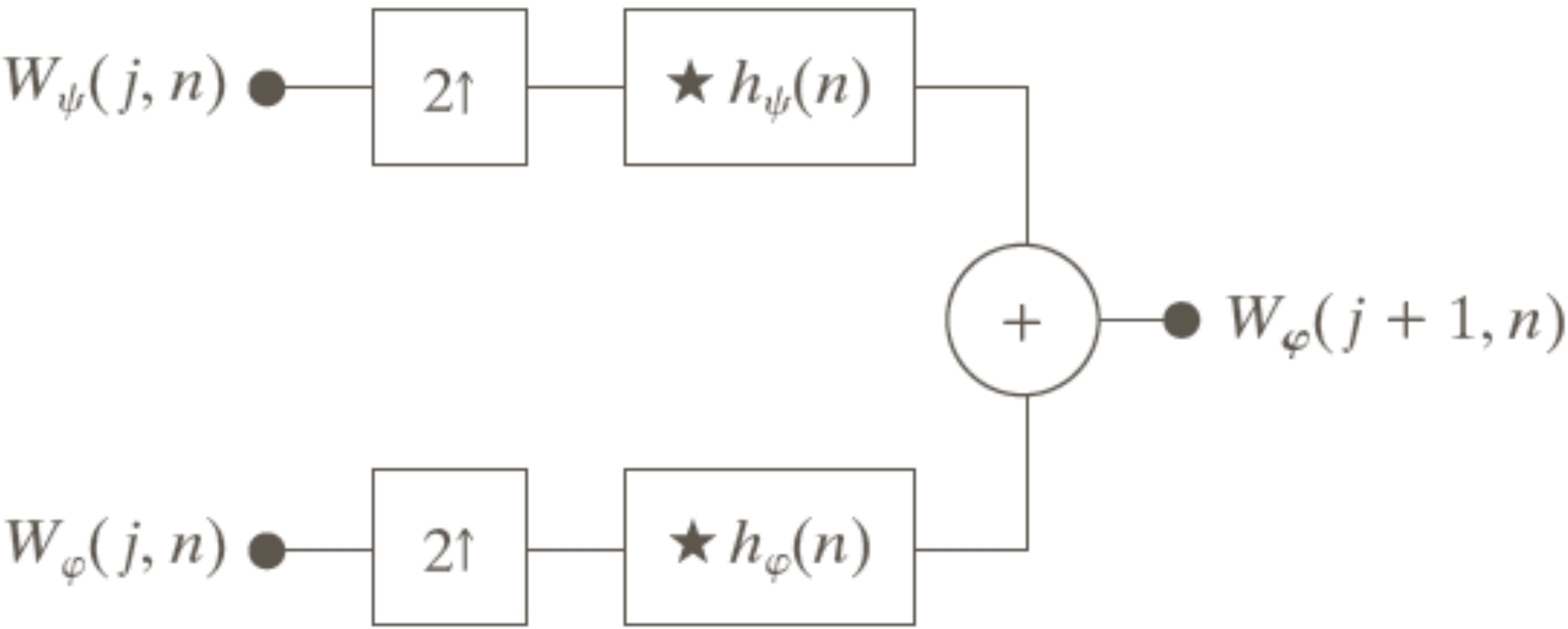
$$W^{\varphi}_{j,k} = h_{\varphi}(-2k) * W^{\varphi}_{j+1,k}$$

$$W^{\psi}_{j,k} = h_{\psi}(-2k) * W^{\varphi}_{j+1,k}$$

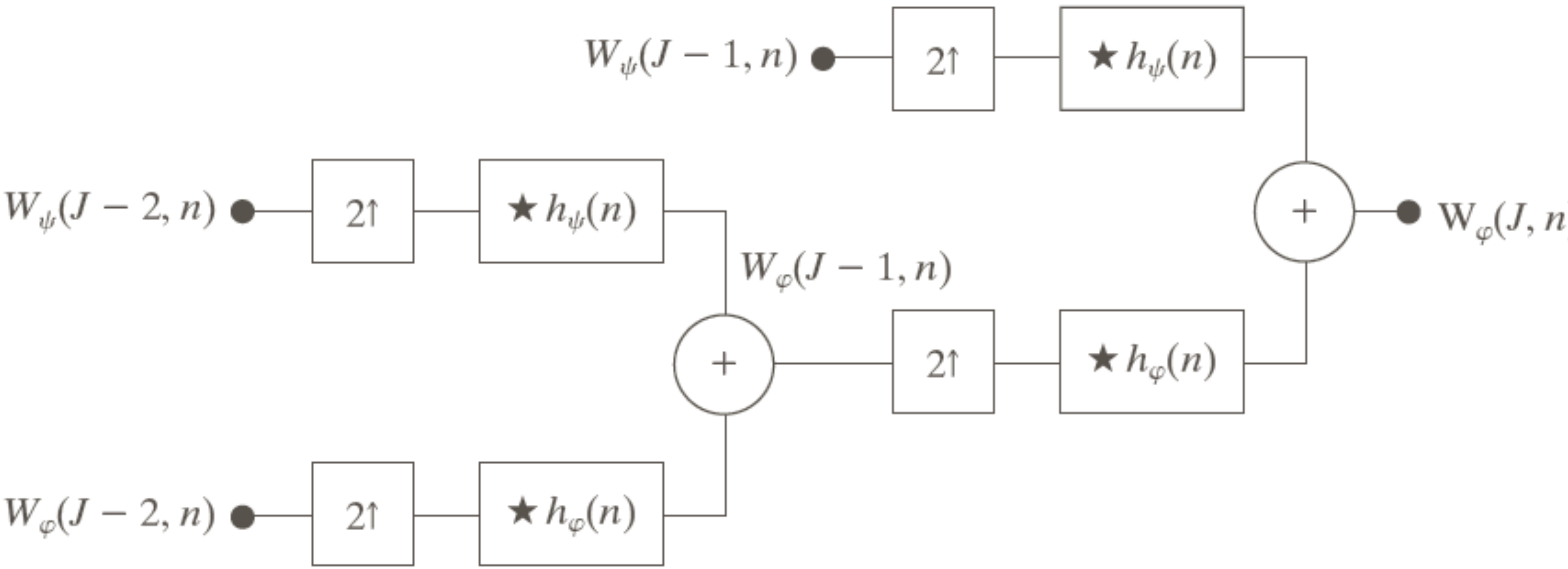
$$\Downarrow$$

$$W^{\varphi}_{j+1,k} = h_{\phi}(k) * W^{\varphi}_{j,k\uparrow 2} + h_{\psi}(k) * W^{\psi}_{j,k\uparrow 2}$$

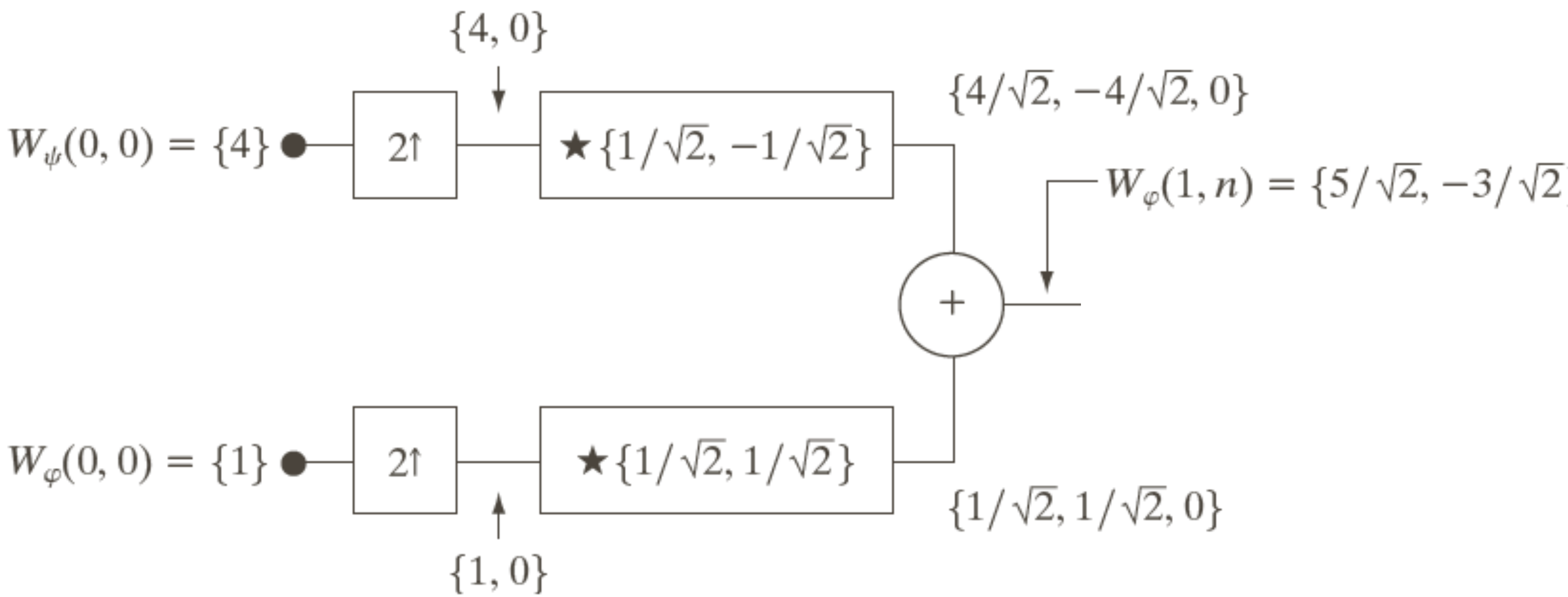
Inverse Discrete Wavelet Transform



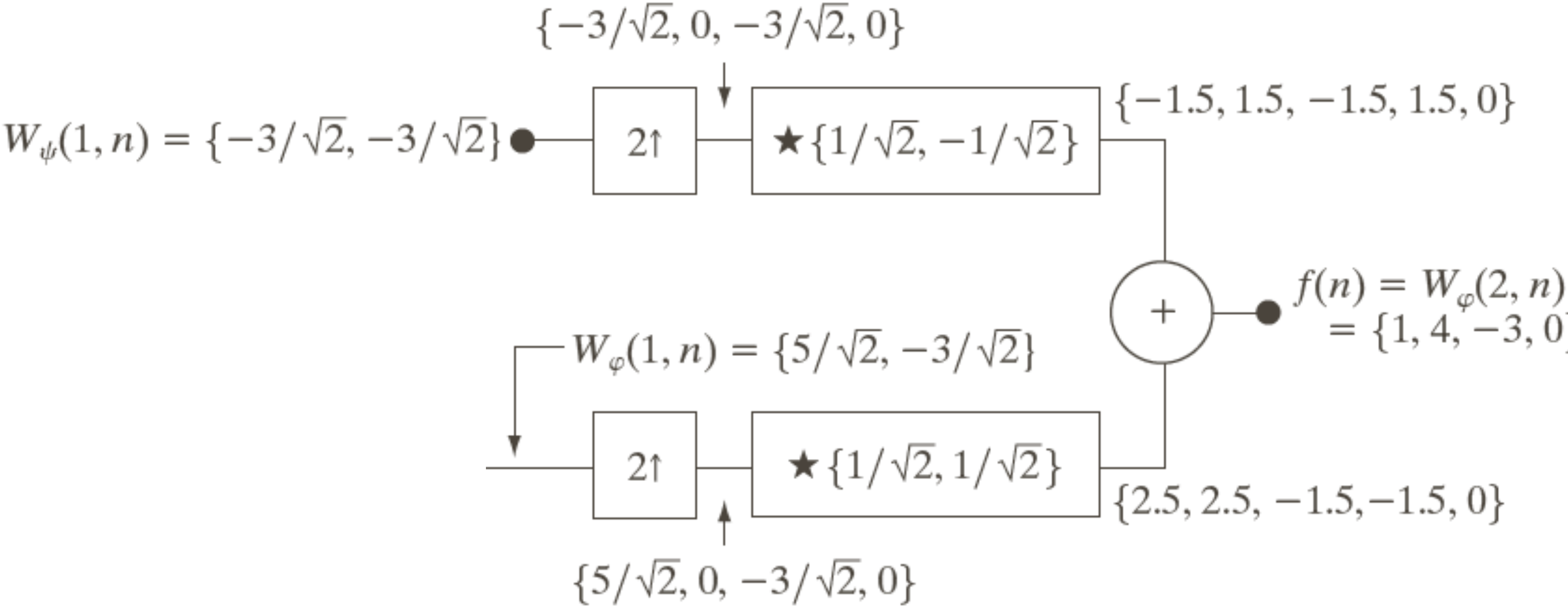
Inverse Discrete Wavelet Transform



Example: Inverse Discrete Wavelet Transform



Example: Inverse Discrete Wavelet Transform



Next Class

- 2D Wavelet Transform (Textbook 7.5)