Outline

- 2D Wavelet Transform (Textbook 7.5)
2D Scaling and Wavelet Functions

\[ \varphi(x, y) = \varphi(x) \varphi(y) \]
2D Scaling and Wavelet Functions

\[ \psi^H(x, y) = \psi(x) \varphi(y) \]

\[ \psi^V(x, y) = \varphi(x) \psi(y) \]

\[ \psi^D(x, y) = \psi(x) \psi(y) \]
2D DWT
2D Scaling and Wavelet Functions

$$\varphi_{j,m,n}(x, y) = 2^{j/2} \varphi(2^j x - m, 2^j y - n)$$

$$\psi^i_{j,m,n}(x, y) = 2^{j/2} \psi^i(2^j x - m, 2^j y - n), \quad i = H, V, D$$

$$\varphi(x, y) = \varphi(x)\varphi(y)$$

$$\psi^H(x, y) = \psi(x)\varphi(y)$$

$$\psi^V(x, y) = \varphi(x)\psi(y)$$

$$\psi^D(x, y) = \psi(x)\psi(y)$$
2D Discrete Wavelet Transform

\[ W_\varphi(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \varphi_{j,m,n}(x, y) \]

\[ W_\psi(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi_{j,m,n}(x, y) \]

\[ f(x, y) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_\varphi(j_0, m, n) \varphi_{j_0,m,n}(x, y) \]

\[ + \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=0}^{\infty} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_\psi^i(j, m, n) \psi^i_{j,m,n}(x, y) \]
2D Discrete Wavelet Transform

\[ \varphi(x, y) = \varphi(x) \varphi(y) \]

\[ \psi^H(x, y) = \psi(x) \varphi(y) \]

\[ \psi^V(x, y) = \varphi(x) \psi(y) \]

\[ \psi^D(x, y) = \psi(x) \psi(y) \]

\[ \varphi_{j,m,n}(x, y) = 2^j \varphi(2^j x - m, 2^j y - n) \]

\[ \psi_{j,m,n}^i(x, y) = 2^j \psi^i(2^j x - m, 2^j y - n), \quad i = H, V, D \]

\[ W_\varphi(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \varphi_{j,m,n}(x, y) \]

\[ W_\psi(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi_{j,m,n}(x, y) \]

\[ f(x, y) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_\varphi(j_0, m, n) \varphi_{j_0,m,n}(x, y) \]

\[ + \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_\psi^i(j, m, n) \psi_{j,m,n}^i(x, y) \]
Recall:

\[
W_\varphi(j, n) = \sum_k h_\varphi(k - 2n) W_\varphi(j + 1, k)
\]
Fast 2D Wavelet Transform

1D case:
\[ W_\varphi(j, m) = \sum_k h_\varphi(k - 2m)W_\varphi(j + 1, k) \]

2D case:
\[ W_\varphi(j, m, n) = \sum_l h_\varphi(l - 2n) \left[ \sum_k h_\varphi(k - 2m)W_\varphi(j + 1, k, l) \right] \]
Inverse 2D DWT
Uncertainty Principle

Energy spread of a function and its Fourier transform CANNOT BE simultaneously arbitrarily small.
Example: Uncertainty Principle

**FIGURE 4.13** (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the \( t \)-axis, so the spectrum is more “contracted” along the \( \mu \)-axis. Compare with Fig. 4.4.
Example: Uncertainty Principle

**FIGURE 4.52** Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

**FIGURE 4.53** Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.
1D Case: Gabor -- Ville -- Wigner

- Decompose a signal
- over elementary waveforms
- that have a minimal spread in the time-frequency plane
1D Case: Time-Frequency Plane

A Wavelet Tour of Signal Processing
Stéphane Mallat, Academic Press 1999 (2nd edition)

Figure 1.1: Time-frequency boxes ("Heisenberg rectangles") representing the energy spread of two Gabor atoms
Gabor Wavelets

\[ g_{j,k,u}(x) = 2^{j/2} g(2^j x - k)e^{jux} \]

- Gabor generator
- Shift
- Rotation
- Support
- Frequency