Effect of Correlation Structure Model on Geotechnical Reliability-based Serviceability Limit State Simulations

Jonathan C. Huffman
Senior Project Engineer and Graduate Student, Foundation Engineering, Inc. and Oregon State University, Corvallis, USA

Armin W. Stuedlein
Assistant Professor and Loosley Faculty Fellow, Oregon State University, Corvallis, USA

ABSTRACT: Reliability-based serviceability limit state (SLS) models for foundation elements commonly employ a two-parameter load-displacement model to relate imposed displacements to a particular load or vice versa. Numerous studies have shown that the load-displacement model parameters tend to be correlated; subsequently, considerable effort is required to appropriately model the correlation structure for Monte Carlo-based reliability simulations. This paper uses copula theory, a database of high quality full-scale loading tests of spread footings on aggregate pier (i.e., stone column) reinforced clay, and a recently developed ultimate limit state (ULS) model to investigate the effect of various correlation structures on the probability of exceeding the SLS. "Lumped" load and resistance factors, which conveniently relate the portion of mobilized resistance to a given footing displacement, accounting for uncertainty in the applied load, ULS resistance of the clay subgrade, allowable displacement, and footing size, generated using various assumed correlation structure models are compared. Additionally, the penalty in reliability associated with ignoring the bivariate correlation is explored for comparison. This study demonstrates that selection of the appropriate copula represents a critical task in the development of calibrated reliability-based geotechnical SLS design approaches.

1. INTRODUCTION

Geotechnical serviceability limit state (SLS) design for foundations is performed to limit displacements for a given structure to an acceptable level. Numerous SLS procedures have been recently developed that use a two-parameter load-displacement model to relate the imposed displacements to a particular load or vice versa. Such functions have been developed for both shallow foundations (e.g., Uzieli and Mayne 2011, 2012; Huffman and Stuedlein 2014, 2015) and deep foundations (e.g., Phoon and Kulhawy 2008, Dithinde et al. 2011, Li et al. 2013). Full-scale foundation loading tests are preferably used to calibrate the load-displacement models, and the observed dispersion of the loading test data are then used to develop the reliability-based limit state function. Many of these recent studies have noted that the parameters characterizing the selected load-displacement model are correlated. Effort is therefore required to appropriately characterize and model the correlation structure and account for it in reliability-based design procedures.

Where previous studies by Huffman and Stuedlein (2014, 2015) use selected bivariate load-displacement model parameter correlation structure to construct "packaged" SLS design methodology, this paper expands the previous work by investigating the impact of using several copula models of various degrees of fitting quality on reliability calibrations. The impact on reliability calibrations are evaluated using a database of full-scale loading tests of stone column- (or aggregate pier-) reinforced clay. First, an empirically-derived ultimate limit state (ULS) model is reviewed, followed by the presentation of the selected SLS function. Next, the SLS calibration procedure is described within the Monte Carlo simulation framework, which allowed the back-calculation of a ‘lumped” load and resistance factor that conveniently relates the portion of mobilized resistance to a given footing displacement, accounting for uncertainty in the applied bearing pressure, ULS resistance of the clay subgrade, allowable displacement, and footing size. The effect of the correlation structure between the bearing pressure-displacement model parameters is then investigated using well-known copula functions, namely, the Gaussian, Frank, Clayton, and Gumbel copulas. This study demonstrates that the selection of the appropriate copula represents a critical task in the development of appropriately calibrated reliability-based geotechnical SLS design approaches with significant impact to the allowable bearing pressure of shallow foundations.

2. SELECTED LIMIT STATE MODELS

This study uses a database of footing loading tests on aggregate pier-reinforced clay described by Stuedlein and Holtz (2013, 2014) and Huffman and Stuedlein (2014) to demonstrate the role of correlation structure model on the computed reliability for the SLS. Stuedlein and Holtz (2013) used the database to develop a bearing capacity (i.e., ULS) function. Subsequently, Huffman and Stuedlein (2014) developed an SLS model for allowable immediate displacement. Stuedlein and Holtz (2013) developed the ULS function using multiple nonlinear regression analyses to provide a predicted bearing capacity, $q_{ult,p}$, from the capacities interpreted, $q_{ult,i}$, from the loading tests. The predicted capacity of an isolated, rigid spread footing supported on aggregate pier improved clay is (in kPa; Stuedlein and Holtz 2013):

$$\ln(q_{ult,p}) = b_0 + b_1 S_r + b_2 a_r + b_3 d_f S_r + b_4 \tau_m^{-1} + b_5 \tau_m \quad (1)$$

where S_r is the slenderness ratio of the aggregate pier(s), equal to $S_r = L_p/d_p$ (i.e., ratio of pier length to pier diameter), a_r is the area replacement, equal to the ratio of the pier area to the foundation footprint, d_f is the footing embedment depth (in meters), and τ_m is the matrix soil shear mass participation factor, given as $\tau_m = s_u/a_r$, where s_u is the undrained shear strength of the matrix soil (in kPa). The best-fit model coefficients are b_0 = 4.756, b_1 = 0.013, b_2 = 1.914, b_3 = 0.070, b_4 = 13.71, and b_5 = 0.005. Equation (1) is characterized with a mean bias (i.e., average ratio of $q_{ult,i}$ to $q_{ult,p}$) of 1.00 and coefficient of variation (COV) in point bias of 13.1 percent (Stuedlein and Holtz 2013).

Huffman and Stuedlein (2014) used Eq. (1) as a reference capacity, q_{ref}, to formulate a SLS power law describing immediate displacement, δ, of footings on aggregate pier-reinforced clay. The best-fit SLS model for the displacement-dependent mobilized resistance, q_{mob}, was determined equal to:

$$q_{mob} = c_1 \left(\delta / B' \right)^{c_2} q_{ref} \quad (2)$$

where B' = the equivalent footing diameter. For square footings, B' is defined as the diameter that provides the same area as that of a square footing (Mayne and Poulos 1999). Fitting coefficient c_1, and exponent, c_2, were determined for each of the loading tests in the database using least squares optimization.

3. MONTE CARLO-BASED PERFORMANCE FUNCTION SIMULATION

The performance function, P, equal to the margin of safety that limits the probability of failure, p_f, to a pre-determined magnitude, was used to form the probabilistic framework for incorporating the observed or assumed uncertainty in the SLS function. The general framework is given by (Baecher and Christian 2003, Phoon 2008):

$$p_f = \Pr(R - Q < 0) = \Pr(P < 0) \quad (3)$$

where R = the resistance, and Q = the load.

For the current study, the resistance, R, is estimated from Eq. (2) and can be incorporated as follows:

$$p_f = \Pr\left(c_1\left(\delta_a/B'\right)^{c_2} q_{ult,p} - q_{app} < 0\right)$$

$$= \Pr\left(c_1\left(\delta_a/B'\right)^{c_2} < \frac{q_{app}}{q_{ult,p}}\right) \qquad (4)$$

where q_{app} = the applied bearing pressure, associated with an allowable displacement, δ_a. The reference capacity, q_{ref}, was replaced with the predicted capacity, $q_{ult,p}$, in Eq. (4).

Normalized unit mean random variables for $q_{ult,p}$ and q_{app}, each associated with an observed or assumed distribution (e.g., Uzielli and Mayne 2011), can be incorporated into Eq (4) for ease of simulation as:

$$p_f = \Pr\left(c_1\left(\delta_a/B'\right)^{c_2} < \frac{q_{app,n} * q_{app}^*}{q_{ult,n} * q_{ult}^*}\right)$$

$$= \Pr\left(c_1\left(\delta_a/B'\right)^{c_2} < \frac{1}{\psi_q}\frac{q_{app}^*}{q_{ult}^*}\right) \qquad (5)$$

where $q_{app,n}$ and $q_{ult,n}$ equal the nominal values of the applied bearing pressure and ultimate resistance, respectively, and q^*_{app} and q^*_{ult} represent the normalized random values. By dividing $q_{app,n}$ by $q_{ult,n}$, Eq. (5) becomes unitless and allows the introduction of ψ_q, used to represent a combined (i.e., "lumped") load and resistance factor. Thus, p_f can then be solved in terms of ψ_q based on a predetermined allowable displacement and given footing size.

Monte Carlo simulations (MCS) were used to solve p_f and the associated reliability index, β, using prescribed magnitudes of ψ_q ranging from 1 to 20. Table 1 summarizes the pertinent characteristic distributions estimated or assumed for the MCS. Each simulation included approximately $5x10^6$ randomly-generated load and resistance variables (c_1, c_2, δ_a, B', q^*_{ult} and q^*_{app}) to solve the performance function. The MCS was repeated approximately 3,600 times in order to estimate β and p_f for different combinations of δ_a, B', COV(q^*_{app}), COV(δ_a), and ψ_q for each correlation structure model.

Once calibrated using the MCS, the allowable bearing pressure, q_{allow}, satisfying the given serviceability limit state of immediate displacement for spread footings supported on aggregate pier reinforced clay may be computed using (Huffman and Stuedlein 2014):

$$q_{allow} = \frac{1}{\psi_q}\left(c_1\left(\delta_a/B'\right)^{c_2}\right) q_{ult,p} \qquad (6)$$

The resistance parameters (q^*_{ult}, c_1 and c_2) used for the MCS were established using the results of the loading test database as described earlier.

Table 1. Summary of resistance, load, and displacement parameters used in the Monte Carlo Simulations (adapted from Huffman and Stuedlein 2014).

Parameter	Nominal Value	COV (%)	Distribution
c_1	3.088	40.7	lognormal
c_2	0.454	23.3	lognormal
q^*_{ult}	1.00	13.1	lognormal
q^*_{app}	1.00	10, 20	lognormal
δ_a (mm)	2.5, 5, 7.5, 10, 12.5, 15, 20, 25, 30, 37.5, 50, 75, 100, 112.5, 125, 150, 187.5, 200, 225, 250, 300, 400, 500, 600	0, 20, 40, 60	lognormal
B' (m)	0.5, 1.0, 1.5, 2.0, 2.5, 3.0	2	normal

Lognormal distributions were selected to represent each of the resistance parameters based on Anderson-Darling (1952) goodness-of-fit tests evaluated with a significance of 5 percent.

The normalized applied bearing pressure parameter q^*_{app}, used with the MCS was established based on accepted loading scenarios consistent with national codes (e.g., AASHTO 2012) and similar reliability analyses performed by others (e.g., Phoon and Kuhawy 2008, Uzielli and Mayne 2011, and Li et al. 2011).

The range in allowable displacement, δ_a, and equivalent footing diameter, B', were selected to provide a normalized allowable displacement (i.e., δ_a/B') between 0.005 and 0.20, capturing the typical range in spread footing design. However, it should be noted some values of δ_a included herein are larger than typical design scenarios. A range in the COVs was assumed for δ_a (see Table 1) owing to the lack of a clear consensus associated with allowable structure displacements, and provides user flexibility (Huffman and Stuedlein 2014).

4. USE OF COPULA THOERY TO MODEL CORRELATION STRUCTURE

The model parameters associated with the selected SLS power law model showed moderate to strong dependence, which must be accounted for to avoid biases that can influence the reliability simulations (e.g., Phoon and Kulhawy 2008; Uzielli and Mayne 2011, 2012; Tang et al. 2013). Such dependence was observed between the c_1 and c_2 parameters, as shown in Figure 1, and associated with a sample Kendall's Tau correlation coefficient, ρ_τ, of 0.43. Dependence between other variables included in the reliability simulations were also investigated, but none were observed (Huffman and Stuedlein 2014).

To evaluate the influence of correlation structure model on the reliability calibrations, bivariate Gaussian, Frank, Clayton and Gumbel copula functions were investigated. Copula functions can be used to describe the probable values of one variable given the values of another, and a full description of the distribution of the each variable is established by coupling the results of the copula analyses to the marginal distribution of each variable (Nelson 2006).

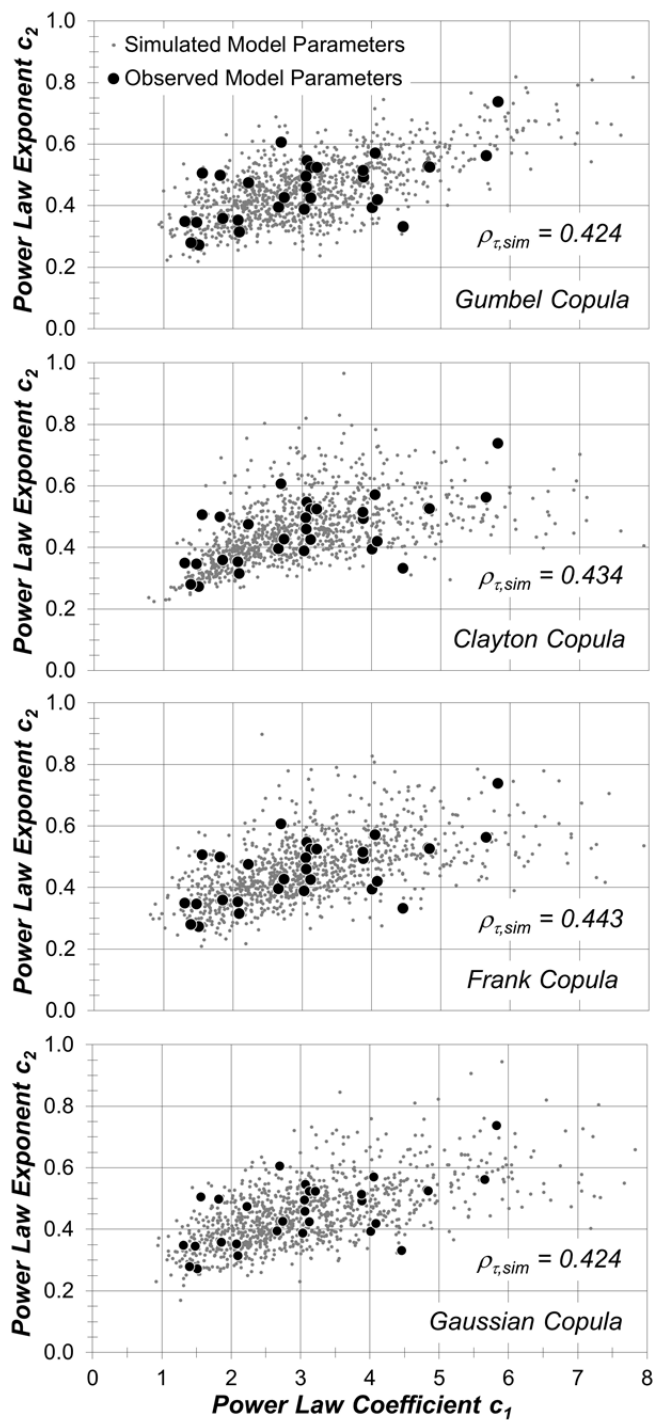

Figure 1. Comparison of 1,000 c_1-c_2 pairs simulated using various copula models with the sample pairs derived from the load test database.

Copula functions, each with its own formulation for associating ranked variables, allow the description of different correlation structures. The bivariate Gaussian copula models an elliptical correlation (positive or negative correlation) between the two parameters being investigated, whereas the Frank copula models a uniform linear correlation (positive or negative). The Clayton and Gumbel copulas model positive correlation only, with the Clayton copula providing for greater tail dependence at small values and the Gumbel copula providing for greater tail dependence at larger values. Each of these copulas are calibrated with a copula parameter, θ, which is a function of the sample ρ_τ, to describe the relative strength of the correlation.

Copula functions assume uniformly distributed marginal distributions in rank [0,1] space. Therefore, standardized rank values u_1 and u_2 of the power law model parameters (c_1 and c_2), were calculated by dividing the rank values by the total number of values in the dataset. The u_1 and u_2 values were used with ρ_τ to establish the copula parameter, θ, and the copula probability functions. The relationship between two variables (e.g., c_1 and c_2) and a two-parameter copula probability function, C_{c_1,c_2}, is determined by fitting to ρ_τ using (e.g., Nelson 2006, Li et al. 2013):

$$\rho_\tau(c_1, c_2) = 4\int_0^1\int_0^1 C_{c_1,c_2}(u_1,u_2)\,dC_{c_1,c_2}(u_1,u_2) - 1 \quad (7)$$

The best-fit copula was evaluated using the Akaike information criteria (AIC; Akaike 1974) and Bayesian information criteria (BIC; Schwarz 1978). The probability functions for the different copulas investigated herein and the best-fit copula parameters determined from Eq. (7) are summarized in Table 2.

The lowest AIC and BIC values were realized using the Gumbel copula, indicating that it provided the best fit among the selected copula types for modeling the correlation structure between bearing pressure-displacement model parameters. As a result, the Gumbel copula was used in the reliability calibrations reported by Huffman and Stuedlein (2014). The degree of fit between the sample c_1-c_2 and 1,000 simulations for each copula model is shown Figure 1. The simulated c_1-c_2 pairs appear to closely correspond to the observed values for each copula investigated as shown in Figure 1 and by comparison of ρ_τ. However, the selection of the copula model strongly impacts the reliability of the allowable bearing pressure, as described subsequently.

5. EFFECT OF CORRELATION STRUCTURE ON THE CALIBRATED LOAD AND RESISTANCE FACTOR

The MCS of the performance function resulted in smooth relationships between the lumped load and resistance factor, ψ_q, and the reliability

Table 2. Selected copula functions, copula parameters, and goodness-of-fit outcomes.

Copula	**Copula Function, $C(u_3, u_4)$**	**Copula Parameter, θ**	**AIC**	**BIC**
Gaussian	$\Phi_\theta\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)$	0.626	-12.9	-11.5
Frank	$-\frac{1}{\theta}\ln\left[1 + \frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right]$	4.593	-12.0	-10.6
Clayton	$\left(u_1{}^{-\theta} + u_2{}^{-\theta} - 1\right)^{-1/\theta}$	1.510	-11.0	-9.6
Gumbel	$\exp\left\{-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right\}$	1.755	-14.4	-13.0

Table 3. Summary of best-fit coefficients for Eq. (8), valid for $\beta > 0$ and $\psi_q \leq 10$.

COV (δ_a)	COV (q^*_{app})	a	b	c	d
0.0	0.10	0.100	2.549	0.996	2.543
0.2	0.10	0.096	2.478	0.972	2.465
0.4	0.10	0.064	2.212	0.935	2.359
0.6	0.10	0.059	2.041	0.846	2.048
0.0	0.20	0.079	2.365	0.946	2.408
0.2	0.20	0.080	2.319	0.923	2.330
0.4	0.20	0.066	2.153	0.879	2.183
0.6	0.20	0.061	1.994	0.803	1.921

index, β, as a function of the allowable immediate displacement, δ_a, and the equivalent footing diameter, B'. Thus, the calibrated ψ_q may be set equal to (Huffman and Stuedlein 2014):

$$\psi_q = \exp\left[\frac{\beta - c\ln\left(\delta_a/_{B'}\right) - d}{a\ln\left(\delta_a/_{B'}\right) + b}\right] \quad (8)$$

where a, b, c, and d are fitting parameters that vary with the assumed COV(δ_a) and COV(q_a). Table 3 provides the fitting parameters for the best-fit Gumbel copula; similar parameters were generated for other copula models but were not presented so as to prevent their use, which would not be desirable as discussed subsequently.

The correlation structure of the c_1-c_2 pairs is critical for reliability simulations as the larger the deviation in the simulated pairs from the sample pairs, the greater the dispersion in the normalized bearing pressure-displacement response, leading to overly-conservative results in the reliability analyses. The effect of copula model selection on the lumped load and resistance factor is evaluated by comparing ψ_q generated from the MCS for a given probability of exceeding the serviceability limit state.

Consider for example a typical design scenario for a footing with an equivalent diameter of 1 m. The goal is to identify the appropriate ψ_q for a given β or p_f with Eq. (8) and then use Eq. (6) to compute the bearing

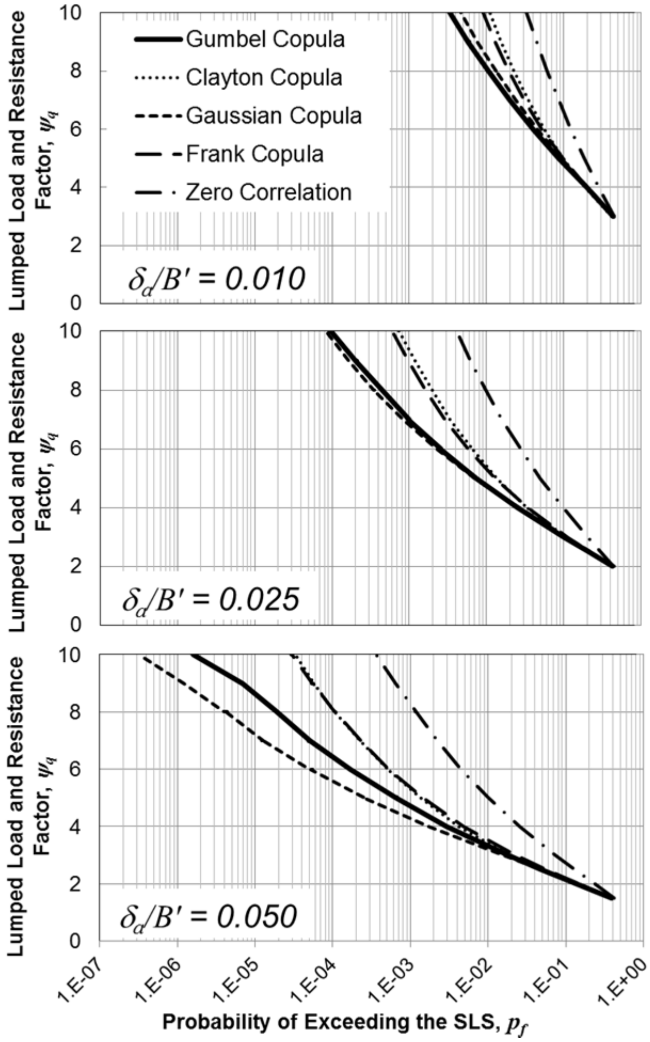

Figure 2. Impact of correlation structure model selection on the lumped load and resistance factor as a function of probability of exceeding the serviceability limit state.

pressure associated with the allowable immediate displacement. For this example, it was assumed that COV(δ_a) = 0 and COV(q_{app}) = 10 percent.

Figure 2 presents the variation of the calibrated load and resistance factor simulated using each of the four copulas, and the case with an assumed zero correlation between model parameters, with the probability of exceeding the allowable immediate displacement for normalized allowable displacements of 0.010, 0.025, and 0.050. This corresponds to allowable immediate displacements of 10, 25, and 50 mm. In general, the probability of exceeding the SLS decreases as the amount of allowable displacement increases for each correlation structure model due to the greater deviation in modeled bearing pressure-displacement response at smaller displacements than those at greater displacements.

Figure 2 shows that the probability of exceeding the SLS at a given ψ_q depends strongly on the correlation structure model considered and that ignoring the bivariate dependence of the bearing pressure-displacement models results in the lowest probability. The difference between the ψ_q generated from the best-fit Gumbel copula and the less appropriate copulas increases with decreasing p_f. Because the c_1-c_2 pairs simulated using poorly-fitted copulas result in a greater likelihood that extreme c_1-c_2 pairs will be sampled, and the frequency of softer bearing pressure-displacement curves generated increases, the p_f is greater using the most poorly-fitted copulas than those simulated using the better-fitting copulas. For example, the Gaussian copula generally produced ψ_q in the same order of magnitude as those using the Gumbel copula. Nonetheless, differences between the p_f can be significant for stringent performance cases; consider the 2.5- and 5-fold larger probabilities of exceeding the SLS using the Gumbel copula at ψ_q = 5 and 10, respectively, than those for the Gaussian copula.

Figure 3 presents the variation of allowable bearing pressure associated with each copula model, computed using Eq. (6) and the mean (nominal) parameters c_1 and c_2 (Table 1), with the normalized bearing displacement at probabilities of exceeding the SLS of 10, 1, and 0.1 percent (β = 1.28, 2.33, 3.09, respectively). The allowable bearing pressure is presented as a percentage of the ultimate resistance, and corresponds to the case of COV(δ_a) = 0 and COV(q_{app}) = 10 percent. Figure 3 shows that at high probabilities of exceeding the SLS, there is very little difference in the allowable bearing pressure given that a fitted copula was used to account for the bivariate bearing pressure-displacement model parameters. Ignoring the

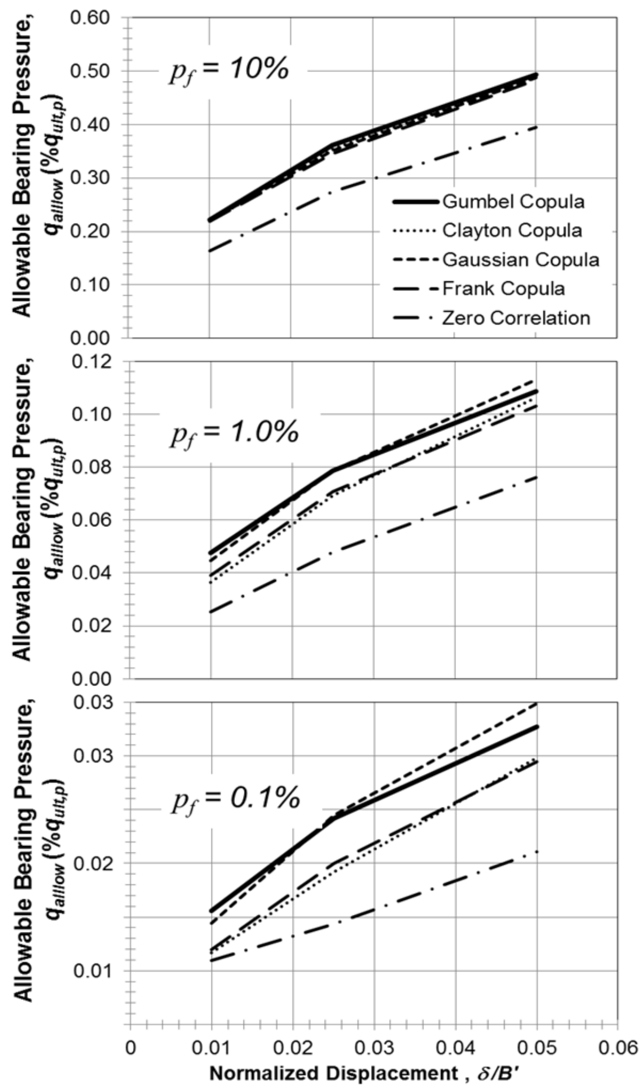

Figure 3. Impact of correlation structure model selection on the allowable bearing pressure considering allowable immediate displacement.

bivariate relationship between c_1 and c_2 resulted in a reduction of bearing pressure ranging from about 20 to 25 percent. However, as the allowable probability of exceeding the SLS decreases, the selection of correlation structure model becomes more important. For example, at p_f = 1 percent, the bearing pressures derived using the relatively well-fitting Gumbel copula are 10 to 20 percent greater than those using the less appropriate Frank and Clayton copulas, depending on the magnitude of allowable normalized displacement. Thus, use of an appropriately fitted copula to capture the correlation structure of the bivariate bearing pressure-displacement model parameters is critical for the optimization of the allowable bearing pressure and cost-effective foundations.

6. CONCLUDING REMARKS

The recognition of the impact that modeling techniques and decisions have on the outcome of reliability-based design procedures is becoming more important and the profession moves to adopt new and harmonize existing codes. Part of the challenge lies with our understanding of correlated model parameters and the importance such correlation has on outcomes that control cost and efficacy, such as footing dimensions or use of ground improvement. This paper explored the sensitivity of a reliability-based serviceability limit state procedure to the selection of correlation structure model that captures the correlation of bivariate bearing pressure-displacement model parameters. The correlation structure models investigated included four different copulas, as well as an assumed zero correlation. The findings developed herein show that the allowable bearing pressure is sensitive to the correlation structure model and that the magnitudes are larger (i.e., more cost-effective) when using the most appropriate copula.

7. REFERENCES

AASHTO (2012) *AASHTO LRFD Bridge Design Specifications, 6th Ed.*, Am. Assn. of State Highway and Trans. Off., Washington, DC.

Akaike, H. (1974) "A New Look at the Statistical Model Identification," *Trans. on Automatic Control*, IEEE, Vol. 19, No. 6, pp. 716-723.

Anderson, T.W., Darling, D.A. (1952) "Asymptotic theory of certain 'goodness-of-fit' criteria based on stochastic processes," *Ann. Math. Stat.*, Vol. 23, No. 2 pp. 193-212.

Baecher, G.B. and Christian, J.T. (2003) Reliability and Statistics in Geotechnical Engineering, John Wiley and Sons, Ltd., , 605 pp.

Dithinde, M., Phoon, K.K., De Wet, M., and Retief, J.V. (2011) "Characterization of Model Uncertainty in the Static Pile Design Formula," *J. of Geotech. and Geoenv. Engrg.*, Vol. 137, No. 1, pp. 70-85.

Huffman, J.C., and Stuedlein, A.W. (2014) "Reliability-Based Serviceability Limit State Design of Spread Footings on Aggregate Pier Reinforced Clay," *J. of Geotech. and Geoenv. Engrg.*, Vol. 140, No. 10, 04014055.

Huffman, J.C., and Stuedlein, A.W. (2015) "Reliability-Based Serviceability Limit State Design for Immediate Settlement of Spread Footings on Clay," *Soils and Foundations*, Vol. 55, No. 4.

Li, D.Q., Tang, X.S., Phoon, K.K., Chen, Y.F., Zhou, C.B. (2013) "Bivariate Simulation Using Copula and its Application to Probabilistic Pile Settlement Analysis," *Int. J. for Num. and Anal. Meth. in Geomech.*, John Wiley & Sons, Vol. 37, No. 6, pp. 597-617.

Mayne, P.W., Poulos, H.G. (1999) "Approximate Displacement Influence Factors for Elastic Shallow Foundations," *J. of Geotech. and Geoenv. Engrg.*, Vol. 125, No. 6, pp. 453-460.

Nelson, R.B. (2006) An Introduction to Copulas. 2nd Ed., Springer, New York, 269 p.

Phoon, K.K. (2008) "Numerical Recipes for Reliability Analysis – A Primer," in *Reliability-Based Design in Geotechnical Engineering: Computations and Applications*, Taylor and Francis, London, pp. 1-75.

Phoon, K.K., and Kulhawy, F.H. (2008) "Serviceability Limit State Reliability-Based Design," in *Reliability-Based Design in Geotechnical Engineering: Computations and Applications*, London, pp. 344-383.

Stuedlein, A.W., and Holtz, R.D. (2012) "Analysis of Footing Load Tests on Aggregate Pier Reinforced Clay," *J. of Geotech. and Geoenv. Engrg.*, Vol. 138, No. 9, pp. 1091-1103.

Stuedlein, A.W. and Holtz, R.D. (2013) "Bearing Capacity of Spread Footings on Aggregate Pier Reinforced Clay." *J. of Geotech. and Geoenv. Engrg.*, Vol. 139, No. 1, pp. 49-58.

Stuedlein, A.W., and Holtz, R.D. (2014) "Disp. of Spread Footings on Aggregate Pier Reinforced Clay," *J. of Geotech. and Geoenv. Engrg.*, 140(1), pp. 36-45.

Schwarz, G. (1978) "Estimating the Dimension of a Model," *Ann. of Stat.*, 6(2), pp. 461-464.

Tang, X.S., Li, D.Q., Rong, G., Phoon, K.K., Zhou, C.B. (2013) "Impact of Copula Selection on Geotechnical Reliability Under Incomplete Probability Information," *Comp. and Geotech.*, Vol. 49, pp. 264-278.

Uzielli, M. and Mayne, P., (2011) "Serviceability Limit State CPT-based Design for Vertically Loaded Shallow Footings on Sand," *Geomech. and Geoeng.*, Vol. 6, No. 2, pp. 91-107.

Uzielli, M. and Mayne, P., (2012) "Load–Displacement Uncertainty of Vertically Loaded Shallow Footings on Sands and Effects on Probabilistic Settlement," *Georisk*, Vol. 6, No. 1, pp. 50–69.