Homework 1
CS 321
Due Date: 10/9/09, 2 PM

1. Search the web and find 3 different applications of finite state automata other than digital logic design, neural networks, and scanners. Give a brief description or example of each application.

   Answer: Search for finite state machines on the web. Example applications include computer games, software specifications, user interfaces, robot control, and others.

2. Section 1.1. Problem 4. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Note $|S|$ is the size of the set S. Show that this is an equivalence relation.

   Answer:

   (a) Reflexive: $|S_1| = |S_1|$

   (b) Symmetric: $|S_1| = a$ and $|S_2| = b$
   $a = b \Rightarrow b = a$
   $\Rightarrow |S_1| = |S_2| \Rightarrow |S_2| = |S_1|$

   (c) Transitive: $|S_1| = a$ $|S_2| = b$ and $|S_3| = c$
   $a = b \wedge b = c \Rightarrow a = c$
   $|S_1| = |S_2| \wedge |S_2| = |S_3| \Rightarrow |S_1| = |S_3|$

3. Section 1.1. Problem 26. Show by induction on n that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \tag{1}$$

   Answer:

   (a) **Base case** $n = 1$ **:** $\frac{1}{1^2} \leq 2 - \frac{1}{1} \quad \Rightarrow \quad 1 \leq 1$ (Correct)

   (b) **Assumption for n:** Assume that $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ is correct

   (c) **Inductive step**

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \overset{??}{\leq} 2 - \frac{1}{n+1}$$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

$$\sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \text{ (by the assumption)}$$

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 - \frac{(n+1)^2 - n}{n(n+1)^2} = 2 - \frac{(n^2+n+1)}{n(n+1)^2} \leq 2 - \frac{(n^2+n)}{n(n+1)^2} = 2 - \frac{1}{n+1}$$

$$\Rightarrow \quad \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$$

4. Section 1.1. Problem 31. Show that $\sqrt{3}$ is not a rational number.

   Hint: Follow the proof that $\sqrt{2}$ is not a rational number in the book.

   Answer:

- Assume that $\sqrt{3}$ is a rational number
  $\Rightarrow \sqrt{3} = \frac{a}{b}$ s.t. a and b have no common divisor.
  $\Rightarrow 3b^2 = a^2$
  $\Rightarrow$ 3 is a divisor of a^2
  $\Rightarrow$ 3 is a divisor of a
  $\Rightarrow 3b^2 = 9k^2$ (we replace a with 3k)
  $\Rightarrow b^2 = 3k^2$
  $\Rightarrow$ 3 is a divisor of b
  $\Rightarrow$ b and a have common divisor **!!Contradiction!!**
  $\Rightarrow \sqrt{3}$ is not a rational number

5. Section 1.2. Problem 1. Use induction on n to show that $|u^n| = n|u|$ for all srings u and integers $n \geq 0$.

   Hint: Use the fact that $u^{i+1} = u^i u$ and $u^0 = \lambda$.

   Answer:

   (a) **Base case** $n = 0$ **:** $|u^0| = \lambda$
   (b) **Assumption for n:** Assume that $|u^n| = n|u|$ is correct
   (c) **Inductive step** $|u^{n+1}| \stackrel{??}{=} (n+1)|u|$
   $|u^{n+1}| = |u^n u| = |u^n| + |u| = n|u| + |u|$ (from assumption) $= (n+1)|u|$

6. Section 1.2. Problem 8. Show that $(L_1 L_2)^R = L_2^R L_1^R$.

   Hint: Use the following fact (problem 2) as given. For any strings u,v over some alphabet $(uv)^R = v^R u^R$. Note that u^R is the reverse of string u. $(L)^R$ is the set of reverses of strings in L.

   Answer: Use induction on the length of the string v

   (a) **Base case** $n = 1$ $(v = a)$ **:** $(uv)^R = (ua)^R = au^R = a^R u^R = v^R u^R$
   (b) **Assumption for n:** Assume this is true for any v such that $|v| <= n$
   (c) **Inductive step** We will show it is true for $|v| = n + 1$.
   Let $v = wa$. Since w is of length n, $(uw)^R = w^R u^R$.

   $(uv)^R = (uwa)^R = (uwa)^R = a(uw)^R$(by def of R given in pg 26)
   $= aw^R u^R$ (by inductive hypothesis)
   $= (wa)^R u^R$ (by def of R given in pg 26)
   $= v^R u^R$ (since v = wa).

7. Section 1.2. Problem 14(a). Give a grammar to generate the language $L_1 = \{a^n b^m : n \geq 0, m > n\}$.
   Answer:

   - $S \rightarrow b$
   - $S \rightarrow Sb$
   - $S \rightarrow aSb$

8. Section 1.2. Problem 14(b). Give a grammar to generate the language $L_2 = \{a^n b^{2n} : n \geq 0\}$.
   Answer:

   - $S \rightarrow aSbb$
   - $S \rightarrow \lambda$