Schema Independent Relational Learning

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ABSTRACT

Learning novel and interesting concepts and relations from relational databases is an important problem with many applications in database systems and machine learning. Relational learning algorithms generally leverage the properties of the database schema to find the definition of the target concept in terms of the existing relations in the database. Nevertheless, it is well established that the same data set may be represented under different schemas for various reasons, such as efficiency, data quality, and usability. Unfortunately, many current learning algorithms tend to vary quite substantially over the choice of schema, both in terms of learning accuracy and efficiency, which complicates their off-the-shelf application. In this paper, we formalize the property of schema independence of relational learning algorithms, and study both the theoretical and empirical dependence of existing algorithms on the common class of vertical (de)composition schema transformations.

We study two sample-based learning algorithms, which learn from sets of labeled examples, and query-based algorithms, which learn by asking queries to a user. For sample-based algorithms we consider the two main algorithm classes: top-down and bottom-up. We prove that practical top-down algorithms are generally not schema independent, while, in contrast, two bottom-up algorithms Golem and ProGolem are schema independent with some modifications. For query-based learning algorithms we show that the vertical (de)composition transformations influence their learning efficiency. We support the theoretical results with an empirical study that demonstrates the schema dependence/independence of several algorithms on existing benchmark data sets under natural vertical (de)compositions.

1. INTRODUCTION

Over the last decade, users’ information needs over relational databases expanded from seeking exact answers to precise queries to discovering and learning interesting and novel relations and concepts [21, 11, 4, 19, 18, 10, 20, 26]. In recent years, the database community has proposed multiple algorithms and systems that leverage the database approaches and techniques to make discovering and learning interesting and novel relations and concepts [21, 11, 4, 19, 18, 10, 20, 26]. In recent years, the database community has proposed multiple algorithms and systems that leverage the database approaches and techniques to make discovering and learning interesting and novel relations and concepts [21, 11, 4, 19, 18, 10, 20, 26].

Unfortunately, many current learning algorithms tend to vary quite substantially over the choice of schema, both in terms of learning accuracy and efficiency, which complicates their off-the-shelf application. In this paper, we formalize the property of schema independence of relational learning algorithms, and study both the theoretical and empirical dependence of existing algorithms on the common class of vertical (de)composition schema transformations. We study two sample-based learning algorithms, which learn from sets of labeled examples, and query-based algorithms, which learn by asking queries to a user. For sample-based algorithms we consider the two main algorithm classes: top-down and bottom-up. We prove that practical top-down algorithms are generally not schema independent, while, in contrast, two bottom-up algorithms Golem and ProGolem are schema independent with some modifications. For query-based learning algorithms we show that the vertical (de)composition transformations influence their learning efficiency. We support the theoretical results with an empirical study that demonstrates the schema dependence/independence of several algorithms on existing benchmark data sets under natural vertical (de)compositions.

<table>
<thead>
<tr>
<th>Original Schema</th>
<th>Alternative Schema</th>
</tr>
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<tbody>
<tr>
<td>student(stud)</td>
<td>student(stud, phase, years)</td>
</tr>
<tr>
<td>inPhase(stud, phase)</td>
<td>professor(prof, position)</td>
</tr>
<tr>
<td>yearsInProgram(stud, years)</td>
<td>publication(title, person)</td>
</tr>
<tr>
<td>professor(prof)</td>
<td>hasPosition(prof, position)</td>
</tr>
<tr>
<td>publication(title, person)</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Fragments of some schemas for UW-CSE database. Primary key attributes are underlined.

Learning algorithms attempt to induce general (approximate) definitions of the target in terms of existing relations [30, 38, 41]. For example, given a database with student and professor information, the goal may be to induce a Datalog definition of a missing relation advisedBy(stud, prof) based on a training set of known student-advisor pairs. Since the space of possible definitions (e.g. all Datalog programs) is enormous, learning algorithms employ heuristics to search for effective definitions, which generally depend on the schema of the database. More generally, statistical relational learning algorithms [19] use the same heuristic mechanisms for structure learning, which in turn renders them schema dependent.

As an example, Table 1 shows some relations from two schemas for the UW-CSE database over students and professors, which is used as a common relational learning benchmark [41]. The first and original schema was designed by relational learning experts and is generally discouraged in database community as it delivers poor usability and performance in query processing without providing any advantages in terms of data quality in return [2, 39]. A database designer may use a schema closer to the alternative in Table 1. We used the classic learning algorithm FOIL [38] to induce a definition of advisedBy(stud, prof) for each of the two schemas, resulting in two very different definitions. The original schema yielded a more accurate definition based on co-authorship information, while the alternative schema led to a definition based on information about TA and teaching assignments.

Generally, there is no canonical schema for a particular set of content in practice and people often represent the same information in different schemas [2, 16, 39]. For example, we have also observed that researchers have used different schemas to represent the Mutagenesis database, which is another well-known benchmark in the field of relational learning [44]. People may choose to represent their data in one schema or another for several reasons. For example, it is generally easier to enforce integrity constraints over highly normalized schemas [2]. On the other hand, because more normalized schemas usually contain many relations, they are hard to understand and maintain [39]. It also takes a relatively long
time to answer queries over database instances with such schemas \[2\] \[39\]. Thus, a database designer may sacrifice data quality and choose a more denormalized schema for its data to achieve better usability and/or performance. She may also hit a middle ground by choosing a style of design for some relations and another style for other relations in the schema. Further, as the relative priorities of these objectives change over time, the schema will also evolve.

Thus, users generally have to restructure their databases to some proper schema, in order to effectively use relational learning algorithms, i.e., deliver definitions for the target concepts that a domain expert would judge as correct and relevant. To make matters worse, these algorithms do not normally offer any clear description of their desired schema and database users have to rely on their own expertise and/or do trial and error to find such schemas. Nevertheless, we ideally want our database analytics algorithms to be used by ordinary users, not just experts, who know the internals of these algorithms. Further, the structure of large-scale databases constantly evolve \[11\], and we want to move away from the need for constant expert attention to keep learning algorithms effective.

For similar reasons, current relational learning algorithms will not be well suited for Big Data, which is inherently heterogeneous and evolving as it obtains its content from many different data sources \[45\] \[1\]. Moreover, researchers often use (statistical) relational learning algorithms to solve various important core database problems, such as query processing, schema mapping, entity resolution, and information extraction \[3\] \[47\] \[10\] \[12\] \[18\]. Thus, the issue of schema dependence appears in other areas of database management.

One approach to solving the problem of schema dependence is to run a learning algorithm over all possible schemas for a validation subset of the data and select the schema with the most accurate answers. Nonetheless, computing all possible schemas of a DB is generally undecidable \[16\]. One may limit the search space to a particular family of schemas to make their computation decidable. For instance, she may choose to check only schemas that can be transformed via join and project operations, i.e., vertical composition and decomposition \[2\] \[13\]. However, the number of possible schemas within a particular family for a data set are extremely large. For example, a relational table may have exponential number of distinct vertical decompositions. As many relational learning algorithms need some time for parameter tuning under a new schema \[26\], it may take a prohibitively long time to find the best schema. Since many relational learning algorithms need to access the content of the database, one has to transform the underlying data to the desired schema, which may not be practical for a large and/or constantly evolving database. Another possible approach is to define a universal schema to which all possible schemas can be transformed and use or develop algorithms that are effective over this schema. The experience gained from the idea of universal relation \[28\], indicates such schemas may not always exist \[2\]. Users also have to transform their databases to the universal schema, which may be quite complex and time-consuming considering the intricacies associated with defining such a representation, especially for large and/or constantly evolving databases.

The inherent contradiction between generality and effectiveness has been observed in machine learning and statistics \[42\]. If a learning algorithm is tuned too thoroughly for its training datasets, it will be very effective on those datasets, but may be much less effective elsewhere. We argue that the same parallel can be drawn for algorithms that use the structural details of schemas to learn novel concepts and relationships. Developers design these heuristics according to their observations of the structural properties of entities or relationships over schemas that experts deem more natural. With limited time and resources, it is not feasible to check if these heuristics capture the desired properties for other schemas. Hence, relational learning algorithms may also face the danger of over-fitting to schemas – akin to a focus on syntax rather than deeper semantic measures.

In this paper, we introduce the novel property of schema independence i.e., the ability to deliver the same answers regardless of the choices of schema for the same data, for relational learning algorithms. We propose a formal framework to measure the amount of schema independence of a relational learning algorithm for a given family of transformations. We also analyze and compare the schema independence of popular relational learning algorithms to find the characteristics of more schema independent heuristics. We also leverage concepts from database literature to design schema independent algorithms. To the best of our knowledge, the property of schema independence has not been introduced or explored for relational learning algorithms. Our contributions:

- We introduce and formally define the property of schema independence and explore its benefits to relational learning algorithms. We define the property of schema independence for a relational learning algorithm as its ability to return the same answer over transformations that modify the database schema and preserve its information content. We study schema independence for two types of relational learning frameworks: 1) sample-based learning, e.g. \[30\] \[38\] \[41\] \[31\], which learn the target relation using some training data, and 2) query-based learning algorithms, e.g. \[25\] \[43\] \[3\], which learn the target relation by asking queries to some oracle, e.g., database user.

- We analyze the property of schema independence for a popular family of sample-based learning algorithms, called top-down relational learning algorithms \[30\] \[38\] \[41\]. We show that this family of algorithms is not schema independent under vertical (de)composition transformations.

- We explore the property of schema independence for another family of widely used sample-based algorithms called bottom-up algorithms. We formally analyze two typical algorithms from this family: Golem \[32\] and ProGolem \[33\]. We prove that these these algorithms are schema dependent under vertical (de)composition. We extend these algorithms and prove our extensions of are schema independent under vertical (de)composition.

- We explore the schema independence of some typical query-based relational learning algorithms \[25\] \[33\]. We prove that the number of queries this algorithms needs for successful depends on the schema used to represent the data. Our result is strong because it further shows that any reasonably good query-based algorithm will require drastically more queries under some vertical (de)composition. We also prove that the resource requirements of these algorithms can grow exponentially under vertical (de)compositions.

- We empirically study the schema dependence of popular relational learning algorithms under vertical (de)composition. Our empirical results generally confirm our theoretical results and indicate that transforming schema considerably affects the effectiveness, efficiency, and query complexities of well-known relational learning algorithms. Because ProGolem is more efficient than Golem, we evaluate the effectiveness of the extended ProGolem algorithm using a widely used benchmark dataset. Our results show that the extended ProGolem algorithm is as effective as the original version of ProGolem.

This paper is organized as follows. Section 2 describes the background. Section 3 formally defines the property of schema inde-
dependence for sample-based relational learning algorithms. Sections 5 and 6 explore the schema independence of top-down and bottom-up relational learning algorithms, respectively. Section 7 defines schema independence for query-based algorithms and explores their schema independence. Section 8 contains our empirical results and Section 9 concludes the paper. The proofs for our theoretical results are in the appendix.

2. BACKGROUND

2.1 Related Work

The architects of relational model have argued for logical data independence, which oversimplifying a bit, means that an exact query should return the same answers no matter which logical schema is chosen for the data [2]. In this paper, we extend the principle of logical data independence for relational learning algorithms. The property of schema independence also differs with the idea of logical data independence in a subtle but important issue. One may achieve logical data independence by an affordable amount of experts’ intervention, such as defining a set of views over the database [2]. However, it generally takes more time and much deeper expertise to find the proper schema for a relational learning algorithm, particularly for database applications that contain more than a single learning algorithm [21]. Hence, it is less likely to achieve schema independence via expert’s intervention.

Database researchers have applied techniques from query optimization to create usable systems for tuning the parameters of unstructured learning algorithms [16]. Schema of the database, however, is not a tuning parameter of a learning algorithm. We also focus on structured learning algorithms. Further, the authors in [25] try a reasonable subset of possible values for learning parameters to find the desired settings for the algorithm. Nevertheless, as explained in Section 1, this approach cannot be successfully applied to address the issue of schema dependence. Finding a subset of relevant features from the data, is an important step in deploying learning algorithms [4]. When applied in this context, the idea of representation independence prefers features that are more robust against representational variations in the underlying database. Researchers have realized the need to transform, i.e., wrangle, data as an important and widely used operation in data preparation and have developed usable systems for data transformation [24]. We address the same underlying problem but propose a very different approach: making the data analytics algorithm independent of representation. Researchers have analyzed the stability of some (unstructured) learning algorithms against relatively small perturbations in the data [24, 17, 2]. We also seek to instill robustness in learning algorithms, but we are targeting robustness in a new dimension: robustness in the face of variations in the schema of data.

There is a large body of work on converting a database represented under one schema to another one without modifying its information content [22, 15, 29, 8]. We build on this work by exploring the sensitivity of relational learning algorithms to such transformations. Researchers have defined other types of schema transformations [6]. A notable group is schema mappings in the context of data exchange, which are defined using tuple generating dependencies between source and target schemas [15]. This group of transformations may modify the information content of and/or introduce incomplete information to a database. Nevertheless, for the property of schema independence, the original and transformed databases should contain essentially the same information.

 Researchers have defined the property of design independence for keyword query processing over XML documents [46]. We extend this line of work by introducing and formally exploring the property of schema independence for relational learning algorithms. We focus on supervised learning algorithms and their schema independence properties over relational data model.

2.2 Basic Definitions

Let Attr be a (countably) infinite set of symbols that contains the names of attributes [2]. The domain of attribute A is a countably infinite set of values (i.e. constants or objects) that A may contain. We assume that all attributes share a single domain dom. A relation is a finite subset of Attr. We use the terms predicate and relation interchangeably. A tuple over relation R is a total map from the set of attributes in R to dom. The relation instance I_R of relation R is a finite set of tuples. A constraint restricts the properties of data stored in a database. Examples of constraints are functional dependencies (FD) and inclusion dependencies (IND). FD A → B in relation R, where A, B ⊆ R, states that the values of attribute A uniquely determine the values of attributes B in each tuple in every relation instance I_R. An IND between attribute C ∈ R and D ∈ S, for relation S, denoted as \( H(C) \subseteq S(D) \), states that in all instances of I_R and I_S, values of attribute C in any tuple of I_R must also appear in attribute D of some tuple of I_S. A schema is a pair R = \( (R, \Sigma) \), where R is a finite set of relations and \( \Sigma \) is a finite set of constraints. An instance of schema R is a mapping \( \mathcal{I} \) over R that associates each relation R ∈ R to a relation instance I_R. An atom is a formula in the form of \( R(u_1, \ldots, u_n) \) where R is a relation, n is the number of attributes in R, and each \( u_i \), 1 ≤ i ≤ n, is a variable or constant. A literal is an atom, or the negation of an atom. A definite Horn clause (Horn clause or clause, for short) is a finite set of literals that contains exactly one positive literal. The positive literal is called the head of the clause, and the set of negative literals is called the body. A clause has the form:

\[ T(u_1) \leftarrow L_1(u_1), \ldots, L_n(u_n). \]

An ordered clause is a clause where the order and duplication of literals matter. A Horn expression is a set of Horn clauses. A Horn definition is a Horn expression with the same predicate in the heads of all clauses. A Horn definition to define new target relations that are not in the current schema. In this work, we will use Horn definitions to define new target relations that are not in the current schema. Thus, the heads of all clauses in such definitions will be the target relation. The literal associated with the target relation does not contain any constant.

3. FRAMEWORK

3.1 Relational Learning

Relational learning can be viewed as a search problem for a hypothesis that deduces the training data, following either a top-down or bottom-up approach. Top-down algorithms [38, 30] start from the most general hypothesis and employ specialization operators to get more specific hypotheses. A common specialization operator is the addition of a new literal to the body of a clause. On the other hand, bottom-up algorithms [32, 33, 31] start from specific hypotheses that are constructed based on ground training examples, and use generalization operators to search the hypothesis space. Generalization operators include inverse resolution, relative least general generalization, asymmetric relative minimal generalization, among others. Therefore, a generic relational learning algorithm can be seen as a sequence of steps, where in each step an operator is applied to the current hypothesis.

Inductive Logic Programming (ILP) is the subfield of machine learning that performs relational learning by learning first-order
Examples 3.1. Given background knowledge B, positive examples E⁺, negative examples E⁻, and a target relation T, the ILP task is to find a definition H for T such that:

- ∀p ∈ E⁺, H ∩ B ⊨ p (completeness)
- ∀p ∈ E⁻, H ∩ B ⊭ p (consistency)

In the following sections we provide concrete definitions of several relational learning algorithms.

Example 3.2. Consider using a relational learning algorithm and the UW-CSE database with the original schema shown in Table 1 to learn a definition for the target relation collaborated(X,Y), which indicates that person X has collaborated with person Y. The algorithm may return the following definition:

\[ \text{collaborated}(X, Y) \leftarrow \text{publication}(P, X), \text{publication}(P, Y). \]

This is a complete and consistent definition with respect to the training data, and indicates that two persons have collaborated if they have been co-authors.

In this paper, we study relational learning algorithms for Horn definitions. We denote the set of all Horn definitions over schema \( \mathcal{R} \) by \( \mathcal{HD}_R \). This set can be very big, which means that algorithms would need a lot of resources (e.g., time and space) to explore all definitions. However, in practice, resources are limited. For this reason, algorithms accept parameters that either restrict the hypothesis space or restrict the search strategy. For instance, an algorithm may consider only clauses whose number of literals are fewer than a given number, or may follow a greedy approach where only one clause is considered at a time. Let the parameters for a learning algorithm be a tuple of variables \( \theta = (\theta_1, ..., \theta_r) \), where each \( \theta_i \) is a parameter for the algorithm. We denote the parameter space by \( \Theta \), and it contains all possible parameters for an algorithm. We denote the hypothesis space (or language) of algorithm \( A \) over schema \( \mathcal{R} \) with parameters \( \theta \) as \( \mathcal{L}_{R, \theta}^A \). Note that not all parameters affect the hypothesis space. For instance, a parameter setting the search strategy to greedy impacts how the hypothesis space is explored, but does not restrict the hypothesis space. The hypothesis space \( \mathcal{L}_{R, \theta}^A \) is a subset of \( \mathcal{HD}_R \) [30][38][19], and each member of \( \mathcal{L}_{R, \theta}^A \) is a hypothesis.

Clearly, there is a trade-off between computational resources used by an algorithm and the size of its hypothesis space. The hypothesis space is restricted so that the algorithm can be used in practice, with the hope that the algorithm will find a consistent and complete hypothesis.

Example 3.3. Continuing Example 3.2 consider restricting the hypothesis space to clauses whose number of literals are fewer than a given number, which we call clause-length. Assume that we are now interested in learning a definition for the target relation collaboratedProf(X,Y), which indicates that professor X has collaborated with professor Y, under the original schema. If we set clause-length = 5, the learning algorithm is able to learn the complete and consistent definition

\[ \text{collaboratedProf}(X, Y) \leftarrow \text{professor}(X), \text{professor}(Y), \text{publication}(P, X), \text{publication}(P, Y). \]

However, if we set clause-length = 3, the previous definition is not in the hypothesis space of the algorithm. Therefore, the algorithm is not able to learn this definition or any other complete and consistent definition.

3.2 Schema Independence of Relational Learning Algorithms

3.2.1 Mapping Database Instances

One may view a schema as a way of representing background knowledge used by relational learning algorithms to learn the definitions of target relations. Intuitively, in order to learn essentially the same definitions over schemas \( \mathcal{R} \) and \( \mathcal{S} \), we should make sure that \( \mathcal{R} \) and \( \mathcal{S} \) represent basically the same information. Let us denote the set of database instances of schema \( \mathcal{R} \) as \( \mathcal{I}(\mathcal{R}) \). In order to compare the ability of \( \mathcal{R} \) and \( \mathcal{S} \) to represent the same information, we would like to check whether for each database instance \( I \in \mathcal{I}(\mathcal{R}) \) there is a database instance \( J \in \mathcal{I}(\mathcal{S}) \) that contains basically the same information as \( I \). We adapt the notion of equivalency between schemas in the database literature to precisely state this idea [22][10].

Given schemas \( \mathcal{R} \) and \( \mathcal{S} \), a transformation is a (computable) function \( \tau : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{S}) \). For brevity, we write transformation \( \tau \) as \( \mathcal{R} \to \mathcal{S} \). Transformation \( \tau \) is invertible iff it is total and there exists a transformation \( \tau^{-1} : \mathcal{S} \to \mathcal{R} \) such that the composition of \( \tau \) and \( \tau^{-1} \) is the identity mapping on \( \mathcal{I}(\mathcal{R}) \), that is \( \tau^{-1}(\tau(I)) = I \) for \( I \in \mathcal{I}(\mathcal{R}) \). The transformation \( \tau^{-1} \) may or may not be total. We call \( \tau^{-1} \) the inverse of \( \tau \) and say that \( \tau \) is invertible. If transformation \( \tau \) is invertible, one can convert every instance \( I \in \mathcal{I}(\mathcal{R}) \) to an instance \( J \in \mathcal{I}(\mathcal{S}) \) and reconstruct \( I \) from the available information in \( J \). Schemas \( \mathcal{R} \) and \( \mathcal{S} \) are information equivalent via transformation \( \tau : \mathcal{R} \to \mathcal{S} \) iff \( \tau \) is bijective. Informally, if two schemas are equivalent, one can convert the databases represented using one of them to the other without losing any information. Hence, one can reasonably argue that equivalent schemas essentially represent the same information. Our definition of information equivalence between two schemas is more restricted than the ones proposed in [22][10]. We assume that in order for schemas \( \mathcal{R} \) and \( \mathcal{S} \) to be information equivalent via \( \tau \), \( \tau^{-1} \) has to be total. Although more restricted, this definition is sufficient to cover the transformations discussed in this paper. Employing a framework that adheres to the ones used in [22][10] is future work.

Example 3.4. In addition to the functional dependencies shown in Table 1, let the following inclusion dependencies hold over the relations of original schema in this table: student[stud] ⊆ inPhase[stud], inPhase[stud] ⊆ student[stud], yearsInProgram[stud] ⊆ student[stud], student[stud] ⊆ yearsInProgram[stud], professor[prof] ⊆ hasPosition[prof], hasPosition[prof] ⊆ professor[prof]. One may join relations student, inPhase, and yearsInProgram and join relations professor and hasPosition to map each instance of the original schema to an instance of the alternative schema. Further, each instance of the alternative schema can be mapped to an instance of the original schema by projecting relation student to relations student, inPhase, and yearsInProgram and projecting relation professor to relations hasPosition and professor. Hence,
these schemas are information equivalent.

### 3.2.2 Mapping Definitions

Let $\mathcal{H}_R$ be the set of all Horn definitions over schema $R$. In order to learn semantically equivalent definitions over schemas $R$ and $S$, we should make sure that the sets $\mathcal{H}_R$ and $\mathcal{H}_S$ are equivalent. That is, for every definition $h_R \in \mathcal{H}_R$, there is a semantically equivalent Horn definition in $\mathcal{H}_S$, and vice versa. If the set of Horn definitions over $R$ is a superset or subset of the set of Horn definitions over $S$, it is not reasonable to expect a learning algorithm to learn semantically equivalent definitions over $R$ and $S$.

Let $\mathcal{L}_R$ be a set of Horn definitions over schema $R$ such that $\mathcal{L}_R \subseteq \mathcal{H}_R$. Let $h_R \in \mathcal{L}_R$ be a Horn definition over schema $R$ and $I \in \mathcal{I}(R)$ be a database instance. The result of applying a Horn definition $h_R$ to database instance $I$ is the set containing the head of all instantiations of $h_R$ for which the body of the instantiation belongs to $\mathcal{I}(R)$. We denote the result of $h_R$ over $I$ by $h_R(I)$.

**Definition 3.5.** Transformation $\tau : R \rightarrow S$ is definition preserving w.r.t. $\mathcal{L}_R$ and $\mathcal{L}_S$ iff there exists a total function $\delta_\tau : \mathcal{L}_R \rightarrow \mathcal{L}_S$ such that for every $h_R \in \mathcal{L}_R$ and $I \in \mathcal{I}(R)$, $h_R(I) = \delta_\tau(h_R)(\tau(I))$.

Intuitively, Horn definitions $h_R$ and $\delta_\tau(h_R)$ deliver the same results over all corresponding database instances in $R$ and $S$. We call function $\delta_\tau$ a definition mapping for $\tau$. Transformation $\tau$ is definition bijective w.r.t. $\mathcal{L}_R$ and $\mathcal{L}_S$ iff $\tau$ and $\tau^{-1}$ are definition preserving w.r.t. $\mathcal{L}_R$ and $\mathcal{L}_S$.

If $\tau$ is definition bijective w.r.t. equivalent sets of Horn definitions, one can rewrite each Horn definition over $R$ as a Horn definition over $S$ such that they return the same results over all corresponding database instances of $R$ and $S$, and vice versa. We call these definitions equivalent. We use the operator $\equiv$ to show that two definitions are equivalent.

### 3.2.3 Relationship Between Information Equivalence and Definition Bijective Transformations

In order for a learning algorithm to learn equivalent definitions over schemas $R$ and $S$, where $\tau : R \rightarrow S$, it is reasonable for $\tau$ to be information equivalent via $\tau$ and $\tau$ to be definition bijective w.r.t. $\mathcal{H}_R$ and $\mathcal{H}_S$. Information equivalence guarantees that the learning algorithm takes as input the same background knowledge. A definition bijective transformation ensures that the learning algorithm can output equivalent Horn definitions over both schemas. Nevertheless, it may be hard to check both conditions for given schemas. Next, we extend the results in [16] to find the relationship between the properties of information equivalence and definition bijective transformations.

In this paper, we consider only transformations that can be written as sets of Horn definitions. We call these Horn transformations. Vertical composition/decomposition are examples of Horn transformations.

**Example 3.6.** The transformation from the original schema to the alternative schema in Table 1 can be written as the following set of Horn definitions:

- $\text{student}(X, Y, Z) \leftarrow \text{student}(X), \text{inPhase}(X, Y), \text{yearsInProgram}(X, Z)$.
- $\text{professor}(X, Y) \leftarrow \text{professor}(X), \text{hasPosition}(X, Y)$.
- $\text{publication}(X, Y) \leftarrow \text{publication}(X, Y)$.

Assume that transformation $\tau : R \rightarrow S$ and its inverse $\tau^{-1} : S \rightarrow R$ are Horn transformations. Clearly, the head of each Horn definition in $\tau^{-1}$ will be a relation in $R$. Let $h_R$ be a Horn definition in $\mathcal{H}_R$. The composition of $h_R$ and $\tau^{-1}$, denoted by $h_R \circ \tau^{-1}$, is a Horn definition that belongs to $\mathcal{H}_S$, created by applying $h_R$ to the head predicates of clauses in $\tau^{-1}$ [2]. That is, $h_R \circ \tau^{-1}(J) = h_R(\tau^{-1}(J))$, for all $J \in \mathcal{I}(S)$. We prove the following proposition similar to Theorem 3.2 in [16].

**Proposition 3.7.** Given schemas $R$ and $S$, if transformation $\tau : R \rightarrow S$ is a Horn transformation and it is invertible, then $\tau$ is definition preserving w.r.t. $\mathcal{H}_R$ and $\mathcal{H}_S$.

**Proof.** Suppose that transformation $\tau : R \rightarrow S$ is a Horn transformation and it is invertible. We define a function $\delta_\tau : \mathcal{H}_R \rightarrow \mathcal{H}_S$ to be $\delta_\tau(h_R) = h_R \circ \tau^{-1}$ for any $h_R \in \mathcal{H}_R$. We know that $\delta_\tau(h_R) \in \mathcal{H}_S$. Furthermore, for any $h_R \in \mathcal{H}_R$ and $I \in \mathcal{I}_R$, $h_R(I) = h_R(\tau^{-1}(\tau(I))) = (h_R \circ \tau^{-1})(\tau(I)) = \delta_\tau(h_R)(\tau(I))$. Thus, $\delta_\tau$ is a definition mapping for $\tau$ and $\tau$ is definition preserving w.r.t. $\mathcal{H}_R$ and $\mathcal{H}_S$.

Intuitively, if transformation $\tau : R \rightarrow S$ is an invertible Horn transformation, then any Horn definition in $\mathcal{H}_R$ can be rewritten as a Horn definition in $\mathcal{H}_S$ such that they return the same results over equivalent database instances. In the proof of Proposition 3.7, the definition mapping function that maps members of $\mathcal{H}_R$, such as $h_R$, to members of $\mathcal{H}_S$ is $h_R \circ \tau^{-1}$. According to Proposition 3.7, if schemas $R$ and $S$ are information equivalent via Horn transformation $\tau$, for all $h_R \in \mathcal{H}_R, h_R \circ \sigma^{-1} \in \mathcal{H}_S$, and for all $h_S \in \mathcal{H}_S, h_S \circ \sigma^{-1} \in \mathcal{H}_R, \tau$ is definition bijective w.r.t. $\mathcal{H}_R$ and $\mathcal{H}_S$.

**Example 3.8.** Let $R$ be the original schema and $S$ be the alternative schema in Example 3.4. Let $\tau : R \rightarrow S$ be the join operator, and $\tau^{-1} : S \rightarrow R$ be the projection operator, which is the inverse of join. Because of Proposition 3.7, $\tau$ is definition bijective w.r.t. $\mathcal{H}_R$ and $\mathcal{H}_S$.

In this paper, we consider only the Horn transformations that are both invertible and definition bijective w.r.t. sets of Horn definitions.

### 3.2.4 Schema Independence

The hypothesis space determines the set of possible Horn definitions that the algorithm can explore. Therefore, the output of a learning algorithm depends on its hypothesis space. In Example 3.3, we showed that an algorithm is able to learn a definition for a target relation with some hypothesis space but not in another more restricted space. In order for an algorithm to learn semantically equivalent definitions for a target relation over schemas $R$ and $S$, it should have equivalent hypothesis spaces over $R$ and $S$. We call this property hypothesis invariance. Let $\Theta$ be the parameter space for algorithm $A$.

**Definition 3.9.** Algorithm $A$ is hypothesis invariant under transformation $\tau : R \rightarrow S$ iff $\tau$ is definition bijective w.r.t. $\mathcal{L}_R^{\Theta}$ and $\mathcal{L}_S^{\Theta}$, for all $\Theta \in \Theta$.

Let $\Gamma$ be a set of transformations. We say that algorithm $A$ is hypothesis invariant under $\Gamma$ if it is hypothesis invariant under $\tau$, for all $\tau \in \Gamma$.

We now define the notion of schema independence for relational learning algorithms over a set of bijective transformations. We define a relational learning algorithm as a function $A(I, E, \theta)$ to $\mathcal{L}_R^{\Theta}$. That is, taking as input a database instance $I$, training examples $E$, and parameters $\theta \in \Theta$, the algorithm outputs a hypothesis that belongs to $\mathcal{L}_R^{\Theta}$.
Definition 3.10. Algorithm A is schema independent under bijective transformation $\tau : R \rightarrow S$ iff $\tau$ is definition bijective w.r.t. $HD_R$ and $HD_S$ and for all $I \in \mathcal{I}(R)$ and $J \in \mathcal{I}(S)$, all $\theta$, and target relation $T$, we have:

- $A(I, E, \theta) \equiv \delta_r(A(\tau(I), E, \theta))$, where $\delta_r$ is the definition mapping for $\tau$.
- $A(J, E, \theta) \equiv \delta_{\tau^{-1}}(A(\tau^{-1}(J), E, \theta))$, where $\delta_{\tau^{-1}}$ is the definition mapping for $\tau^{-1}$.

Again, we say that algorithm A is schema independent under the set of transformations $\Gamma$ if it is schema independent under $\tau$, for all $\tau \in \Gamma$. Note that if an algorithm is schema independent under transformation $\tau$, then it is hypothesis invariant under $\tau$. In other words, hypothesis invariance under a set of transformations is a necessary condition for an algorithm to be schema independent under the same set of transformations. Note that it is possible for an algorithm to not be schema independent, but be hypothesis invariant. In such cases, the cause of schema independence must necessarily be related to the search process of the algorithm, rather than hypothesis representation capacity.

Example 3.11. Consider the original schema and the alternative schema in Table 3.16. The original schema is the result of a vertical decomposition of the alternative schema. Consider the learning algorithm FOIL. If the target relation is collaboratedProf(X,Y), then it learns non-equivalent definitions under these schemas. Under the original schema, it learns a definition based on co-authorship:

$$advisedBy(X,Y) \leftarrow student(X), professor(Y),\text{publication}(P,X),\text{publication}(P,Y).$$

On the other hand, under the alternative schema, FOIL learns a definition based on courses taught by the professor in which the student has been TA:

$$advisedBy(X,Y) \leftarrow course(C,Y,X,T,L).$$

Therefore, FOIL is not schema independent.

In this paper, we consider only generic transformations, which treat data values as essentially uninterpreted objects [23]. Generic transformations are usually allowed to use a bounded number of constants. However, the transformations considered in this paper use no constants. Considering this type of transformations is reasonable as relational learning algorithms also treat data values as uninterpreted objects.

4. VERTICAL (DE)COMPOSITION

There are a wide variety of information-preserving transformations between relational schemas [23][24]. It will take more space than a single paper to explore the behavior of relational learning algorithms over all such transformations. In this paper, we explore the schema independence of relational learning algorithms under vertical composition/decomposition transformations [2][35]. We select this group of transformations because we have observed several instances of these transformations in relational learning research papers and systems. Section 4 presented one of these cases. Further, they are widely used in relational databases as a database designer may decompose and/or compose their relations to achieve the desired trade-off between efficiency, degree of normalization and data quality, and schema readability.

Vertical composition and decomposition may be done in the presence of functional and/or join dependencies in the schema [2]. Due to the limited space, we focus only on vertical composition and decomposition that involve functional dependencies. Since our analysis in this paper mainly leverages the structure of the transformed schema rather than the properties of its dependencies, we believe that our results may extend for the vertical composition and decompositions that involve join dependencies. A more careful analysis of this case is a subject for future work.

Following [36], we define vertical decomposition as follows. Let $FD_R$ denote the closure FDs in schema $R$. We denote relation $R$ as $R(\bar{A})$ where $\bar{A}$ is the set of attributes in $R$. If both INDs $R_1[A] \subseteq R_2[B]$ and $R_2[B] \subseteq R_1[A]$ are in $R$, we denote them as $R_1[A] = R_2[B]$ for brevity. We call such IND an IND with equality.

Definition 4.1. A vertical decomposition (decomposition for short) of schema $R$ with single relation $R(\bar{A})$ is a schema $S$ with relations $S_1(\bar{B}_1), \ldots, S_n(\bar{B}_n)$, $n \geq 1$, such that:

- $\bar{A} = \cup_{1 \leq i \leq n} S_i(\bar{B}_i)$.
- $FD_R = FD_S$.
- $\bar{A} \supseteq \cap_{1 \leq i \leq n} S_i(\bar{B}_i) \neq \emptyset$.
- $\cap_{1 \leq i \leq n} S_i(\bar{B}_i)$.
- Let $\bar{C}$ be $\cap_{1 \leq i \leq n} S_i(\bar{B}_i)$, the inclusion dependencies $S_i[\bar{C}] = S_j[\bar{C}]$, $1 \leq i,j \leq n$ are in schema $S$.

If $n = 1$ in Definition 4.1, then schema $\bar{R}$ will remain unchanged. As we will see below, the last condition in Definition 4.1 is needed to ensure that $S$ does not contain more information than $\bar{R}$ [22][36]. We define vertical decomposition of a schema with more than one relation as the set of vertical decomposition of all its relations. Table 3.16 depicts an example of decomposition. Relation student in the alternative schema is decomposed into relations student, inPhase, and yearsInProgram in the original schema. One may also rename the attributes after decomposing the schema. Our results will hold if such renaming will be applied after a decomposition. According to Corollary 4.3.2 in [36], every decomposition is bijective.

A vertical composition (composition for short) is the inverse of a decomposition. Because decomposition is bijective, composition is also bijective. We define a composition/decomposition of a schema as a finite set of applications of composition or decomposition to the schema. Hence, a decomposition/composition may decompose some relations in the schema, compose some relations, and leave some intact. This transformation reflects the modifications one may apply on a schema: compose some relations to improve performance, decompose some to achieve quality and/or readability, and leave some unchanged. Because both composition and decomposition are bijective, composition/decomposition is bijective. Because composition/decompositions are expressed by projection and natural join [2], they are Horn transformations and generic. Hence, they are definition bijective.

5. TOP-DOWN ALGORITHMS

Top-down relational learning algorithms follow a covering approach [38][39]. An algorithm that uses a covering approach constructs one clause at a time. After building a clause, the algorithm adds the clause to the hypothesis, discards the positive examples covered by the clause, and moves on to learn a new clause. Algorithm 1 sketches a generic relational learning algorithm that follows a covering approach. The strategy followed by the LearnClause procedure depends on the nature of the algorithm.
algorithms, the LearnClause procedure in Algorithm 1 searches the hypothesis space from general to specific, by using a refinement (specialization) operator. The refinement operator can perform two syntactic operations. The first operation is to substitute the variables in the literals of the clause with fresh variables, other used variables, or constants. The second operation is to add a new literal to the clause.

### Algorithm 1: Generic relational learning algorithm following a covering approach.

**Input**: Database instance \( I \), positive examples \( E^+ \), negative examples \( E^- \)

**Output**: A set of Horn definitions \( H \)

\[
\begin{align*}
H & \leftarrow \emptyset; \\
U & \leftarrow E^+; \\
\text{while } U \text{ is not empty} & \text{ do} \\
C & \leftarrow \text{LearnClause}(I, E^+, E^-); \\
H & \leftarrow H \cup C; \\
U & \leftarrow U - \{c \in U | H \land I = c\}; \\
\text{end}
\end{align*}
\]

The hypothesis space in top-down algorithms can be seen as a refinement graph, that is a rooted directed acyclic graph in which nodes represent clauses and each arc is the application of a basic refinement operator. The basic strategy of top-down algorithms consists of starting from the most general clause, which corresponds to the root of the refinement graph, and repeatedly refining it until it does not cover any negative example. Figure 1 shows fragments of the refinement graph for learning the definition of \( \text{collaborated}(X,Y) \). Because of space constraints, we do not show the head of the clause \( \text{collaborated} \) in any node of the refinement graph in Figure 1 but its root.

The strategy of constructing and searching the refinement graph varies between different top-down algorithms. For instance, FOIL [38, 35] is an efficient and popular top-down algorithm that follows a greedy best-first search strategy. In this section, we analyze the schema independence properties of FOIL. However, the results that we show in this section hold for all top-down algorithms no matter which search strategy they follow.

The refinement graph for most schemas, even the ones with a relatively small number of relations and attributes, may grow significantly [38, 30]. Hence, the construction and search over the refinement graph may become too inefficient to be practical. To be used in practice, FOIL restricts its search space, i.e. hypothesis space. We call the number of literals in a clause its length. A common method is to restrict the maximum length of each clause in the refinement graph [38, 30]. Intuitively, because composition/decompositions modify the number of relations in a schema, equivalent clauses over the original and transformed schemas may have different lengths. Hence, this type of restrictions may result in different hypothesis spaces. One may like to fix this problem by choosing different values for the maximum lengths over the original and transformed schemas. For example, one may pick a smaller value to bound the clause lengths over the schemas with smaller number of relations. The following theorem proves that it is not possible to achieve equivalent hypothesis spaces over the original and transformed schemas by restricting the maximum length of clauses no matter what values are used over the original and transformed schemas.

**Theorem 5.1.** FOIL is not hypothesis invariant under vertical composition/decomposition.

Now, we describe a method of restricting hypothesis space for FOIL that achieves hypothesis invariance. FOIL constructs its hypothesis space by starting from an empty clause and gradually adding a new relation or substituting the variables in the current clause. Let \( \text{FOIL} \) learn a target relation over schema \( S \) with relations \( S_i, 1 \leq i \leq n \). Given relation \( S_i \) in \( S \), we call the set of all relations \( S_i \) such that there is an IND \( S_i[B] = S_j[C] \) in \( S \), the inclusion class of \( S_i \). We make the following modifications to FOIL. First, right after adding a new relation \( S_i \) to the current clause, FOIL adds all relations in its inclusion class to the clause. Second, given \( S_i \) and \( S_j \) appear in the current clause and we have IND \( S_i[B] = S_j[C] \) is in \( S \), FOIL assigns the same variables to attributes \( B \) and \( C \). Finally, we restrict the hypothesis space by limiting the maximum number of inclusion classes in each clause.

**Proposition 5.2.** The modified FOIL is hypothesis invariant under composition/decomposition.

One problem with the modified FOIL, and other similarly modified top-down algorithms for that matter, is that it has to evaluate clauses with rather large number of relations. Since most these clauses are already minimal, the algorithm may need to join large number of relations to evaluate each candidate clause. Hence, the learning may be very slow and not practical over relatively large databases. Further, FOIL traverses the refinement graph, evaluate a set candidate clauses in the graph, and returns the most promising clause. Hence, to be schema independent, the algorithm must evaluate clauses at the same order over equivalent schemas. Let \( \tau : R \rightarrow S \) be a composition/decomposition. If FOIL generates and evaluates clause \( h_R \) before clause \( h_S \) over schema \( R \), it must generate and evaluate the clause \( h_S \) before the clause \( h_S \) over \( S \). One of the operations of (modified) FOIL for generating new clauses is assigning variables to attributes in the current clause. It is not clear how FOIL can assign variables to attributes such that it maintains the same order of generating clauses over \( R \) and \( S \) without strong assumptions about the schema, such as strong universal relation and unique attribute role assumptions [2]. Hence, the generate and test approach used in top-down algorithms like FOIL is generally at odds with schema independence.

## 6. BOTTOM-UP ALGORITHMS

In this section, we consider bottom-up algorithms. Unlike top-down algorithms, bottom-up algorithms search the hypothesis space from specific to general. They usually construct a specific hypothesis taking as seed one training example, and then they apply generalization operators on one or more of these hypotheses. More specifically, given a positive example, bottom-up algorithms usually try to find the most specific clause in the hypothesis space that covers the example, relative to the background knowledge. This is called the saturation or bottom clause. In this section, we analyze and propose a modification to the algorithm for bottom clause construction given by [30]. We then analyze two popular algorithms, namely Golem [32] and ProGolem [33].
6.1 Bottom Clause Construction

Let $\bot_e$ be the bottom clause associated with example $e$, relative to the background knowledge, i.e., database instance, $B$. Then, $\bot_e$ is the most specific clause such that $B \cup \bot_e \models e$, where $\models$ is a deductive inference operator (e.g. resolution). Therefore, the bottom clause contains all information that is relevant to both the example and the background knowledge. The bottom clause can be computed by inverting the normal deductive proof process. Because the resolution operator is complete, the bottom clause can be computed by using the inverse resolution operator. However, employing inverse resolution is highly inefficient. Inverse entailment was proposed to overcome this issue. An algorithm for computing bottom clauses using inverse entailment is given in [30].

Because the resolution operator is complete, the bottom clause can be computed by inverting the normal deductive proof process. Therefore, the bottom clause can be computed by using the inverse resolution operator. However, employing inverse resolution is highly inefficient. Inverse entailment was proposed to overcome this issue [30]. An algorithm for computing bottom clauses using inverse entailment is given in [30]. This algorithm assigns variables to constants. Then the literals added to the bottom clause may contain variables and constants. We call these free literals. Given a seed example, this algorithm first adds the free literal corresponding to the example to the head of the bottom clause. It then adds free literals to the body of the clause in an iterative manner. The database instance is used to determine which literals to add.

**Example 6.1.** Consider the database instance shown in Table 2 and the example collaborated(John, Jake). The algorithm keeps a function that maps constants to variables. When it sees an already used constant, it uses the previously assigned variable. The algorithm first adds the free literal collaborated(V1, V2) to the head of the bottom clause, where it assigned V1 to John and V2 to Jake. Next, it adds literals to the body of the clause in an iterative manner. For instance, in iteration 1 it may add all literals where constants John and Jake appear. Then, the resulting bottom clause after iteration 1 would be

\[
collaborated(V_1, V_2) \leftarrow \text{professor}(V_1), \text{student}(V_2), \\
\text{hasPosition}(V_1, V_3), \text{inPhase}(V_2, V_4), \\
\text{publication}(V_5, V_6), \text{publication}(V_7, V_8).
\]

This algorithm can generate very long clauses after multiple iterations. For a large database, a bottom clause may include hundreds or thousands of literals. This would result in a long running time, and therefore would not be useful in practice. Therefore, we should avoid generating long bottom clauses. A common method is to restrict the maximum depth of any term in the bottom clause. The depth of a variable $X$, denoted by $d(X)$, is 0 if it appears in the head of the clause, otherwise it is $\min_{V \in \cup_X} d(V) + 1$, where $\cup_X$ are the variables in literals in the body of the clause containing $X$. The depth of a literal is the maximum depth of the variables appearing in the literal. The depth of a clause is the maximum depth of the literals appearing in the clause. In the bottom clause construction algorithm, in iteration $i$ only literals of depth at most $i$ are added to the clause.

**Example 6.2.** The following clause has depth 1:

\[
collaborated(X, Y) \leftarrow \text{publication}(P, X), \text{publication}(P, Y).
\]

If we set the maximum depth parameter to 1, then under the Original Schema, the clause shown in Example 6.2 would not be in the hypothesis language, as it contains variables that have depth 2. On the other hand, under the new schema, the clause presented above would be in the hypothesis language. Therefore, an algorithm that uses this bottom clause construction algorithm with parameter depth would not be hypothesis invariant.

We propose the following modifications to the algorithm for generating bottom clauses so that it produces equivalent bottom clauses for the same example, relative to equivalent instances of information equivalent schemas. The algorithm follows the normal procedure of bottom clause construction, where at each iteration, it selects a relation and adds one or more free literals of that relation to the bottom clause. The algorithm keeps a function that maps constants to variables, so that the same variable is assigned to the same constant every time. The literals that are added to the clause are based on the tuples for this relation in the database.

When adding literals, the algorithm applies a modified version of the Chase algorithm to the bottom clause with regard to the available INDs with equality in the schema as follows. Assume that the algorithm is generating the bottom clause relative to $I$, which is an instance of schema $\mathcal{R}$. Assume that the algorithm selects relation $R_i \in \mathcal{R}$ and adds a literal $L$ to the bottom clause based on some tuple $t_i$ of relation $R_i$. Let $\mathbf{L}$ be an inclusion class in $\mathcal{R}$, which contains $R_i$. For each constraint $R_i[A_i] = R_k[A_k]$ between the members of $\mathbf{L}$, the algorithm checks all tuples of relation $R_k$ that share join attributes with $t_i$. For each tuple, the algorithm creates a free literal. If a constant in a tuple has been already seen, then it uses the variable that was assigned to that constant. If the constant has not been seen, it assigns a fresh new variable. If the literal is not redundant with any literal already in the clause, then it is added to the clause. Two literals are redundant if they have the same relation name and the same old variables, and only differ in fresh new variables. If the literal is redundant, then it is ignored. The algorithm ensures that the corresponding attributes in $A_i$ and $A_k$ are assigned the same variables. Because this version of Chase algorithm is terminal and enforces the available INDs with equality

---

**Table 2: Sample database for UW-CSE Original Schema.**

| student(Jake) | inPhase(Jake,PreQuals) | publication(A.Jake) |
| professor(Mary) | hasPosition(Mary,Assistant) | publication(B.Mary) |
| professor(John) | hasPosition(John,Associate) | publication(A.John) |
| student(Sara) | inPhase(Sara,PostQuals) | publication(B.Sara) |

**proposition.**

\[
collaboratedWithPerson(X, Y) \leftarrow \text{publication}(P_1, X|Y) \\
\text{publication}(P_1, Z), \text{publication}(P_2, Z), \text{publication}(P_2, Y).
\]

This clause has depth 2.

The algorithms that rely on bottom clauses can be heavily influenced by the literals that appear in the bottom clauses. To achieve schema independence, these algorithms should get equivalent bottom clauses associated with the same example, relative to equivalent instances of information equivalent schemas. If this is not the case, then algorithms would not be hypothesis invariant. Unfortunately, using the depth parameter does not result in equivalent bottom clauses associated with the same example, relative to equivalent instances of information equivalent schemas. This is because different schemas require different depth values for equivalent clauses.

**Example 6.3.** Consider a new schema where we have a new relation coauthor(title, person1, person2), which indicates that person1 and person2 are coauthors in publication title. Then we could have the following clause for the collaboratedWithPerson relation, which has depth $I$

\[
collaboratedWithPerson(X, Y) \leftarrow \text{coauthor}(P_1, Z), \text{coauthor}(P_2, Y, Z).
\]

If we set the maximum depth parameter to 1, then under the Original Schema, the clause shown in Example 6.2 would not be in the hypothesis language, as it contains variables that have depth 2. On the other hand, under the new schema, the clause presented above would be in the hypothesis language. Therefore, an algorithm that uses this bottom clause construction algorithm with parameter depth would not be hypothesis invariant.
to the clause, the resulting clause is equivalent to the input clause \( \text{rlgg} \). We call this procedure \textit{chase}.

As explained above, the bottom clauses may get too long. Then, we also propose a modification of the algorithm so that the stopping condition is based on an alternative parameter called \textit{maxvars}. This parameter indicates the maximum number of (distinct) variables in a bottom clause before starting a new iteration of the algorithm. That is, at the end of each iteration the algorithm checks how many (distinct) variables are contained in the bottom clause. If this number is less than the parameter \textit{maxvars}, then the algorithm continues to the next iteration. On the other hand, if the number of variables is greater than or equal to \textit{maxvars}, the algorithm stops. Therefore, \textit{maxvars} does not indicate the exact number of variables in the clause. Instead, it is used as a stopping criterion in the algorithm.

In the following Lemma, we show that this algorithm delivers equivalent bottom clauses associated with the same example, relative to equivalent instances of information equivalent schemas. Let \( \tau : \mathcal{R} \to \mathcal{S} \) be a composition/decomposition. Let \( I \) and \( J \) be instances of \( \mathcal{R} \) and \( \mathcal{S} \), respectively, such that \( \tau(I) = J \).

\begin{lemma}
If \( \bot_{e, I} \) and \( \bot_{e, J} \) are bottom clauses associated with \( e \) relative to \( I \) and \( J \), respectively, generated by the algorithm described above, then \( \bot_{e, I} \equiv \bot_{e, J} \).
\end{lemma}

### 6.2 Golem

In this section, we consider a bottom-up learning algorithm called Golem \cite{6.4}. Golem, like other learning algorithms, follows a covering approach, as the one shown in Algorithm \ref{alg:Golem}. Golem’s \textit{LearnClause} procedure follows a bottom-up approach, which is based on the relative least general generalization (rlgg) operator.

Given clauses \( C_1 \) and \( C_2 \), the least general generalization (lgg) of \( C_1 \) and \( C_2 \) is the clause \( C \) that is more general than \( C_1 \) and \( C_2 \), but the least general such clause. The notion of generality is defined by \( \theta \)-subsumption. Therefore, clause \( C \) is more general than \( C_1 \) if and only if \( C \theta \)-subsumes \( C_1 \) (and similarly for \( C_2 \)). This notion of generality gives a computable generality relation. Further, the lgg of two clauses is unique. Because of the lack of space, for further details we refer the readers to \cite{2}.

Consider a schema \( \mathcal{R} \) and database instance \( I \in \mathcal{I}_R \). Let \( e \) be a positive example for some target relation \( T \). A special case of a bottom clause or saturation, used in Golem, is the one where all literals in the body of the clause are grounded. Then, the saturation of \( e \) relative to \( I \), denoted \( \bot_{e, I} \), is the most specific clause \( e \mapsto I' \) such that \( I' \subseteq I \) contains all ground atoms in \( I \) that are somehow related to \( e \). By somehow related, we mean that the ground atoms are linked to \( e \) by some chain of ground atoms. The saturation can be computed using the approach described previously, by only allowing ground literals in the clause and restricting the maximum number of constants instead of the maximum number of variables. Therefore, the saturations for \( e \) relative to \( I \) and \( J \) are equivalent and have an “equivalent” order. The operator that computes the lgg for a pair of saturations is called \textit{rlgg}. The lgg of a set of saturations is defined via pairwise operations, that is

\[
\text{lgg}(\{C_1, \ldots, C_n\}) = \text{lgg}(\text{lgg}(\{C_1, \ldots, C_{n-1}\}), C_n)
\]

The order of pairwise lgg does not matter as the lgg operator is commutative and associative.

Given a database instance \( I \) and training examples \( E^+ \) and \( E^- \), Golem’s \textit{LearnClause} procedure learns a clause that covers as many positive examples as possible and no negative examples. The procedure first computes the saturation for each example \( e \in E^+ \). The set of all saturated clauses is \( S = \{ \bot_{e, I} : e \in E^+ \} \). Then it tries to find a subset \( S' \) of \( S \) such that the clause \( C = \text{lgg}(S') \) covers many positive examples and no negative examples. Therefore, Golem’s \textit{LearnClause} procedure approach is to find the largest subset of saturated clauses \( S' \) whose lgg covers no negative examples. Algorithm \ref{alg:Golem} sketches this procedure. This algorithm uses the function \textit{covers} defined as \( \text{covers}(I, C, E) = \{ e \in E \mid I \land C \models e \} \). Intuitively, the algorithm first finds a set of pairs to examples to generalize. These examples are picked greedily according to coverage. It then greedily includes new examples into the generalization as long as no negative examples are covered.

### Algorithm 2: Golem’s \textit{LearnClause} procedure.

\begin{algorithm}
\caption{Golem’s \textit{LearnClause} procedure.}
\begin{algorithmic}
\State \textbf{Input} : Database instance \( I \), positive examples \( E^+ \), negative examples \( E^- \).
\State \textbf{Output}: A clause \( C^* \) that covers as many positive examples and no negative examples.

\begin{algorithmic}
\State \( U = E^+ \);
\State \( C = \{ \text{lgg}(\bot_{e, I}, \bot_{e', I}) \mid e, e' \in U, \text{covers}(C, I, N) = \emptyset \} \);
\While \( C \) is not empty
\State \( C^* = \text{argmax}_{C \subseteq C} \text{covers}(C, I, U) \);
\State \( C = \text{reduce}(C^*) \);
\State \( U = U \cup \text{covers}(C^*, I, U) \);
\State \( C = \{ \text{lgg}(C^*, \bot_{e, I}) \mid e \in U, \text{covers}(C, I, N) = \emptyset \} \);
\EndWhile
\State \textbf{return} \( C^* \)
\end{algorithmic}
\end{algorithm}
\end{algorithm}

Golem may generate definitions that contain very large clauses. The reason is that the size of a clause generated by \textit{lgg}(\( C_1 \), \( C_2 \)), where \( C_1 \) and \( C_2 \) are clauses, is bounded by \(|C_1| \cdot |C_2|\). Therefore, the size of a clause resulting from pairwise \textit{lgg} operations can grow exponentially with the number of clauses from which it is generalized. This results in exponential running time. For this reason, Golem \cite{6.4} employs some techniques to prune clauses in the resulting hypothesis. Pruned clauses should be shorter than the original clauses, but still express the desired hypothesis. In this paper, we consider a pruning approach called \textit{negative reduction}. This technique consists of permanently removing literals of a clause if after their removal, the clause does not cover any negative example. More formally, let \( C' = \text{reduce}(C) \) be the negative reduction operation such that if \( C \) is a consistent clause and \( C' \subseteq C \) is a pruned clause, \( C' \) is also consistent. Given that a good set of negative examples are available, this technique is effective and efficient \cite{6.4}.

\begin{lemma}
The \textit{rlgg} operator is schema independent w.r.t. vertical composition/decomposition transformations.
\end{lemma}

\begin{theorem}
Golem is schema independent w.r.t. vertical composition/decomposition transformations.
\end{theorem}

### 6.3 ProGolem

ProGolem is another learning algorithm that follows a bottom-up approach \cite{6.3}. Unlike Golem, it is based on the asymmetric minimal generalization (armg) operator, which is another generalization operator. As the previous algorithms, ProGolem follows a covering approach similar to Algorithm \ref{alg:Golem}. Given the bottom clause associated with an example, ProGolem’s \textit{LearnClause} procedure uses a greedy beam search to select the best clause generated by the \textit{armg} operator with respect to the bottom clause. The \textit{armg} operator uses other positive examples to generalize the input bottom clause, so that the resulting clause covers these examples, as well as the seed positive example. The clauses kept in the
beam correspond to different examples used for generalizing the bottom clause. The beam search requires an evaluation function to score clauses in the beam. We select an evaluation function that is agnostic of the schema used, such as coverage, which is simply the number of positive examples covered by the clause minus the number of negative examples covered by the clause. We already showed that we are able to get equivalent bottom clauses associated with the same example, relative to equivalent instances of information equivalent schemas. Therefore, in order for ProGolem to be schema independent, we must show that the armg operator is schema independent.

ProGolem considers bottom clauses as ordered clauses. This means that the order of the bottom clauses can have an impact in the result of the algorithm. Therefore, to ensure that the algorithm is schema independent, we must force clauses to have an equivalent order. Let $\tau : R \rightarrow S$ be a composition/decomposition. Let $I$ and $J$ be instances of $R$ and $S$, respectively, such that $\tau(I) = J$. Let $\perp_{e,I}$ and $\perp_{e,J}$ be the bottom clauses associated with example $e$ relative to $I$ and $J$, respectively. Then, $\perp_{e,I}$ and $\perp_{e,J}$ have an equivalent order if given that for $R_i, R_j \in R$, the relations in the inclusion class of $R_i$ appear before the relations in the inclusion class of $R_j$ in $\perp_{e,I}$ iff the relations in the inclusion class of $\tau(R_i)$ appear before the relations in the inclusion class of $\tau(R_j)$ in $\perp_{e,J}$. Note that the order of relations within the inclusion classes of $\tau(R_i)$ and $\tau(R_j)$ in $\perp_{e,J}$ does not matter as long as all relations in $\tau(R_i)$ appear before relations in $\tau(R_j)$. Therefore, this order is a fixed partial order between relations in $R$.

One may use the content of $I$ to establish a partial order between inclusion classes in $R$, which is preserved under composition/decomposition. Let us define the natural join over an inclusion class in schema $R$ as the join of all relations in $R$ using their attributes that appear in INDs with equality. According to Definition 4.1, $\tau$ does not join the relations from different inclusion classes in $R$. It also introduces a new IND with equality only when it decomposes a relation in $R$. Further, it eliminates INDs with equality from $R$ only if it joins some relations within an inclusion class in $R$. Hence, we may define a bijective mapping $M$ between all inclusion classes in $R$ and $S$ such that the natural join over inclusion class $L$ in $R$ is equal to the natural join of $M(L)$ in $S$ for all corresponding instances of $R$ and $S$. Hence, one may use the natural joins of inclusion classes to define an order between inclusion classes in a database, which is preserved over all composition/decomposition of the database. In the rest of this section, we assume that equivalent bottom clauses have an equivalent order.

The details for the armg operator are given in $\[33]$. The cardinality of the output clause from the armg operator is restricted by the cardinality of the bottom clause used to construct it. This clause will cover the seed example, as well as other examples selected in the beam search. Therefore, employing the armg operator is significantly more efficient than the rlgg operator, as resulting clauses grow polynomially, instead of exponentially, with the number of examples. The algorithm for constructing ARMGs is given in Algorithm $\[3]$. Given the bottom clause associated with a positive example, the algorithm constructs the ARMG by dropping literals from the body of the clause until another positive example is covered. The dropped literals are called blocking atoms $\[33]$. The following definition employs the $\perp$ operator, where $x \perp y$ means that $y$ is provable from $x$.

**Definition 6.7.** Let $B$ be background knowledge, $E^+$ the set of positive examples, $e \in E^+$ and $\perp_{e} = T \leftarrow L_1, \ldots , L_n$ be a definite ordered clause. $L_i$ is a blocking atom if and only if $i$ is the least value such that for all $\theta$, $e = T\theta$, $B \perp (L_1, \ldots , L_i)\theta$.

**Algorithm 3: ARMG algorithm.**

<table>
<thead>
<tr>
<th>Input</th>
<th>Bottom clause $\perp_{e}$, positive example $e'$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>An ARMG in $\text{armg}(e'</td>
</tr>
<tr>
<td>$\perp$</td>
<td>$\perp_{e} = T \leftarrow L_1, \ldots , L_n$;</td>
</tr>
<tr>
<td>while</td>
<td>there is a blocking atom $L_i$ w.r.t. $e'$ in the body of $\perp$ do</td>
</tr>
<tr>
<td></td>
<td>Remove $L_i$ from $\perp$;</td>
</tr>
<tr>
<td></td>
<td>Remove atoms from $\perp$ which are not head-connected;</td>
</tr>
<tr>
<td>end</td>
<td>Return $\perp$;</td>
</tr>
</tbody>
</table>

**Theorem 6.8.** The armg operator is schema independent w.r.t. composition/decomposition.

7. QUERY-BASED ALGORITHMS

In this section, we consider query-based learning algorithms, which learn exact definitions by asking queries to an oracle $\[25][30][43][6]$. These type of algorithms have been recently used in various areas of database management, such as finding schema mappings and designing usable query interfaces $\[10][8]$. Queries can be of multiple types, however the most common types are equivalence queries and membership queries. In equivalence queries (EQ), the learner presents a hypothesis to the oracle and the oracle returns yes if the hypothesis is equal to the target relation definition, otherwise it returns a counter-example. In membership queries (MQ), the learner asks if an example is a positive example, and the oracle answers yes or no. Query-based algorithms are theoretically evaluated by their query complexity – the asymptotic number of queries asked by the algorithm. Particularly, we evaluate the lower and upper bounds on the query complexity of these algorithms. A good query-based algorithm is the one that does not ask many queries, as the more queries are asked, the more resources are required. For instance, in the query interface described by $\[3]$, the oracle is the user. Therefore, the smaller number of questions asked to the user makes the interface more usable.

In this paper, we focus on the $A2$ algorithm by Khardon $\[25]$, a query-based learning algorithm that learns function-free, first-order Horn expressions. The reasons for choosing this algorithm are three fold: i) $A2$ is representative of query-based learning algorithms that work on the relational model, ii) there is an implementation of the algorithm $\[6]$, iii) $A2$ is a generalization to the relational model of a classic query-based propositional algorithm $\[5]$. Because query-based algorithms follow a different learning model, in this section we follow a different approach by analyzing the impact of schema transformations on the query complexity of learning algorithms. If a query-based algorithm is schema independent, it should be able to learn exact definitions with asymptotically the same number of queries under equivalent schemas. For this purpose, we compare the lower bound on the query complexity of these algorithms against the upper bounds on their query complexities over their composition/decomposition. We argue that if the lower bound under one of the schemas is greater than the upper bound under another schema, then the algorithm is not schema independent. Of course, this is not a desirable property, as this means that the choice of representation has a huge impact on the performance of the algorithm. However, we prove that algorithms such as $A2$ suffer from this property.

**Theorem 7.1.** Let $\Omega(f)x$ and $O(g)x$ be the lower bound and upper bound, respectively, on the query complexity of $A2$ for all
target relations under schema $R$. Then, there is a composition/decomposition of $R$, $S$, such that $\Omega(f) \gg O(g)$.

The lower bound of $A2$ is actually the Vapnik-Chervonenkis dimension (VC-Dim) of the hypothesis language that consists of function-free, first-order Horn expressions. Therefore, we have proven in Theorem 7.1 that there are cases where the lower bound on the query complexity of any algorithm under this hypothesis language is greater than the upper bound on the query complexity of $A2$. This means that any algorithm that is as good as $A2$ (does not ask more queries than $A2$) is highly dependent on the schema details.

In query-based algorithms, the running time is dependent on the number of queries asked to the oracle. For instance, the running time of the $A2$ algorithm is polynomial in the upper bound on the query complexity and $n^k$ [25]. Parameters $n$ and $k$ are not dependent on the schema. Therefore, as in other families of algorithms, the running time of $A2$ is exponential in the maximum arity and linear in the number of relations. This results in the algorithm taking longer on schemas that contain relations with large arity.

### 8. EMPIRICAL RESULTS

#### 8.1 Sample-based algorithms

We evaluate the average-case schema independence of three popular relational learning algorithms: FOIL [38], Progol [30], and ProGolem [53]. Progol is a top-down algorithm similar to FOIL, but performs a non-greedy search over the hypothesis space, which is restricted by some parameters. We emulate both FOIL and Progol using Aleph, a well known Inductive Logic Programming (ILP) system. We use the names A-FOIL and A-Progol to indicate the results of these systems. ProGolem is implemented in GILPS, another ILP system. The configurations of Aleph and GILPS systems are in Appendix B.

We have used UW-CSE and IMDb DBs for our experiments. We represent the UW-CSE DB using four equivalent schemas: the original schema, its transformed 4th normal form, and two denormalized schemas, which are all shown in Table 3. The schemas contain the required FD and INdS explained in Section 4. For the UW-CSE dataset, we use 939 tuples and 46 positive examples. We generate negative examples using the closed-world assumption, and then sample these to obtain twice as many negative examples as positive examples. We learn the relation advisedBy(stud,prof), which indicates that student stud is advised by professor prof.

We also run experiments using a subset of the IMDb. We obtain movies that were high grossing in their opening weekend by querying the IMDb website. We then obtain information such as actors, directors, genres, countries, etc. for each movie from the JMDB database, which contains the same information found in the IMDb website, in relational format. We use a sample of 249160 tuples from JMDB. We represent IMDb using two schemas: the original JMDB schema and an alternative schema called “Single Lookup”, where all attributes shown in the summary box of each movie in the IMDb website are in a single relation. These schemas are shown in Table 4. We hypothesize that to improve the response time of showing the summary information about a movie, a database designer may compose all the attributes of the summary box in one relation to avoid expensive joins. We leave other relations in the original JMDB schema intact. We enforce the required INdS constraints on the original schema to make the transformation an information preserving composition/decomposition. We learn the relation femaleActor(person), which indicates that person is a female. We sample the training data and obtain 250 positive examples and 500 negative examples.

We evaluate the precision, recall, and running time of the algorithms as the average over 5-fold cross validation. Precision is the proportion of true positives against all the positive results and recall is the proportion of true positives against all positive examples. Tables 5 and 6 show the results for the UW-CSE and IMDb DBs, respectively. These results show that the algorithms are not generally schema independent as the effectiveness measures vary quite widely across different schemas. ProGolem appears to be more robust than others. For instance, ProGolem generates equivalent definitions across all schemas of UW-CSE. On the other hand, A-FOIL and A-Progol show large differences in precision and recall across

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Metric</th>
<th>Original</th>
<th>4NF</th>
<th>Denorm. 1</th>
<th>Denorm. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-FOIL</td>
<td>Precision</td>
<td>0.89</td>
<td>0.90</td>
<td>0.51</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.73</td>
<td>0.44</td>
<td>0.55</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td>Time (s)</td>
<td>1.33</td>
<td>1.48</td>
<td>14.86</td>
<td>1.78</td>
</tr>
<tr>
<td>A-Progol</td>
<td>Precision</td>
<td>0.84</td>
<td>0.56</td>
<td>0.49</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.89</td>
<td>0.73</td>
<td>0.40</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td>Time (s)</td>
<td>3.39</td>
<td>2.84</td>
<td>21.10</td>
<td>3.17</td>
</tr>
<tr>
<td>ProGolem</td>
<td>Precision</td>
<td>0.86</td>
<td>0.86</td>
<td>0.86</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.92</td>
<td>0.92</td>
<td>0.92</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td>Time (s)</td>
<td>9.82</td>
<td>9.66</td>
<td>23.10</td>
<td>9.66</td>
</tr>
</tbody>
</table>

Table 5: Results of learning relations over UW-CSE.

---

1http://www.cs.ox.ac.uk/activities/machlearn/Aleph/aleph.html
2http://www.doc.ic.ac.uk/~jcs06/GILPS/
3http://www.imdb.com
4http://www.doc.ic.ac.uk/ jcs06/GILPS/
different schemas for both datasets. This is because ProGolem relies less on heuristic guidance compared A-FOIL and A-Progol. ProGolem is also more effective than A-FOIL and A-Progol over UW-CSE. It is also significantly more precise on IMDb than top-down algorithms, but has a lower recall. Our hypothesis is that this happens because bottom-up algorithms are based on generalization operators that take bigger search steps.

A particularly notable observation is that the performance of all learning algorithms is decreased when using the Denormalized 1 schema of the UW-CSE DB and the Single Lookup schema of the IMDb. In A-FOIL and A-Progol, using this schema results in very bad precision and recall. However, this is not the case for schema Denormalized 2. This is because the Denormalized 2 schema contains the relation coauthor(title,prof,stud), which is highly predictive for the target relation. Finally, we can see that the runtime of all learning algorithms is increased when using the Denormalized 1 schema and the Single Lookup schema. Such an increase in runtime would likely discourage users who use more denormalized schemas, unless they have prior knowledge of which relations can be predictive of the target relation.

In order to achieve schema independence, we have modified the bottom clause construction algorithm of ProGolem, as described in Section 6. We call this algorithm M-ProGolem. We added a new parameter called maxvars. In this algorithm, in iteration $i$ all literals that contain variables of depth at most $i$ are added to the bottom clause. With the partial order assumption, this algorithm is schema independent. Therefore, all schemas obtain the same precision and recall, and only their running time varies. It may not be clear to the user how to set the maxvars parameter. One approach is to experiment with a few reasonable values and evaluate them in terms of accuracy and running time. Table 7 shows the results for running M-ProGolem over UW-CSE with modified ProGolem.

### Table 6: Results of learning relations over IMDb.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Metric</th>
<th>Original</th>
<th>Single Lookup</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-FOIL</td>
<td>Precision</td>
<td>0.63</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.43</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>Time (m)</td>
<td>0.88</td>
<td>20.49</td>
</tr>
<tr>
<td>A-Progol</td>
<td>Precision</td>
<td>0.61</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.42</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>Time (m)</td>
<td>5.91</td>
<td>24.55</td>
</tr>
<tr>
<td>ProGolem</td>
<td>Precision</td>
<td>0.89</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.34</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>Time (m)</td>
<td>5.21</td>
<td>79.47</td>
</tr>
<tr>
<td>M-ProGolem</td>
<td>Precision</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>Time (m)</td>
<td>59.34</td>
<td>59.34</td>
</tr>
</tbody>
</table>

### Table 7: Results of learning relations over UW-CSE with modified ProGolem.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Metric</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-ProGolem</td>
<td>Precision</td>
<td>0.86</td>
<td>0.86</td>
<td>0.86</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>Recall</td>
<td>0.92</td>
<td>0.92</td>
<td>0.91</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td>Time (s)</td>
<td>36.43</td>
<td>37.81</td>
<td>102.75</td>
<td>481.66</td>
</tr>
</tbody>
</table>

### 8.2 Query-based algorithms

We use the LogAn-H system [6], which is an implementation of the A2 algorithm [25]. Specifically, we use the interactive algorithm with automatic user mode. In this mode, the system is told the Horn definition to be learned, so that it can act as an oracle. Then the algorithm’s queries are answered automatically until it learns the exact definition. We generated random Horn definitions over the alternative schema of the UW-CSE DB, shown in Table 6. The only parameter for generating each clause in a definition is the number of variables in the clause. To generate the head of each clause, we created a new relation of random arity, where the minimum arity is 1 and the maximum arity is the maximum arity of the relations in the alternative schema. The body of each clause can be of any length as long as the number of variables in the clause is equal to the specified parameter and all variables appearing in the head relation also appear in any relation in the body. The body of the clause is composed of randomly chosen relations, where each relation can be the head relation (allowing for recursive clauses) or any relation in the input schema. Head and body relations are populated with variables, where each variable is randomly chosen to be a new (until reaching the input number of variables) or already used variable. Clauses cannot contain function or constant symbols.

After generating each random Horn definitions over the alternative schema, we transformed these expressions to the original schema simply doing vertical decomposition to each of the clauses in an expression. Then, we minimized all definitions using the Homomorphism theorem and the Chase algorithm with the functional and inclusion dependency constraints [2]. We varied the number of clauses in a definition to be between 1 and 5, each containing between 4 and 8 variables. We generated 50 random definitions for each setting, getting a total of 250 expressions for each number of variables. The A2 algorithm takes as input the target expression and the signature. The signature consists of the names of all relations in the input schema and the head relation, as well as the arity of each relation. We ran the LogAn-H system with the original definition over the alternative schema and the transformed definition over the original schema, and recorded the number of queries required to learn each definition. In these experiments, we report the query complexity – number of equivalence queries (EQs) and membership queries (MQs) – of the A2 algorithm.

The number of EQs and MQs asked by the algorithm under the original and alternative schemas is presented in Figure 2. The average number of EQs required by the A2 algorithm over both schemas is constant for different number of variables. However, this is not the case for MQs. Particularly, we can see that the number of MQs increased with the more decomposed schema, that is the original schema.

![Figure 2: Average number of membership queries required by the A2 algorithm.](image)

### 9. CONCLUSION

We formally defined the novel property of schema independence for relational learning algorithms, which states that the output of
these algorithms should not depend on the schema used to represent their input databases. We prove that current popular relational learning algorithms are not schema independent. We used the dependencies in the schema to extend current bottom-up learning algorithms and proved that the resulting algorithms are schema independent. Our empirical results on benchmark and real datasets validated our theoretical results and showed that the proposed algorithms are as effective as the current relational learning algorithms.

10. ACKNOWLEDGMENTS
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11. REFERENCES
APPENDIX

A. PROOFS

Proof of Theorem 5.1

Proof. Let \( \tau : R \rightarrow S \) be a composition/decomposition. Without loss of generality, we assume that (distinct) relations \( R_1(A, B, C) \) and \( R_2(D, B, E) \) belong to schema \( R \). We also assume that \( \tau \) decomposes \( R_1 \) to \( S_1(A, B) \) and \( S_2(B, C) \) and \( R_2 \) to \( S_3(D, B) \) and \( S_4(B, E) \), respectively. Let \( l \) be the maximum clause length \( \theta = (l) \) be the parameter settings for FOIL. Without loss of generality we set the value of \( \theta \) to 2. Let \( L_{R,\theta}^{\delta} \) and \( L_{A,\theta}^{\delta} \) be the hypothesis languages over \( R \) and \( A \), respectively. Let \( T(X, Y) \) be the target relation. Consider hypothesis \( h_R \)

\[
T(X, Y) \leftarrow R_1(X, Z, W), R_2(Y, Z, V).
\]

over schema \( R \). The mapped hypothesis \( \delta_r(h_R) \) is:

\[
T(X, Y) \leftarrow S_1(X, Z), S_3(Z, W), S_4(Z, Y), S_5(Z, W).
\]

Because \( h_R \) is minimal and all literals in the body of the clause in \( \delta_r(h_R) \) are different, \( \delta_r(h_R) \) is also minimal. Hypothesis \( h_R \) is in the hypothesis language \( L_{R,\theta}^{\delta} \) because \( \text{clauselength}(H_R) \leq 2 \). However, hypothesis \( \delta_r(h_R) \) is not in the hypothesis language \( L_{A,\theta}^{\delta} \) because \( \text{clauselength}(H_R) > 2 \). There may be other definitions semantically equivalent to \( \delta_r(h_R) \), however they will have clause length greater than or equal to \( \text{clauselength}(\delta_r(h_R)) \). Thus, they will not be in \( L_{A,\theta}^{\delta} \) either. Therefore, hypothesis spaces \( L_{R,\theta}^{\delta} \) and \( L_{A,\theta}^{\delta} \) are not equivalent. Now, let us use another parameter setting \( \theta' \) for schema \( A \) where the value of \( \theta' \) is 4 so that the hypothesis \( \delta_{r'}(h_R) \) becomes a member of \( L_{R,\theta}^{\delta} \). In this case, the hypothesis following will also be a member of \( L_{A,\theta}^{\delta} \)

\[
T(X, Y) \leftarrow S_1(X, Z), S_3(X, W), S_5(X, T), S_6(X, Y).
\]

However, the equivalent hypothesis to this hypothesis over \( R \) is:

\[
T(X, Y) \leftarrow R_1(X, Z, V_1), R_2(X, V_1, V_2), R_3(X, T, V_3), R_4(X, Y, V_4)\]

where \( V_i, 1 \leq i \leq 4 \) are fresh variables. Since this hypothesis over \( R \) is minimal, one has to change \( l \) over \( R \) to 4 to achieve equivalent hypothesis spaces over \( R \) and \( S \). Hence, we have to alternate between the parameter settings over \( R \) and \( S \) without any stopping condition. Thus, there is not any fixed values for the parameters to ensure the hypothesis equivalence over schemas \( R \) and \( S \). \( \square \)

Proof of Proposition 5.2

Proof. Let \( \tau : R \rightarrow S \) be a composition/decomposition. Without loss of generality, let \( R_1 \) and \( R_2 \) be relations in \( R \) such that either \( R_1 \) or \( R_2 \) participate in only one IND \( R_1[A] = R_2[B] \). \( R_1 \) and \( R_2 \) will always appear together in all clauses of the hypothesis space of modified FOIL. Without loss of generality, we set the maximum number of inclusion classes for modified FOIL over \( R \) to 1. Let \( e \) be the set of clauses that contain a single occurrence of \( R_1 \) and a single occurrence of \( R_2 \). The set \( e \) is a subset of the hypothesis space of modified FOIL over \( R \). Let us assume that \( \tau \) maps relation \( R_1 \) to a set of relations \( S_1 \) and \( R_2 \) to a relation \( S_2 \) \( \notin S_1 \). There is an INDS with equality between every pair of relations in \( S_1 \). Further, because composition/decomposition preserves INDS, there is an IND with equality between \( S_1 \) and \( S_2 \). Hence, all clauses in the hypothesis space of modified FOIL over \( S \) that contain a relation in \( S_1 \) will have the rest of relations in \( S_1 \) and \( S_2 \). We call this set of clauses \( e' \). Because composition/decompositions do not introduce any new IND with equality, these clauses do not contain any other relation and satisfy the restriction on the maximum number of inclusion classes. Because the same variables will be assigned to the attributes that participate in the INDS with equality, there is a bijection \( M \) between all clauses in \( e \) and \( e' \) such that \( M(e) \) and \( e \) are equivalent. The proof extends for all definitions built over these clauses. \( \square \)
that if \( \tau \) is bijective, \( \Sigma_S \) contains inclusion dependencies between the join attributes of \( S_1, \ldots, S_m \). Let \( r_1 = R(a_1, \ldots, a_k) \) and \( r_2 = R(a'_1, \ldots, a'_k) \) be two ground atoms in \( I \). Then, \( \tau(r_1) = S_i(t_1), \ldots, S_m(t_m) \) and \( \tau(r_2) = S_i(t'_1), \ldots, S_m(t'_m) \) are ground atoms in \( J \), where \( t_i \) and \( t'_i \), \( 1 \leq i \leq m \), are tuples. Then, the \( \text{lgg} \) of ground atoms \( r_1 \) and \( r_2 \) is defined as

\[
\text{lgg}(r_1, r_2) = R(\text{lgg}(a_1, a'_1), \ldots, \text{lgg}(a_k, a'_k))
\]

By applying transformation \( \tau \), this is equivalent to

\[
S_i(s_1), S_2(s_2), \ldots, S_m(s_m) = \text{lgg}(S_i(t_1), S_i(t'_1)), \ldots, \text{lgg}(S_m(t_m), S_m(t'_m)) = \text{lgg}(\tau(r_1), \tau(r_2))
\]

\[\square\]

**Proof of Theorem 6.6**

**Proof.** Golem follows a covering approach. Any algorithm that follows a covering approach is schema independent if its LearnClause procedure is schema independent. Golem’s LearnClause procedure consists in finding the largest subset of saturated clauses whose \( \text{lgg} \) covers no negative examples.

Let \( \tau : \mathcal{R} \rightarrow \mathcal{S} \) be a bijective transformation that is a vertical composition/decomposition between schemas \( \mathcal{R} = (\mathcal{R}, \Sigma_\mathcal{R}) \) and \( \mathcal{S} = (\mathcal{S}, \Sigma_\mathcal{S}) \). The algorithm first finds a set candidate clauses, which are \( \text{rlgg} \)s between pairs of examples. These clauses are picked greedily according to coverage. We have shown in Proposition 6.5 that the \( \text{rlgg} \) operator is schema independent w.r.t. vertical composition/decomposition transformations. The coverage of equivalent clauses under \( \mathcal{R} \) and \( \mathcal{S} \) must be the same, as it only depends on the set of examples and not on the schema. Therefore, the set of candidate clauses is the same under \( \mathcal{R} \) and \( \mathcal{S} \).

The algorithm then finds the clause with best coverage. Again, this depends only on the set of examples and not on the schema. It then reduces this clause and finds other candidate clauses by generalizing this clause with other positive examples. Because the \( \text{rlgg} \) operator is schema independent and the set of positive examples is the same under both schemas, the resulting set of candidate clauses must be the same. This procedure iterates until there are no more candidate clauses. We now show that negative reduction of two equivalent clauses under equivalent schemas delivers equivalent pruned clauses. Let \( h_C \) and \( h_S \) be definitions for target relation \( T \) under schemas \( \mathcal{R} \) and \( \mathcal{S} \), respectively. Without loss of generality, assume that \( h_C \) and \( h_S \) contain the single clause \( C_R \) and \( C_S \), respectively.

Let \( C'_R \) and \( C'_S \) be pruned clauses such that \( \text{reduce}(C_R) = C'_R \) and \( \text{reduce}(C_S) = C'_S \). We assume that relations in \( C_R \) and \( C_S \) are considered for removal in a fixed order. That is, if for relations \( R_1, R_2 \in \mathcal{R}, R_1 \) is considered before \( R_2 \), then relations \( \tau(R_1) \) are considered before relations \( \tau(R_2) \) under schema \( \mathcal{S} \). We explain how to compute this order in Section 6.3.

Assume that relation \( R \) is in clause \( C_R \). Then, because \( C_R \subseteq C'_S \), relations \( S_1, \ldots, S_m \) must be in clause \( C'_S \). We consider two cases: 1) \( R \in C'_R \) or 2) \( R \notin C'_R \). We show that if case 1 occurs, then \( S_1, \ldots, S_m \in C'_R \) and if case 2 occurs, then \( S_1, \ldots, S_m \notin C'_S \).

Case 1: Assume that some relation \( S_i \in \{ S_1, \ldots, S_m \} \) is not in \( C'_S \). This means that \( S_i \) was dropped without making the clause inconsistent. However, we know that there are inclusion dependencies in \( \Sigma_S \) between \( S_i \) and other relation(s) in \( S_1, \ldots, S_m \). Therefore, we can get an equivalent clause by applying \( \text{chase}_{\Sigma_S}(C_S') \), which will contain \( S_i \). Therefore, we have that \( C'_R \equiv C'_S \).

Case 2: Assume that some relation \( S_i \in \{ S_1, \ldots, S_m \} \) is in \( C'_S \). If we were able to drop relation \( R \) from \( C_R \) without making it inconsistent, because \( \tau(R) = S_1, \ldots, S_m \), then we should be able to drop all relations \( S_1, \ldots, S_m \) without making \( C_S \) inconsistent. Therefore, we have that \( C'_R \equiv C'_S \). \( \square \)

**Proof of Theorem 6.8**

**Proof.** The armg operator is computed by the ARMG algorithm. Let \( \tau : \mathcal{R} \rightarrow \mathcal{S} \) be a bijective transformation that is a vertical composition/decomposition between schemas \( \mathcal{R} = (\mathcal{R}, \Sigma_\mathcal{R}) \) and \( \mathcal{S} = (\mathcal{S}, \Sigma_\mathcal{S}) \). Let \( R \in \mathcal{R} \) be a relation in \( \mathcal{R} \) such that \( \tau(R) = S_1, \ldots, S_m, 1 \leq m \leq |\mathcal{S}| \). Let \( I \) and \( J \) be instances of \( \mathcal{R} \) and \( \mathcal{S} \), respectively, such that \( \tau(I) = J \). Because of Corollary 4.3.2 in [36], we know that if \( \tau \) is bijective, \( \Sigma_S \) contains inclusion dependencies between the join attributes of \( S_1, \ldots, S_m \).

We first show that we get equivalent results for transformation \( \tau \), which is a vertical decomposition. Assume the input to the ARMG algorithm under schema \( \mathcal{R} \) is the bottom clause \( \bot_{e,1} \) and positive example \( e' \). The bottom clause \( \bot_{e,1} \) is the ordered clause \( C_R \).

Assume it has the form

\[
T(w) \leftarrow L_1(u_1), \ldots, L_{i-1}(u_{i-1}), R(u_i), L_{i+1}(u_{i+1}), \ldots, L_n(u_n),
\]

where \( w \) is a variable tuple, \( L_j \) is a literal with predicate symbol in \( \mathcal{R} \), and \( u_j \) is a free tuple, \( 1 \leq j \leq n \). Because \( \tau \) is a bijective Horn transformation, according to Proposition 6.7, \( \tau \) is definition preserving w.r.t. \( \mathcal{H}D_\mathcal{R} \) and \( \mathcal{H}D_\mathcal{S} \). This means that clause \( \delta_\tau(C_R) \) exists that is equivalent to \( C_R \). Then, \( C_S = \text{chase}_{\Sigma_S}(\delta_\tau(C_R)) \) is given by

\[
T(w) \leftarrow L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), S_1(v_1), \ldots, S_m(v_m), L'_{i+1}(u'_{i+1}), \ldots, L'_{n'}(u'_{n'}),
\]

where \( v_j \) is a free tuple, \( 1 \leq j \leq m \), and \( \tau(L_1(u_1)), \ldots, L_{i-1}(u_{i-1}), L_{i+1}(u_{i+1}), \ldots, L_n(u_n) \) = \( L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), L'_{i+1}(u'_{i+1}), \ldots, L'_{n'}(u'_{n'}) \). That is, all literals in the body of \( \delta_\tau \rightarrow \mathcal{R} \) are replaced by the equivalent literals in schema \( \mathcal{S} \).

Assume that \( R(u_i) \) is a blocking atom w.r.t. \( e' \) in \( C_R \). Then for all \( \theta \) such that \( e' = T(w)\theta \), we have that

\[
I \vdash (L_1(u_1), \ldots, L_{i-1}(u_{i-1}), R(u_i))\theta
\]

By definition, we know that \( R(u_i) \) is the first literal with relation name \( R \) that is a blocking atom. Then, no literal in \( L_1(u_1), \ldots, L_{i-1}(u_{i-1}) \) has relation name \( R \) and is a blocking atom. According to the ARMG algorithm, \( R(u_i) \) must be removed from \( C_R \).

Then, the resulting clause, which we denote \( C''_R \), has the following form

\[
T(w) \leftarrow L_1(u_1), \ldots, L_{i-1}(u_{i-1}), L_{i+1}(u_{i+1}), \ldots, L_n(u_n).
\]
Now consider clause $\overrightarrow{C_S}$. If no literals of $S_1(v_1), \ldots, S_m(v_m)$ are blocking atoms w.r.t. $e'$ in $\overrightarrow{C_S}$, then we would have that
\[
J \models (L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), S_1(v_1), \ldots, S_m(v_m))\theta
\]
However, this is equivalent to
\[
I \models (L_1(u_1), \ldots, L_{i-1}(u_{i-1}), R(u_i))\theta
\]
which we know is not true. Therefore at least one literal of $S_1(v_1), \ldots, S_m(v_m)$ must be a blocking atom.

Assume the blocking atom w.r.t. $e'$ in $\overrightarrow{C_S}$ is $S_k(v_k)$. Then, we have that
\[
J \not\models (L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), S_1(v_1), \ldots, S_k(v_k), S_j(v_j))\theta
\]
Let clause $\overrightarrow{C_S}$ be the result of removing blocking atoms w.r.t. $e'$ or atoms that are not head-connected in $S_1(v_1), \ldots, S_m(v_m)$ from $\overrightarrow{C_S}$. Because $\tau$ is a bijective transformation, at least one literal of $S_1(v_1), \ldots, S_m(v_m)$ must share join attributes with $S_k(v_k)$. Let $S_j(v_j)$, $1 \leq j \leq m$, $j \neq k$, be this literal. Then, inclusion dependency $S_k[A_k] = S_j[A_j]$ must be in $\Sigma_S$, where $A_k$ and $A_j$ are join attributes.

Because $S_k(v_k)$ is a blocking atom w.r.t. $e'$, the ARMG algorithm removes it from $\overrightarrow{C_S}$. After $S_k(v_k)$ is removed, there are three possible cases: i) $S_j(v_j)$ is not head-connected, ii) $S_j(v_j)$ is head-connected but not a blocking atom, or iii) $S_j(v_j)$ is a head-connected blocking atom.

If $S_j(v_j)$ is not head-connected, then, according to the ARMG algorithm, it would be removed from $\overrightarrow{C_S}$. Therefore, $S_j(v_j)$ would not be in $\overrightarrow{C_S}$. Now we will show that if $S_j(v_j)$ is head-connected, then it must be a blocking atom w.r.t. $e'$. Assume $S_j(v_j)$ is not a blocking atom w.r.t. $e'$, and $S_k(v_k)$ has already been removed from $\overrightarrow{C_S}$. Then, we would have that
\[
J \models (L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), S_j(v_j), S_k(v_k))\theta
\]
Because of inclusion dependency $S_k[A_k] = S_j[A_j]$, there must be a ground atom $S_\ell(t)$ in $J$, where $t$ is a tuple that shares join attributes with $S_j(v_j)\theta$. If this is the case, then $\overrightarrow{C_S}$ should contain a literal $S_k(v'_k)$ such that $S_k(v'_k)\theta = S_\ell(t)$, as $\overrightarrow{C_S}$ is a bottom clause. Also, $(S_j(v_j), S_k(v'_k))$ must be provable from $J$, as the bottom clause was generated relative to $J$ using inverse entailment, which is a complete logical system when the language is restricted to Datalog programs. Then, the following must hold
\[
J \models (L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), S_j(v_j), S_k(v_k))\theta
\]
Therefore, $S_k(v_k)$ is not a blocking atom. However, we know that $S_k(v_k)$ is different from $S_k(v_k)$ because $S_k(v_k)$ is a blocking atom. Let the result of joining $S_k(v'_k)$ and all literals in $S_j(v_j)$ and $\ldots, S_m(v_m)$ except $S_k(v_k)$ be literal $R(u'_i)$. This literal is different from the original literal $R(u_i)$, which was the blocking atom that we removed from $\overrightarrow{C_S}$. $R(u'_i)$ is also in $\overrightarrow{C_S}$ and appears after $R(u_i)$, as $R(u_i)$ is the first encountered blocking atom. Because $R(u_i)$ and $R(u'_i)$ are different, $S_j(v_j)$ must be a blocking atom w.r.t. $e'$ in $\overrightarrow{C_S}$ and must be removed by the ARMG algorithm.

Removing $S_j(v_j)$ from $\overrightarrow{C_S}$ would cause another literal in $S_1(v_1), \ldots, S_m(v_m)$ to be removed, and so on. Then, all literals $S_1(v_1), \ldots, S_m(v_m)$ would be removed from $\overrightarrow{C_S}$, resulting in clause $\overrightarrow{C_S}$ given by
\[
T(w) \leftarrow L'_1(u'_1), \ldots, L'_{i-1}(u'_{i-1}), L'_{i+1}(u'_{i+1}), \ldots, L'_n(u'_n).
\]
Therefore, $\overrightarrow{C_S}$ is equivalent to $\overrightarrow{C_R}$.

We now show that we also get equivalent results for transformation $\tau^{-1}$, which is a vertical composition. Assume the input to the ARMG algorithm under schema $\Sigma$ is the bottom clause $\perp_{i,j}$ and positive example $e'$. The bottom clause $\perp_{i,j}$ is the ordered clause $\overrightarrow{C_S}$, which was shown above. Then, $\overrightarrow{C_R} = chase_{\Sigma_R}(\delta_{\tau^{-1}}(\overrightarrow{C_S}))$ is given by
\[
T(w) \leftarrow L_1(u_1), \ldots, L_{i-1}(u_{i-1}), R(u_i), L_{i+1}(u_{i+1}), \ldots, L_n(u_n).
\]
Assume that $S(u_k)$, $1 \leq k \leq m$, is a blocking atom w.r.t. $e'$ in $\overrightarrow{C_S}$. We have shown that if this is the case, all literals $S_1(v_1), \ldots, S_m(v_m)$ would be removed from $\overrightarrow{C_S}$, resulting in clause $\overrightarrow{C_R}$. Because we have that $\tau^{-1}(S_1(v_1), \ldots, S_m(v_m)) = R(u_i)$, then $R(u_i)$ must also be a blocking atom w.r.t. $e'$ in $\overrightarrow{C_R}$. Then, it would be removed from $\overrightarrow{C_R}$, resulting in clause $\overrightarrow{C_R}$, which is equivalent to $\overrightarrow{C_S}$.

**Proof of Theorem 7.1**

**Proof.** Let $R$ and $S$ be two definition equivalent schemas. Schema $\mathcal{R} = (R, \Sigma)$ contains the single relation $R(A_1, \ldots, A_t)$. Assume that $t > 2$ and there are $l - 1$ functional dependencies $A_1 \rightarrow A_i$, $2 \leq i \leq l$, in $\Sigma$. Let $S = (S, \Omega)$ be a vertical decomposition of schema $\mathcal{R}$, such that relation $R(A_1, \ldots, A_t) \in \mathcal{R}$ is decomposed into $l - 1$ relations in $S$ in the form of $S(A_1, A_i), 2 \leq i \leq l$. For each relation $S(A_1, A_i) \in S$, $\Omega$ contains the functional dependency $A_1 \rightarrow A_i$. For each set of relations $S(A_1, A_i)$, $2 \leq i \leq l$, $\Omega$ also contains 2 $(l - 1)$ inclusion dependencies in the form of $S_2 \cdot A_1 \subseteq S_2 \cdot A_1$, and $S_2 \cdot A_2 \subseteq S_2 \cdot A_2$, $2 < j < l$. Let $p_S$ be the number of relations in schema $\mathcal{R}$, $a_S$ be the largest arity of any relation in $\mathcal{R}$, $k_S$ be the largest number of variables in a clause, and $m_S$ be the number of clauses in the definition of the target relation over $\mathcal{R}$. We define $p_S, a_S, k_S$, and $m_S$ analogously. The largest number of constants (i.e. objects) in any example is denoted by $n$. Parameter $n$ is a constraint on the answers of the oracle, therefore it is independent of the hypothesis space and the schemas. Because the number of relations in $\mathcal{R}$ is $p_S = 1$ and the maximum arity is $a = a_S$, then the maximum number of relations in $\mathcal{S}$ is $p_S = a = a_S$. We also have that $a_S = 2$.

Let $\mathcal{L}$ be the hypothesis language that consists of the subset of Horn definitions that contain a single clause in which no self-joins are allowed. All definitions in $\mathcal{L}$ under schema $\mathcal{R}$ have the form
\[
T(u) \leftarrow R(x_1, x_2, \ldots, x_k).
\]
where $T$ is the target relation and $u$ is a subset of $\{x_1, x_2, \ldots, x_k\}$. Any clause in a definition $H_R \in \mathcal{L}$ under schema $\mathcal{R}$ has at most $l$ distinct variables, which corresponds to the arity of relation $R$. Therefore $k_S = l$. As schema $\mathcal{S}$ is a vertical decomposition of schema $\mathcal{R}$, and no self-joins are allowed in $\mathcal{L}$, the definition $\delta(H_R) = H_S \in \mathcal{L}$ also has at most $l$ variables. We will use $k = k_S$ to denote the upper bound on $k_S$ and $k_S$. Because definitions in $\mathcal{L}$ consist of a single clause, then the maximum number of clauses in a definition $m = 1$. In general, $m_S = m_S$ because $\mathcal{S}$ is a vertical decomposition of $\mathcal{R}$.

The upper bound on the number of EQs and MQs in the $A2$ algorithm is $\Omega(m^2 pk^{a+3k} + nmpk^{a+1})$, and the lower bound is $\Omega(mpk^a)$ [23]. In order to prove our theorem, the following should hold for $\mathcal{R}$ and $\mathcal{S}$
\[
\Omega(mpk^a)^a > O(m^2 pk^{a+3k} + nmpk^{a+1})(k_S)^{2+k_S}
\]
where the left side of the inequality is the lower bound on the query complexity under schema \( R \) and the right side is the upper bound on the query complexity under schema \( S \). The operator \( > \) means that \( A2 \) will always ask asymptotically more queries under schema \( R \) than under schema \( S \). We have that \( k_R \) and \( k_S \) are bounded by \( k \) and \( m \) is the same for both schemas. We can also ignore \( n \) as it is independent of the hypothesis space and the schemas. Therefore, by canceling out some terms, the previous inequality can be rewritten as
\[
\Omega(k^n) > O(m(a - 1)k^{2+3k} + (a - 1)k^{2+k})
\]
The first term in the upper bound dominates the second term, then we have \( \Omega(k^n) > O(m(a - 1)k^{2+3k}) \). Assuming that \( m = 1 \), as in \( L \), we get \( \Omega(k^n) > O((a - 1)k^{2+3k}) \). This inequality holds for sufficiently large \( k \) and \( a \).

B. SUPPLEMENT EMPIRICAL SETTING AND RESULTS

Aleph and GILPS Configuration: In Aleph, we use the following configuration: noise = 100\%, minpos = 10, search = heuristic, evalfn = compression, nodes = 10000, clauselength = 5, and \( i = 2 \). We use the default values for the rest of the parameters except for openlist (beam), which we set to 1 to emulate FOIL and \( inf \) to emulate Prolog. In ProGolem, we use the default configuration, except for the following parameters: noise = 100\% and \( i = 1 \).