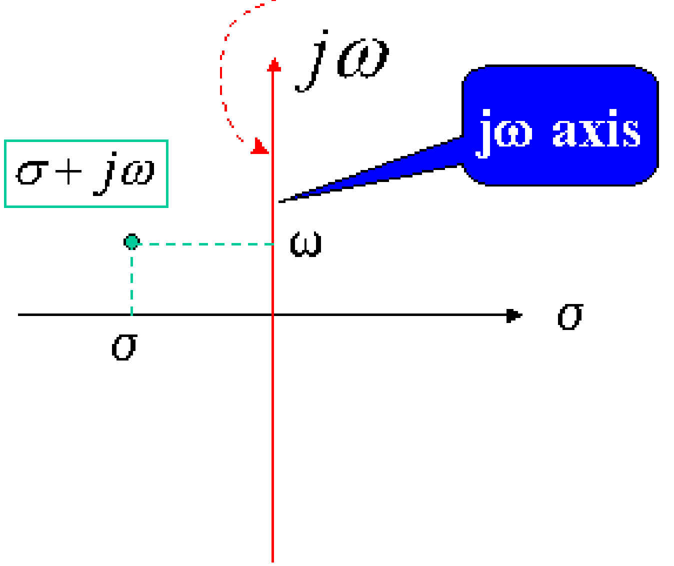
The Laplace Transform

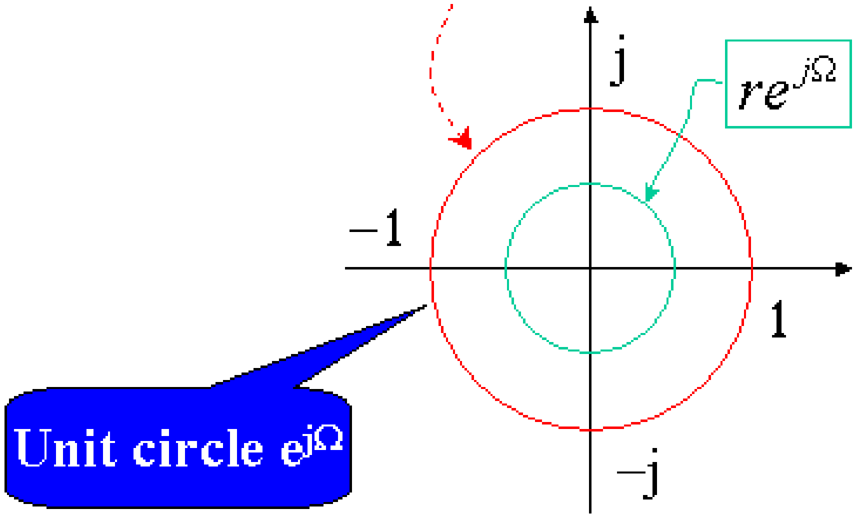
- Continuous-time signals

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$



- Discrete-time signals

$$x[n] \overset{DTFT}{\longleftrightarrow} X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$



- FT does not exist for signals that are not absolutely integrable.
- More general form: a transform as a function of an arbitrary point in the 2-dimensional plane: Laplace transform.

- DTFT does not exist for signals that are not absolutely summable.
- More general form: a transform as a function of an arbitrary circle in the 2-dimensional plane: z-transform.

Definition of the Laplace transform

- FT of $x(t)$: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$.
- Replacing $j\omega$ with a point in the 2-D complex plane $s = \sigma + j\omega$ (*complex freq.*):

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

Definition of the Laplace transform

- $X(s)$ is the Fourier transform of $x(t)e^{-\sigma t}$, a modified version of $x(t)$.
- $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
  is called *the Laplace Transform* of $x(t)$.
- The relationship is expressed with the notation:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s).$$

Inverse Laplace transform

- The inverse Fourier transform of $x(t)e^{-\sigma t}$ must be $X(\sigma + j\omega)$:

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

- Substituting $s = \sigma + j\omega$ and $d\omega = ds/j$, we get the inverse Laplace Transform:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Two types of Laplace Transforms

- Unilateral LT: $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$.
  - ⋆ Convenient for solving differential equations with initial conditions.
- Bilateral LT: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$.
  - ⋆ Offers insight into the nature of system characteristics such as stability, causality, and frequency response.

Topics to be covered in this chapter

- Properties and inversion of unilateral and bilateral Laplace Transform.
- Region of convergence (ROC).
- Solving differential equations with initial conditions.
- LT methods in circuit analysis.
- Transfer function.
- Causality and stability.
- Frequency response from poles and zeros.

Eigenfunction of an LTI system

- Let $x(t) = e^{st}$ be the input to an LTI system with impulse response $h(t)$.

$$\begin{aligned} y(t) &= ?h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \end{aligned}$$

⋆ *Transfer function:* $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$.

⋆ $y(t) = H(s)e^{st}$: e^{st} is an eigenfunction the LTI system, $H(s)$ is the corresponding eigenvalue.

Eigenfunction of an LTI system (cont.)

For input $x(t) = e^{(\sigma+j\omega)t}$

$$\begin{aligned} y(t) &= H(s)e^{st} = |H(s)|e^{j\phi(s)}e^{st} \\ &= |H(\sigma+j\omega)|e^{\sigma t}e^{j(\omega t+\phi(\sigma+j\omega))} \\ &= |H(\sigma+j\omega)|e^{\sigma t}cos(\omega t+\phi(\sigma+j\omega)) + \\ & \quad j|H(\sigma+j\omega)|e^{\sigma t}sin(\omega t+\phi(\sigma+j\omega)) \end{aligned}$$

- System changes input amplitude by $|H(\sigma+j\omega)|$.
- System changes input phase by $\phi(\sigma+j\omega)$.
- System does not change damping factor σ or input frequency ω.

Convergence

- A necessary condition: $x(t)e^{-\sigma t}$ is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty.$$

  - ⋆ Example: FT of $x(t) = e^t u(t)$ does not exist. ∎
  - ⋆ If $\sigma > 1$, $x(t)e^{-\sigma t} = e^{(1-\sigma)t}u(t)$ is absolutely integrable. The Laplace transform, which is the FT of $x(t)e^{-\sigma t}$ does exist.

Convergence (cont.)

- The range of σ for which the Laplace transform exists is termed the *region of convergence (ROC)*.

- Complex frequency s can be graphically represented in a complex plane, which is termed *the s-plane*. The $j\omega$-axis divides the plane into left and right half planes.

- *If* $x(t)$ *absolutely integrable*, FT can be obtained by setting $\sigma = 0$ as $X(j\omega) = X(s)|_{\sigma=0}$

Poles and zeros

The most commonly encountered form of Laplace transform:

$$
\begin{aligned}
X(s) &= \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0} \\
&= \frac{b_M \prod_{k=1}^{M} (s - c_k)}{\prod_{k=1}^{N} (s - d_k)}
\end{aligned}
$$

Poles and zeros (cont.)

- $\prod_{k=1}^{M}$: product of M terms.
- $\sum_{k=1}^{M}$: sum of M terms.
- c_k are the zeros of $X(s)$, will be denoted in the s-plane with “o” symbol.
- d_k are the poles of $X(s)$, will be denoted in the s-plane with “x” symbol.

Examples

- Read examples 6.1 (p487) and 6.2 (p488).

- Example 1: Problem 6.1(b), find $X(s)$ and ROC for $x(t) = e^{5t}u(-t+3)$.

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-t(s-5)}u(-t+3)dt \\
&= \int_{-\infty}^{3} e^{-t(s-5)}dt, \quad \text{let } l = -t \\
&= \int_{-3}^{\infty} e^{l(s-5)}dl, \quad \text{if } Re(s) < 5 \\
X(s) &= -\frac{e^{-3(s-5)}}{s-5}
\end{aligned}
$$

Examples

- Example 2: Problem 6.2(b), find $X(s)$ and ROC for $x(t) = sin(3t)u(t)$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} sin(3t)u(t)e^{-st}dt \\ &= \int_{0}^{\infty} \frac{1}{2j}\left(e^{j3t} - e^{-j3t}\right)e^{-st}dt, \\ &= \frac{1}{2j}\int_{0}^{\infty}\left[e^{-t(s-j3)} - e^{-t(s+j3)}\right]dt, \quad \text{if } Re(s) > 0, \\ &= \frac{1}{2j}\left(\frac{-1}{-(s-j3)} - \frac{-1}{-(s+j3)}\right) = \frac{3}{s^2+9} \end{aligned}$$