Definition of the Laplace transform

- Bilateral Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Unilateral (or one-sided) Laplace Transform:

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Definition of the Laplace transform (cont.)

- We denote this relationship as $x(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} X(s)$.
- The lower limit 0^- is to include discontinuity and impulse at $t = 0$.
- Determine the unilateral LT of $x(t) = u(t+3)$.

$$
\begin{aligned}
X(s) &= \int_{0^-}^{\infty} x(t)e^{-st}dt \\
&= \int_{0^-}^{\infty} u(t+3)e^{-st}dt = \int_{0^-}^{\infty} e^{-st}dt \\
&= 1/s
\end{aligned}
$$

Unilateral LT examples

- Determine the unilateral LT of $x(t) = u(t-3)$.

$$
\begin{aligned}
X(s) &= \int_{0^-}^{\infty} x(t)e^{-st}dt \\
&= \int_{0^-}^{\infty} u(t-3)e^{-st}dt = \int_{3}^{\infty} e^{-st}dt \\
&= e^{-3s}\frac{1}{s}
\end{aligned}
$$

Properties of the Unilateral Laplace Transform

- Linearity
  - ⋆ Let $x(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} X(s)$ and $y(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} Y(s)$.
  - ⋆ $ax(t) + by(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} aX(s) + bY(s)$.

Properties of the Unilateral Laplace Transform (cont.)

- Time scaling

⋆ $x(at) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right)$ for $a > 0$.

$$
\begin{aligned}
& \int_{0^-}^{\infty} x(at) e^{-st} dt \\
= \; & \int_{0^-}^{\infty} x(l) e^{-(s/a)l} \frac{1}{a} dl = \frac{1}{a} \int_{0^-}^{\infty} x(l) e^{-(s/a)l} dl \\
= \; & \frac{1}{a} X\left(\frac{s}{a}\right), \quad \text{for } \; a > 0
\end{aligned}
$$

Properties of the Unilateral Laplace Transform (cont.)

- Time shift

⋆ $x(t-\tau) \xleftrightarrow{\mathcal{L}_u} e^{-s\tau}X(s)$ for $\tau > 0$. ∎

$$\begin{aligned} & \int_{0^-}^{\infty} x(t-\tau)e^{-st}dt \\ = & \int_{0^-}^{\infty} x(l)e^{-sl}e^{-s\tau}dl \\ = & \; e^{-s\tau}X(s), \quad \text{for} \;\; \tau > 0 \end{aligned}$$

Properties of the Unilateral Laplace Transform (cont.)

- s-domain shift

⋆ $e^{s_0 t}x(t) \overset{\mathcal{L}_u}{\longleftrightarrow} X(s-s_0)$.

$$\begin{aligned} & \int_{0^-}^{\infty} e^{s_0 t}x(t)e^{-st}dt \\ = & \int_{0^-}^{\infty} x(t)e^{-(s-s_0)t}dt \\ = & X(s-s_0) \end{aligned}$$

⋆ Multiplication by a complex exponential in time introduces a shift in complex frequency s.

Properties of the Unilateral Laplace Transform (cont.)

- Convolution

  ⋆ $x(t) * y(t) \overset{\mathcal{L}_u}{\longleftrightarrow} X(s)Y(s)$ when $x(t) = 0, y(t) = 0$ for $t < 0$. ∎

- Differentiation in the s-domain

  ⋆ $-tx(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{d}{ds}X(s)$. ∎

Examples

- Determine the unilateral LT of $x(t) = (-e^{3t}u(t)) * (tu(t))$

  ⋆ $u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{s}$. ▮

  ⋆ $-e^{3t}u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{-1}{s-3}$. ▮

  ⋆ $tu(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{s^2}$. ▮

  ⋆ Thus, $(-e^{3t}u(t)) * (tu(t)) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{-1}{s^2(s-3)}$.

Examples (cont.)

- Example 6.4, p_{493}. see figure 6.7 on the same page. Find the LT of the system output $y(t)$ for the input $x(t) = te^{2t}u(t)$.

  ⋆ Applying $x(t) = u(t) \Rightarrow y(t) = (1 - e^{-t/(RC)})u(t)$. If $x(t) = \delta(t) = \frac{d}{dt}u(t) \Rightarrow$

$$
\begin{aligned}
y(t) &= h(t) = \frac{d}{dt}\left[(1 - e^{-t/(RC)})u(t)\right] \\
&= \frac{1}{RC}e^{-t/(RC)}u(t).
\end{aligned}
$$

Examples (cont.)

- With $RC = 0.2$, $h(t) = 5e^{-5t}u(t)$
- $h(t) = 5e^{-5t}u(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{5}{s+5}$
- $x(t) = te^{2t}u(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{(s-2)^2}$.
- Thus, $Y(s) = X(s)H(s) = \frac{5}{(s-2)^2(s+5)}$.

Properties – Differentiation in the time domain

Suppose that LT of $x(t)$ exists. Find the unilateral Laplace Transform of $dx(t)/dt$.

$$\frac{d}{dt}x(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \int_{0^-}^{\infty} \left(\frac{d}{dt}x(t)\right) e^{-st} dt$$

$$\frac{d}{dt}x(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st} dt$$

If $X(s)$ exists, then $x(t)e^{-st} \to 0$ as $t \to \infty$. Thus,

$$\frac{d}{dt}x(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} sX(s) - x(0^-).$$

Properties – Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{x^{(-1)}(0^-)}{s} + \frac{X(s}{s}$$

where

$$x^{(-1)}(0^-) = \int_{-\infty}^{0^-} x(\tau)d\tau$$

is the area under $x(t)$ from $t = -\infty$ to $t = 0^-$.

Example – Problem 6.5, p_{495}

Determine the unilateral LT of $tu(t)$.

- Method 1:

$$u(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{s}$$

$$-tu(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{d}{ds}\left(\frac{1}{s}\right)$$

$$tu(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{s^2}$$

Example – Problem 6.5, p_{495} (cont.)

- Method 2:

$$tu(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

$$\int_{-\infty}^{t} u(\tau)d\tau \xleftrightarrow{\mathcal{L}_u} \frac{1}{s}\left(U(s) + \int_{-\infty}^{0^-} u(\tau)d\tau\right) = \frac{1}{s^2}$$

Properties – Initial- and final-value theorems

- Initial value: $\lim_{s \to \infty} sX(s) = x(0^+)$
- The initial-value theorem does not apply to rational functions $X(s)$ in which the order of the numerator polynomial is greater than or equal to that of the denominator polynomial order.
- Final value: $\lim_{s \to 0} sX(s) = x(\infty)$
- The final-value theorem applies only if all the poles of $X(s)$ are in the left half of the s-plane, with at most a single pole at $s = 0$.

Example – example 6.6, p_{495}

Determine $x(0^+)$ and $x(\infty)$, if the unilateral Laplace Transform of $x(t)$ is given as

$$X(s) = \frac{7s+10}{s(s+2)}$$

- $x(0^+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{7s+10}{s+2} = 7$
- $x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{7s+10}{s+2} = 5$

Example – Problem 6.6(a), p_{496}

Determine the initial and final values of $x(t)$ corresponding to the unilateral Laplace Transform

$$X(s) = e^{-5s}\left(\frac{-2}{s(s+2)}\right)$$

- $x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{-2}{(s+2)e^{5s}} = 0$
- $x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{-2}{(s+2)e^{5s}} = -1$

Inversion of the unilateral Laplace Transform

- Through definition:

$$
\begin{aligned}
x(t)e^{-\sigma t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega
\end{aligned}
$$

Substituting $s = \sigma + j\omega$ and $d\omega = ds/j$, we get the inverse Laplace Transform:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Inversion of the unilateral Laplace Transform (cont.)

- It requires contour integration.
- Difficult to do in general.
- Focus on rational functions
  - ⋆ Rational function: a ratio of polynomials in s, e.g.

$$X(s) = \frac{B(s)}{A(s)} = \frac{3s+4}{(s+1)(s+2)}.$$

  - ⋆ Proper function: order of $B(s) \leq$ order of $A(s)$.

Inversion of the unilateral Laplace Transform (cont.)

- Many commonly encountered LTI systems can be described by integro-differential equations.
- Their transfer function can be expressed as the ratio of two polynomials in s.

Inversion of the unilateral Laplace Transform (cont.)

- Focus: partial fraction expansion applied on rational Laplace Transform.

  ⋆ Requires knowledge of several basic transform pairs, e.g.,

$$\delta(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} 1$$

$$c_k\delta^k(t) = c_k\frac{d^k}{dt^k}\delta(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} c_k s^k$$

$$A_k e^{d_k t}u(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{A_k}{s - d_k}$$

$$\frac{At^{n-1}}{(n-1)!}e^{d_k t}u(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{A}{(s - d_k)^n}$$

Inversion of the unilateral Laplace Transform (cont.)

- Partial fraction expansion:

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

If $M \geq N$:

$$\begin{aligned} X(s) &= \sum_{k=0}^{M-N} c_k s^k + \tilde{X}(s) \\ \tilde{X}(s) &= \frac{\tilde{B}(s)}{\tilde{A}(s)} \end{aligned}$$

Inversion of the unilateral Laplace Transform (cont.)

- $A(s) = \prod_{k=1}^{N}(s - d_k)$. Poles: $s = d_k$
- If all poles are distinct: $\tilde{X}(s) = \sum_{k=1}^{N} \frac{A_k}{s - d_k}$
- If a pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole:

$$\frac{A_{i_1}}{s - d_i}, \; \frac{A_{i_2}}{(s - d_i)^2}, \; \cdots, \; \frac{A_{i_r}}{(s - d_i)^r}$$

Examples

Example 6.7, p_{497}: find $x(t)$ for:

$$X(s) = \frac{3s+4}{(s+1)(s+2)^2}$$

- $X(s)$ is a proper rational function.
- Poles at $s = -1$ and $s = -2$ (double pole).
- Partial fraction expansion of $X(s)$:

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{A_3}{(s+2)^2}.$$

Examples (cont.)

$$\begin{aligned} A_1 &= X(s)(s+1)\big|_{s=-1} = 1 \\ A_2 &= \frac{d}{ds}\left[X(s)(s+2)^2\right]\bigg|_{s=-2} = -1 \\ A_3 &= X(s)(s+2)^2\big|_{s=-2} = 2 \end{aligned}$$

Thus,

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{2}{(s+2)^2}.$$

Examples (cont.)

Pole at -1: $e^{-t}u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{s+1}$

Pole at -2: $-e^{-2t}u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} -\frac{1}{s+2}$

Double pole at -2: $2te^{-2t}u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \frac{2}{(s+2)^2}$

Thus,

$$x(t) = \left(e^{-t} - e^{-2t} + 2te^{-2t}\right) u(t).$$

Examples (cont.)

- Read Example 6.8, p_{498}
- Problem 6.7(c), p_{498}: find $x(t)$ given $X(s)$ as follows.

$$
\begin{aligned}
X(s) &= \frac{s^2+s-3}{s^2+3s+2} \\
&= 1+\frac{-2s-5}{s^2+3s+2} = 1+\frac{-2s-5}{(s+1)(s+2)} \\
&= 1+\frac{A_1}{s+1}+\frac{A_2}{s+2}
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= X(s)(s+1)\big|_{s=-1} = -3 \\
A_2 &= X(s)(s+2)\big|_{s=-2} = 1 \\
x(t) &= \delta(t) - 3e^{-t}u(t) + e^{-2t}u(t)
\end{aligned}
$$

Inversion of the unilateral LT: complex pole pair

Suppose that $\alpha + j\omega_0$ and $\alpha - j\omega_0$ make up a pair of complex-conjugate poles. The partial-fraction expansion is written as

$$
\begin{aligned}
& \frac{A_1}{s-\alpha-j\omega_0} + \frac{A_2}{s-\alpha+j\omega_0} \overset{\mathcal{L}_u}{\longleftrightarrow} \\
& A_1 e^{\alpha t} e^{j\omega_0 t} u(t) + A_2 e^{\alpha t} e^{-j\omega_0 t} u(t) \\
= \; & e^{\alpha t} u(t) \left(A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t} \right) \\
= \; & e^{\alpha t} u(t) \left((A_1 + A_2) cos(\omega_0 t) + j(A_1 - A_2) sin(\omega_0 t) \right) \\
= \; & C_1 e^{\alpha t} cos(\omega_0 t) u(t) + C_2 e^{\alpha t} sin(\omega_0 t)
\end{aligned}
$$

Inversion of the unilateral LT: complex pole pair (cont.)

where

$$
\begin{aligned}
C_1 &= A_1 + A_2 \\
C_2 &= j(A_1 - A_2)
\end{aligned}
$$

- Read Example 6.9, p_{500}.
- Problem 6.8(a), p_{500}: Find $x(t)$ of

$$X(s) = \frac{3s+2}{s^2+4s+5}$$

Example

- Poles at: $s_1 = -2 + j1,\ s_2 = -2 - j1\ (\alpha = -2,\ \omega_0 = 1)$.
- $X(s)$ expressed as

$$
\begin{aligned}
X(s) &= \frac{A_1}{s+2-j} + \frac{A_2}{s+2+j} \\
A_1 &= \left.\frac{3s+2}{s+2+j}\right|_{s=-2+j} = \frac{3}{2} + j2 \\
A_2 &= \left.\frac{3s+2}{s+2-j}\right|_{s=-2-j} = \frac{3}{2} - j2 \\
C_1 &= 3 (= A_1 + A_2) \\
C_2 &= -4 (= j(A_1 - A_2)) \\
x(t) &= C_1 e^{\alpha t} cos(\omega_0 t) u(t) + C_2 e^{\alpha t} sin(\omega_0 t) u(t) \\
&= 3e^{-2t} cos(t) u(t) - 4e^{-2t} sin(t) u(t)
\end{aligned}
$$

Solving differential equations with initial conditions

- Primary application of unilateral Laplace transform in systems analysis: solving differential equations with initial conditions.

- Initial conditions are incorporated into the solutions as the values of the signal and its derivatives that occur at time zero in the differentiation property.