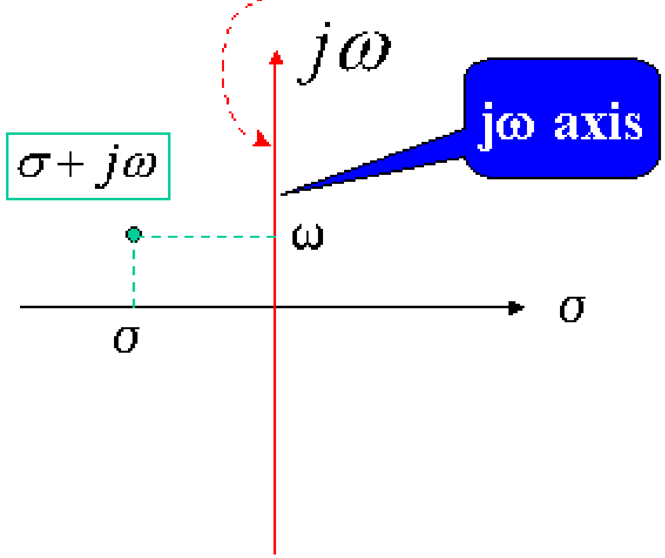
Chapter 7: The z-Transform

- Continuous-time signals

$$x(t) \xleftrightarrow{FT} X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

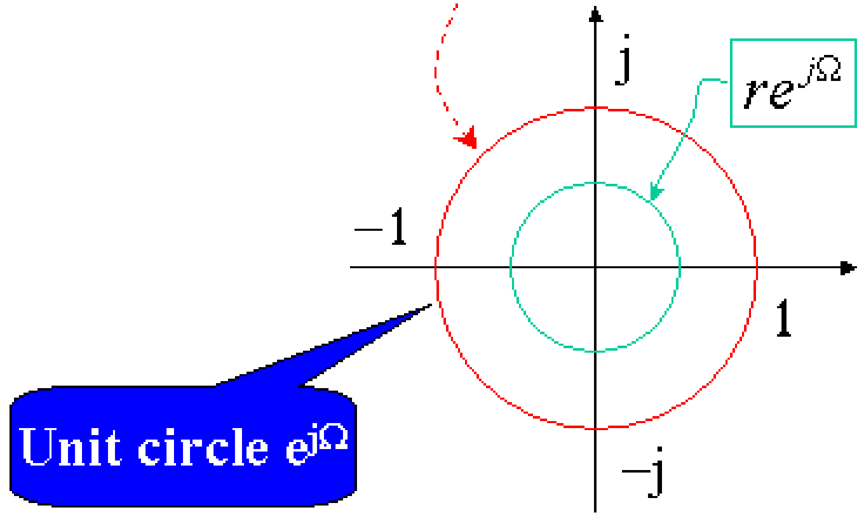


- FT does not exist for signals that are not absolutely integrable.

- More general form: a transform as a function of an arbitrary point in the 2-dimensional plane: Laplace transform.

- Discrete-time signals

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$



- DTFT does not exist for signals that are not absolutely summable.

- More general form: a transform as a function of an arbitrary circle in the 2-dimensional plane: z-transform.

The z-Transform - definition

- Continuous-time systems: $e^{st} \longrightarrow H(s) \Rightarrow y(t) = e^{st}H(s)$
  - ⋆ e^{st} is an eigenfunction of the LTI system $h(t)$, and $H(s)$ is the corresponding eigenvalue.

- Discrete-time systems: $x[n] = z^n \longrightarrow h[n] \Rightarrow y[n]$

$$
\begin{aligned}
y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) = z^n H(z)
\end{aligned}
$$

  - ⋆ z^n is an eigenfunction of the LTI system $h[n]$, and $H(z)$ is the corresponding eigenvalue.

The z-Transform - definition (cont.)

The *transfer function:*

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}.$$

Generally, let $z = re^{j\Omega}$. Then,

$$H(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(h[n] z^{-n} \right) e^{-j\Omega n}.$$

Thus, $H(z)$ is the DTFT of $h[n]r^{-n}$. The inverse DTFT of $H(re^{j\Omega})$ must be $h[n]r^{-n}$.

The z-Transform - definition (cont.)

So we may write

$$h[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega.$$

- $z = re^{j\Omega} \rightarrow dz = jre^{j\Omega}d\Omega$. $d\Omega = \frac{1}{j}z^{-1}dz$.
- As Ω goes from $-\pi$ to π, z traverses a circle of radius r in a counterclockwise direction. Thus, we may write

$$h[n] = \frac{1}{2\pi j}\oint H(z)z^{n-1}dz$$

The z-Transform - definition (cont.)

For an arbitrary signal $x[n]$, the z-transform and inverse z-transform are expressed as

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ x[n] &= \frac{1}{2\pi j} \oint X(z) z^{n-1} dz \end{aligned}$$

We express this relationship between $x[n]$ and $X(z)$ as

$$x[n] \overset{z}{\longleftrightarrow} X(z)$$

The z-Transform - convergence

- A necessary condition for convergence: $\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$ (absolute summability)
- The range of r for which this condition is satisfied is termed the *region of convergence (ROC)*.
- Complex number z is represented as a location in a complex plane, termed the *z-plane*.
- If $x[n]$ is absolutely summable, then the DTFT of $x[n]$ is obtained as

$$X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}}$$

- The contour $z = e^{j\Omega}$ is termed the *unit circle*.

The z-Transform - poles and zeros

The most commonly encountered form of the z-transform is a ratio of two polynomials in z^{-1}, as shown by the *rational function*

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- $\tilde{b} = b_0 / a_0$.
- c_k: *zeros of* $X(z)$. Denoted with the "$\circ$" symbol in the z plane.
- d_k: *poles of* $X(z)$. Denoted with the "$\times$" symbol in the z plane.

The z-Transform - Review of commonly used series

- Geometric series: Let $s_n = a + ar + ar^2 + \cdots + ar^n$, then

$$\begin{aligned} s_n &= \frac{a(1-r^{n+1})}{1-r} \\ \lim_{n\to\infty} s_n &= \frac{a}{1-r}, \quad \text{if } |r| < 1 \end{aligned}$$

Proof:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n \\ rs_n &= \qquad ar + ar^2 + \cdots + ar^n + ar^{n+1} \\ s_n - rs_n &= a(1-r^{n+1}) \\ s_n &= \frac{a(1-r^{n+1})}{1-r} \end{aligned}$$

The z-Transform - Review of commonly used series (cont.)

- Arithmetic progression: Let
  $s_n = a + (a + r) + (a + 2r) + \cdots + (a + nr)$, then

$$
\begin{aligned}
s_n &= \frac{(n+1)(a+(a+nr))}{2} \\
\lim_{n\to\infty} s_n &= \infty, \quad \text{if } \; r > 0 \\
\lim_{n\to\infty} s_n &= -\infty, \quad \text{if } \; r < 0
\end{aligned}
$$

Proof:

$$
\begin{aligned}
s_n &= a + (a + r) + (a + 2r) + \cdots + (a + nr) \\
s_n &= (a + nr) + (a + (n-1)r) + \cdots + (a + r) + a \\
2s_n &= (n+1)(a + (a + nr)) \rightarrow s_n = (n+1)(a + (a + nr))/2
\end{aligned}
$$

The z-Transform - Review of commonly used series (cont.)

- $$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$
- $$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$
- $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
- $$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}$$

The z-Transform - Convergence of commonly used series

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 0$:
  - ⋆ Convergent, if $p > 1$
  - ⋆ Divergent, if $p \leq 1$.

Example:

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$: divergent
- $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$: convergent

The z-Transform - Convergence of commonly used series (cont.)

- Ratio test: Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r$.
  - ⋆ $r > 1$: divergent
  - ⋆ $r < 1$: convergent
  - ⋆ $r = 1$: test gives no information

- Comparison test: Assume $0 \leq a_n \leq b_n, \ \ \forall n$.
  - ⋆ If $\sum b_n$ is convergent $\Rightarrow \sum a_n$ is convergent (For convenience, we use $\sum b_n$ to represent an infinite series in the notes)

Example: Let $a_n = \frac{2n}{3n^3 - 1}, \ \ b_n = \frac{1}{n^2}$.

$\sum b_n$ is convergent. Thus, $\sum a_n$ is convergent because $n \geq 1 \to n^3 \geq 1 \to 3n^3 - 1 \geq 2n^3 \Rightarrow a_n \leq b_n$.

The z-Transform - Convergence of commonly used series (cont.)

- Corollary of comparison test (limiting form): Suppose that $a_n > 0,\ b_n > 0$ and then $\lim_{n\to\infty} \frac{a_n}{b_n} = k > 0$.

$$\sum a_n \ \text{convergent} \overset{iff}{\longleftrightarrow} \sum b_n \ \text{convergent}$$

Example: Let $a_n = \frac{n}{n^2+1},\quad b_n = \frac{1}{n}$.
$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$.
Because $\sum b_n$ is divergent $\Rightarrow \sum a_n$ is divergent.

The z-Transform - Convergence of commonly used series (cont.)

- Necessary condition for convergence of $\sum a_n$:

$$\lim_{n \to \infty} = a_n = 0$$

- It is not a sufficient condition. For example, $a_n = \frac{1}{n}$, $\sum a_n$ is divergent.

The z-Transform - Examples

Determine the z-transform of the following signals and depict the ROC and the locations of the poles and zeros of $X(z)$ in the z-plane:

- $x[n] = \alpha^n u[n]$ (causal signal)
- $x[n] = -\alpha^n u[-n-1]$ (anticausal signal)

For signal $x[n] = \alpha^n u[n]$:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n . \end{aligned}$$

The z-Transform - Examples (cont.)

This infinite series converges to

$$X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}, \quad \text{for} \quad |z| > |\alpha|.$$

For signal $x[n] = -\alpha^n u[-n-1]$:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(-\alpha^n u[-n-1] z^{-n}\right) \\
&= -\sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n \\
&= -\sum_{k=-1}^{-\infty} \left(\frac{\alpha}{z}\right)^k = 1 - \sum_{k=0}^{-\infty} \left(\frac{\alpha}{z}\right)^k \\
&= 1 - \frac{1}{1-z\alpha^{-1}} = \frac{z}{z-\alpha}, \quad \text{for} \quad |z| < |\alpha|.
\end{aligned}$$

The z-Transform - Examples (cont.)

Observations:

- As bilateral Laplace transform, the relationship between $x[n]$ and $X(z)$ is not unique.
- The ROC differentiates the two transforms.
- We must know the ROC to determine the correct inverse z-transform.

The z-Transform - Examples (cont.)

- Read Example 7.4, p_{560}.

- Problem 7.1(c), p_{561}: Determine the z-transform, the ROC, and the locations of poles and zeros of $X(z)$ for the following signal

$$x[n] = -\left(\frac{3}{4}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

Using the results given in the previous two slides:

$$-\left(\frac{3}{4}\right)^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad \frac{z}{z-3/4}$$

$$\left(-\frac{1}{3}\right)^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{z}{z+1/3}.$$

Thus, $X(z) = \dfrac{z}{z-3/4} + \dfrac{z}{z+1/3} = \dfrac{z(2z-5/12)}{(z-3/4)(z+1/3)}$

Properties of the ROC

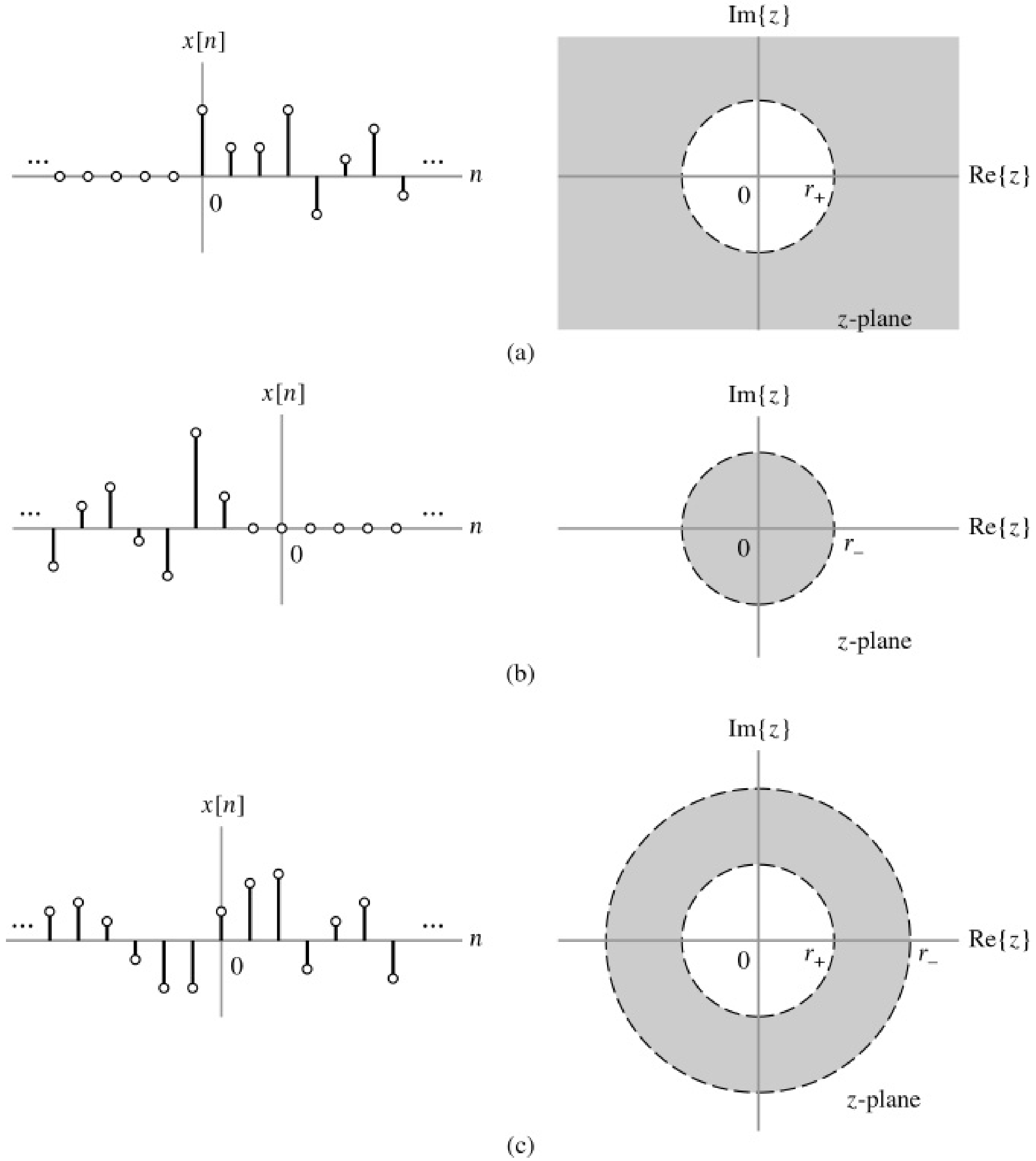
- As the Laplace transform, the ROC cannot contain any poles.
- ROC for a finite-duration signal includes the entire z-plane, except possibly $z = 0$ or $z = \infty$ or both.
- Left-sided sequence: $x[n] = 0$ for $n \geq 0$ (notice the difference between the left-sided signal for Laplace transform).
- Right-sided sequence: $x[n] = 0$ for $n < 0$
- Two-sided sequence: a signal that has infinite duration in both the positive and negative directions.

Properties of the ROC

- RSS: ROC is of the form $|z| > r_+$ ▮
- LSS: ROC is of the form $|z| < r_-$ ▮
- TSS: ROC if of the form $r_+ < |z| < r_-$

where the boundaries r_+ and r_- are determined by the pole locations. See the figure next page.

Properties of the ROC (cont.)



Properties of the ROC - Examples

Example 7.5, identify the ROC associated with the z-transform for each of the following signals.

- $x[n] = (-1/2)^n u[-n] + 2(1/4)^n u[n]$
- $y[n] = (-1/2)^n u[n] + 2(1/4)^n u[n]$
- $w[n] = (-1/2)^n u[-n] + 2(1/4)^n u[-n]$

Properties of the ROC - Examples (cont.)

For $x[n]$, the z-transform is written as

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{0} \left(\frac{-1}{2z}\right)^{n} + 2\sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^{n} \\
&= \sum_{k=0}^{\infty} (-2z)^{k} + 2\sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^{n}
\end{aligned}
$$

- The first sum converges for $|z| < \frac{1}{2}$.
- The second sum converges for $|z| > \frac{1}{4}$.
- Thus, the ROC is $\frac{1}{4} < z < \frac{1}{2}$. Summing the two geometric series:

$$X(z) = \frac{1}{1 + z2} + \frac{2z}{z - 1/4}.$$

Properties of the ROC - Examples (cont.)

Observations:

- The first term on the right side of $z[n]$ is a left-sided sequence. Its ROC is $|z| < r_-$, where r_- is determined by its pole location.
- The second term on the right side of $z[n]$ is a right-sided sequence. Its ROC is $|z| > r_+$, where r_+ is determined by its pole location.

Properties of the ROC - Examples (cont.)

For $y[n]$, both terms are right-sided sequences. Thus, the ROC is $|z| > r_+$, where r_+ is determined by the pole locations.

$$Y(z) = \sum_{n=0}^{\infty} \left(\frac{-1}{2z}\right)^n + 2\sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n$$

The first series converges for $|z| > 1/2$ and the second series converges for $|z| > 1/4$. Thus, the ROC is $|z| > 1/2$, and we write $Y(z)$ as

$$Y(z) = \frac{z}{z + 1/2} + \frac{2z}{z - 1/4}.$$

Properties of the ROC - Examples (cont.)

For $w[n]$, both terms are left-sided sequences. Thus, the ROC is $|z| < r_-$, where r_- is determined by the pole locations.

$$
\begin{aligned}
W(z) &= \sum_{n=-\infty}^{0} \left(\frac{-1}{2z}\right)^n + 2 \sum_{n=-\infty}^{0} \left(\frac{1}{4z}\right)^n \\
&= \sum_{k=0}^{\infty} (-2z)^k + 2 \sum_{k=0}^{\infty} (4z)^k
\end{aligned}
$$

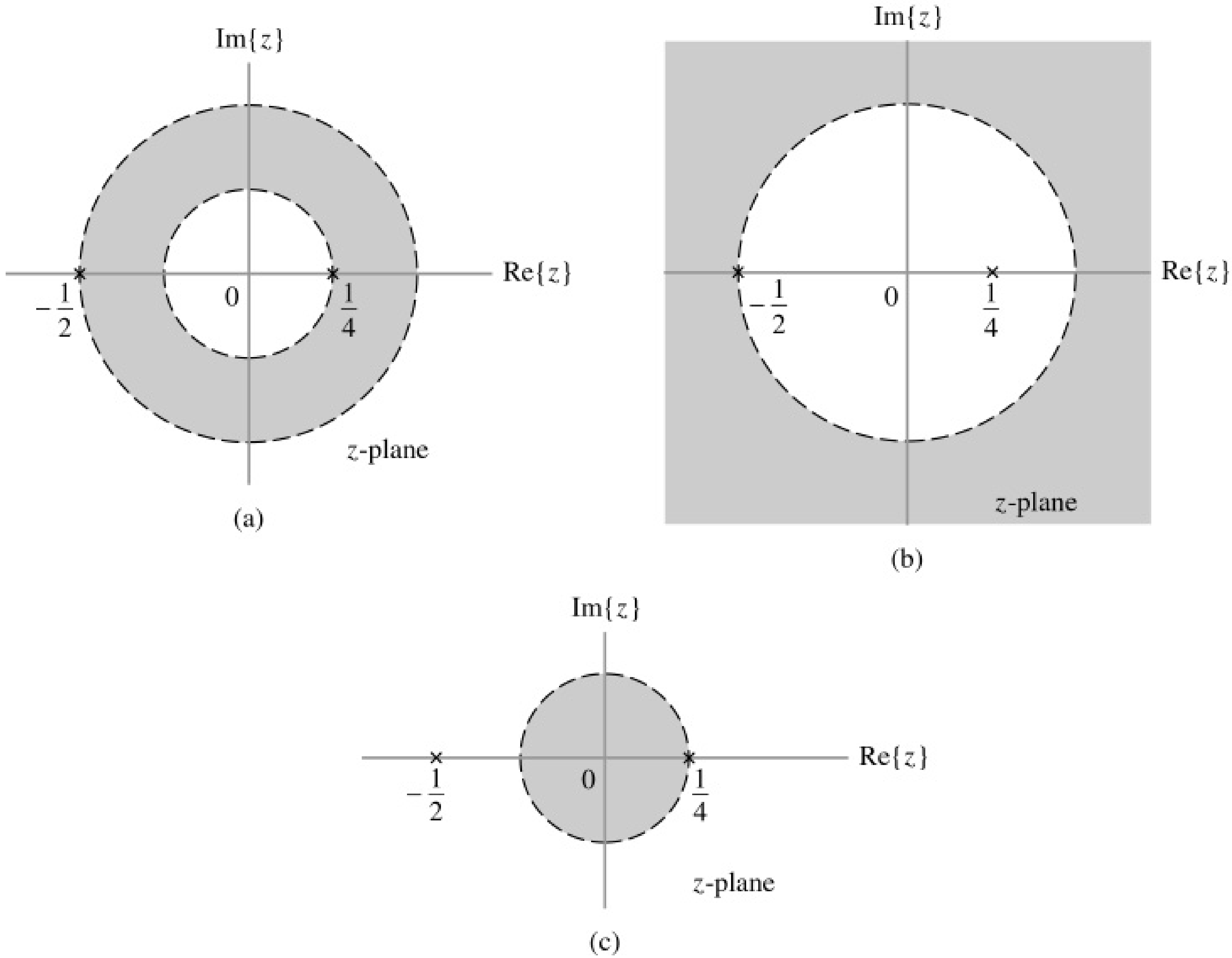
Properties of the ROC - Examples (cont.)

- The first series converges for $|z| < 1/2$.
- The second series converges for $|z| < 1/4$.
- Thus, the ROC is $|z| < 1/4$, and we write $W(z)$ as

$$W(z) = \frac{1}{1+2z} + \frac{2}{1-4z}.$$

The pole locations of sequences $z[n],\ y[n],\ w[n]$ are shown in the figure next slide.

Properties of the ROC (cont.)



Properties of the z-transform

- Linearity
  - ⋆ Let $x[n] \stackrel{z}{\longleftrightarrow} X(z)$ (ROC R_x) and $y[n] \stackrel{z}{\longleftrightarrow} Y(z)$.
  - ⋆ $ax[n] + by[n] \stackrel{z}{\longleftrightarrow} aX(z) + bY(z)$, with ROC at least $R_x \bigcap R_y$
    - ∗ The ROC can be larger than the intersection if one or more terms in $x[n]$ or $y[n]$ cancel each other in the sum.
    - ∗ In the z-plane, this corresponds to a zero canceling a pole that defines one of the ROC boundaries.

Properties of the z-transform (cont.)

Example: Example 7.5, p_{567}. Suppose

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{3}{2}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} X(z) = \frac{-z}{(z-1/2)(z-3/2)}$$

with ROC $1/2 < |z| < 3/2$, and

$$y[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} Y(z) = \frac{-\frac{1}{4}z}{(z-1/4)(z-1/2)}$$

with ROC $|z| > 1/2$. Evaluate the z-transform of $ax[n] + by[n]$, where a and b are constants.

Properties of the z-transform (cont.)

Using the linearity property, we have

$$ax[n] + by[n] \overset{z}{\longleftrightarrow} a\frac{-z}{(z-1/2)(z-3/2)} + b\frac{-\frac{1}{4}z}{(z-1/4)(z-1/2)}$$

Must be careful in determining ROC. In general, the ROC is the intersection of individual ROCs. For some special cases, however, the ROC could be larger. For instance, let $a = b = 1$.

Properties of the z-transform (cont.)

Then,

$$
\begin{aligned}
aX(z) + bY(z) &= \frac{-z}{(z-1/2)(z-3/2)} + \frac{-\frac{1}{4}z}{(z-1/4)(z-1/2)} \\
&= \frac{-\frac{5}{4}z(z-1/2)}{(z-1/4)(z-1/2)(z-3/2)} \\
&= \frac{-\frac{5}{4}z}{(z-1/4)(z-3/2)}
\end{aligned}
$$

The ROC can be verified to be $1/4 < |z| < 3/2$ because the pole-zero cancellation ($z = 1/2$), and the $(1/2)^n u[n]$ no longer presents.

Properties of the z-transform (cont.)

- Time reversal
  - $\star$ $x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right)$ with ROC $\frac{1}{R_x}$.
  - $\star$ If R_x is of the form $a < |z| < b$, the ROC of the reflected signal is $1/b < |z| < 1/a$.

- Time shift
  - $\star$ $x[n - n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z)$ with ROC R_x, except possibly $z = 0$ and $z = \infty$.
  - $\star$ If $n_0 > 0$, z^{-n_0} introduces a pole $z = 0$.
  - $\star$ If $n_0 < 0$, z^{-n_0} introduces a pole $z = \pm\infty$.

Properties of the z-transform (cont.)

- Multiplication by an exponential sequence
  - ⋆ $\alpha^n x[n] \stackrel{z}{\longleftrightarrow} X\left(\frac{z}{\alpha}\right)$ with ROC $|\alpha| R_x$.
  - ⋆ $|\alpha| R_x$ implies that the ROC boundaries are multiplied by $|\alpha|$.
  - ⋆ If $|\alpha| = 1$, then the ROC is unchanged.

- Convolution
  - ⋆ $x[n] * y[n] \stackrel{z}{\longleftrightarrow} X(z)Y(z)$ with ROC at least $R_x \bigcap R_y$.
  - ⋆ The ROC may be larger than the intersection of R_x and R_y if a pole-zero cancellation occurs in the product of $X(z)Y(z)$.

Properties of the z-transform (cont.)

- Differentiation in the z-domain
  - ⋆ $nx[n] \stackrel{z}{\longleftrightarrow} -z\frac{d}{dz}X(z)$, with ROC R_x.
- Read Example 7.8, p_{570}.

Example: Example 7.7, p_{570}. Find the z-transform of

$$x[n] = \left(n\left(\frac{-1}{2}\right)^n u[n]\right) * \left(\frac{1}{4}\right)^{-n} u[-n].$$

Properties of the z-transform: example

- Basic signal of first term: $\left(\frac{-1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z+1/2}$, with ROC $|z| > 1/2$.

- Applying the z-domain differentiation property:

$$
\begin{aligned}
n\left(\frac{-1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} & \quad -z\frac{d}{dz}\frac{z}{z+1/2} \\
= & \quad \frac{-\frac{1}{2}z}{(z+1/2)^2}, \text{ with ROC } |z| > 1/2
\end{aligned}
$$

Properties of the z-transform: example (cont.)

- Applying time reversal property: $\left(\frac{1}{4}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-1/4}$, with ROC $|z| > 1/4$. Thus, $\left(\frac{1}{4}\right)^{-n} u[-n] \overset{1/z}{\longleftrightarrow} \frac{1/z}{1/z-1/4} = \frac{-4}{z-4}$ with ROC $|z| < 4$

- Applying convolution property: $x[n] \overset{z}{\longleftrightarrow} \frac{2z}{(z-4)(z+1/2)^2}$, with ROC $\frac{1}{2} < |z| < 4$.