Inversion of the z-Transform

- Focus on rational z-transform of z^{-1}.
- Apply partial fraction expansion.
- Like bilateral Laplace transforms, ROC must be used to determine a unique inverse z-transform.

Let

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

and assume that $M < N$.

Inversion of the z-Transform (cont.)

If $M \geq N$:

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $\tilde{B}(z)$ has order one less than the denominator polynomial.

- Partial fraction expansion is obtained by factoring the denominator polynomial into a product of first-order terms.

$$\begin{aligned} X(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})} \\ &= \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \quad \text{if all poles } d_k \text{ are distinct} \end{aligned}$$

Inversion of the z-Transform (cont.)

- $A_k(d_k)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}}$, with ROC $|z| > d_k$.

- $-A_k(d_k)^n u(-n-1) \stackrel{z}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}}$, with ROC $|z| < d_k$.

If a pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole:

$$\frac{A_{i_1}}{1-d_i z^{-1}},\ \frac{A_{i_2}}{(1-d_i z^{-1})^2},\ \cdots,\ \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

Inversion of the z-Transform (cont.)

- $A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_i)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}$, with ROC $|z| > d_i$.
- $-A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}$, with ROC $|z| < d_i$.
- ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial-fraction expansion.

Inversion of the z-Transform: Examples

Example 7.9, p_{574}: find the inverse z-transform of

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 - z^{-1})}$$

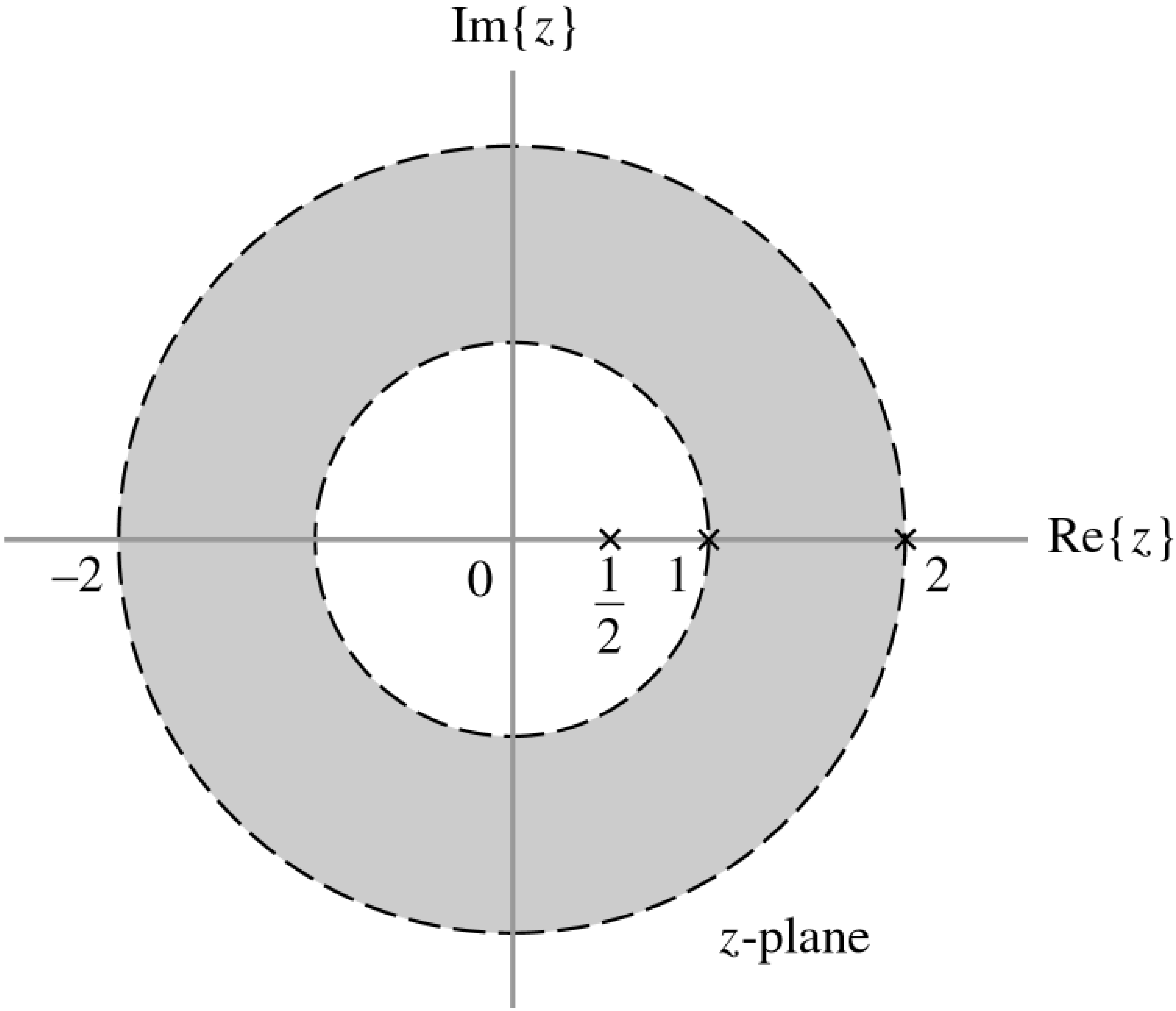
with ROC $1 < |z| < 2$.
Using partial fraction expansion:

$$\begin{aligned} X(z) &= \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - 2z^{-1}} + \frac{A_3}{1 - z^{-1}} \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 2z^{-1}} + \frac{-2}{1 - z^{-1}} \end{aligned}$$

where A_1,A_2, and A_3 are solved the same way as in Laplace transform:

$$A_1 = X(z)(1 - 1/2z^{-1})|_{z=1/2} = 1.$$

Inversion of the z-Transform: Examples (cont.)



Inversion of the z-Transform: Examples (cont.)

Applying the given ROC

- The first term (pole at $z = 1/2$) is a RSS. Thus, $\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}$.
- The second term (pole at $z = 2$) is a LSS. Thus, $-2(2)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{2}{1-2z^{-1}}$.
- The third term (pole at $z = 1$) is a RSS. Thus, $-2(1)^n u[n] \overset{z}{\longleftrightarrow} \frac{-2}{1-z^{-1}}$.

Combining these terms gives

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1] - 2(1)^n u[n].$$

Inversion of the z-Transform: Examples (cont.)

Example 7.10, p_{575}: Find the inverse z-transform of

$$X(z) = \frac{z^3 - 10z^2 - 4z + 4}{2z^2 - 2z - 4}, \quad \text{with ROC } \; |z| < 1$$

- $X(z)$ given in terms of z, instead of z^{-1}.
- $X(z)$ is not a proper function of z^{-1}.

Factoring z^3 from the numerator and $2z^2$ from the denominator gives

$$X(z) = \frac{1}{2} z \left(\frac{1 - 10z^{-1} - 4z^{-2} + 4z^{-3}}{1 - z^{-1} - 2z^{-2}} \right) = \frac{1}{2} z Y(z)$$

Inversion of the z-Transform: Examples (cont.)

- Factor $\frac{1}{2}z$ is easily incorporated using the time-shift property.
- The term in parentheses, $Y(z)$, must be converted into two terms, a polynomial function of z^{-1} and a proper function of z^{-1}, as

$$
\begin{aligned}
Y(z) &= (-2z^{-1}+3) + \frac{-5z^{-1}-2}{(1+z^{-1})(1-2z^{-1})} \\
&= (-2z^{-1}+3) + \frac{1}{1+z^{-1}} - \frac{3}{1-2z^{-1}}, \quad \text{with ROC } \; |z| < 1
\end{aligned}
$$

Inversion of the z-Transform: Examples (cont.)

Thus, we have

$$
\begin{aligned}
X(z) &= \frac{1}{2} z Y(z) \\
Y(z) &= (-2z^{-1} + 3) + \frac{1}{1+z^{-1}} - \frac{3}{1-2z^{-1}} \\
&\quad \text{(apply tables on } p_{784-785}\text{)} \\
y[n] &= -2\delta[n-1] + 3\delta[n] - (-1)^n u[-n-1] + 3(2)^n u[-n-1] \\
x[n] &= \frac{1}{2} y[n+1] \\
&= -\delta[n] + \frac{3}{2}\delta[n+1] - \frac{1}{2}(-1)^{n+1} u[-n-2] + 3(2)^n u[-n-2].
\end{aligned}
$$

The transfer function

- For LTI discrete-time systems with input $x[n]$ and output $y[n]$:
  - ⋆ $y[n] = x[n] * h[n]$
  - ⋆ $Y(z) = X(z)H(z)$, where system transfer function $H(z)$ is viewed as

$$H(z) = \frac{Y(z)}{X(z)}.$$

- In order to uniquely determine the impulse response from the transfer function, must know ROC.

- If ROC is not known, other system characteristics such as stability or casuality must be known.

The transfer function - Examples

Example 7.13, p_{580}: Find the transfer function and impulse of a causal LTI system if the input is

$$x[n] = (-1/3)^n u[n]$$

and the output is

$$y[n] = 3(-1)^n u[n] + (1/3)^n u[n].$$

$$\begin{aligned} X(z) &= \frac{1}{1+(1/3)z^{-1}}, \quad \text{ROC} \;\; |z| > 1/3 \\ Y(z) &= \frac{3}{1+z^{-1}} + \frac{1}{1-(1/3)z^{-1}} \\ &= \frac{4}{(1+z^{-1})(1-(1/3)z^{-1})}, \text{ROC} \;\; |z| > 1 \end{aligned}$$

The transfer function - Examples (cont.)

Thus, the transfer function is obtained as

$$H(z) = \frac{4(1 + (1/3)z^{-1})}{(1 + z^{-1})(1 - (1/3)z^{-1})}, \quad \text{with ROC } |z| > 1$$

Partial-fraction expansion:

$$\begin{aligned} H(z) &= \frac{A}{1 + z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}} \\ &= \frac{2}{1 + z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

Taking inverse z-transform we obtain the system impulse response

$$h[n] = 2(-1)^n u[n] + 2(1/3)^n u[n].$$

The transfer function - Examples (cont.)

Problem 7.8, p_{580}: An LTI system has impulse response $h[n] = (1/2)^n u[n]$. Determine the input to the system if the output if given by $y[n] = (1/2)^n u[n] + (-1/2)^n u[n]$.

- The z-transform of system output

$$Y(z) = \frac{1}{1-(1/2)z^{-1}} + \frac{1}{1+(1/2)z^{-1}}, \quad ROC \;\; |z| > 1/2$$

- System transfer function

$$H(z) = \frac{1}{1-(1/2)z^{-1}}, \quad ROC \;\; |z| > 1/2$$

The transfer function - Examples (cont.)

The z-transform of the system input is

$$
\begin{aligned}
X(z) &= Y(z)/H(z) \\
&= 1 + \frac{1-(1/2)z^{-1}}{1+(1/2)z^{-1}} \\
&= \frac{2}{1+(1/2)z^{-1}} \stackrel{z}{\longleftrightarrow} x[n] = 2(-1/2)^n u[n].
\end{aligned}
$$

The transfer function and difference equation

- For a system described by the difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The transfer function of an LTI system described by a difference equation is a ratio of polynomials in z^{-1}.

- This form of the transfer function is termed a *rational transfer function*.

The transfer function and difference equation - Examples

Example 7.14, p_{581}: Determine the transfer function and the impulse response for the causal LTI system described by the difference equation

$$y[n] - (1/4)y[n-1] - (3/8)y[n-2] = -x[n] + 2x[n-1]$$

Taking z-transform of both sides gives

$$\begin{aligned}
Y(z) - (1/4)z^{-1}Y(z) - (3/8)z^{-2}Y(z) &= -X(z) + 2z^{-1}X(z) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{-1 + 2z^{-1}}{1 - (1/4)z^{-1} - (3/8)z^{-2}} \\
&= \frac{-2}{1 + (1/2)z^{-1}} + \frac{1}{1 - (3/4)z^{-1}} \\
h[n] &= -2(-1/2)^n u[n] + (3/4)^n u[n]
\end{aligned}$$

(causal system)

The transfer function and difference equation - Examples (cont.)

Example 7.15, p_{581}: Find the difference-equation description of an LTI system with transfer function

$$H(z) = \frac{5z+2}{z^2+3z+2}.$$

Dividing both numerator and denominator by z^2, we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5z^{-1}+2z^{-2}}{1+3z^{-1}+2z^{-2}}.$$

Cross multiply and then inverse z-transform:

$$\begin{aligned} Y(z) + 3Y(z)z^{-1} + 2Y(z)z^{-2} &= 5X(z)z^{-1} + 2X(z)z^{-2} \\ y[n] + 3y[n-1] + 2y[n-2] &= 5x[n-1] + 2x[n-2] \end{aligned}$$

System causality and stability

- Similar to continuous-time LTI systems, but there are differences.

- System transfer function $H(z) \stackrel{z}{\longleftrightarrow} h[n]$, system impulse response.

- In order to uniquely determine $h[n]$, must know ROC or other knowledge of the system characteristics.

- Causal system $\rightarrow h[n] = 0$ for $n < 0 \rightarrow H(z)$ is right-sided transform.

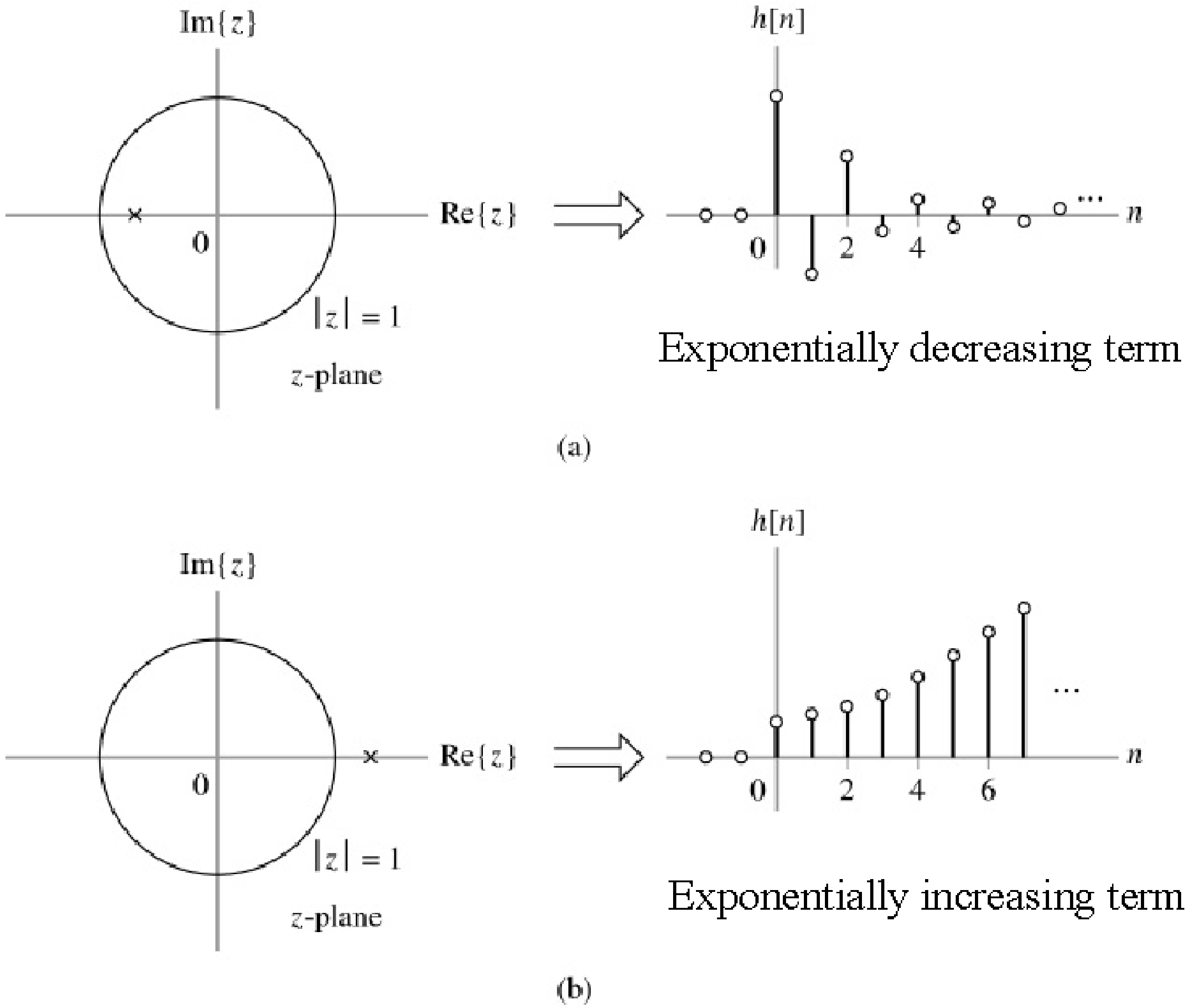
- Stable system $\rightarrow h[n]$ absolutely summable $\rightarrow$ DTFT of $x[n]$ exists $\rightarrow$ ROC includes unit circle ($z = e^{j\Omega}$).

System causality and stability (cont.)

- Assume a pole at $z = d_k$.
  - ⋆ If $|d_k| < 1$ (pole inside unit circle), the pole contributes an exponentially decaying term to the impulse response. ▮
  - ⋆ If $|d_k| > 1$ (pole outside unit circle), the pole contributes an exponentially increasing term to the impulse response. ▮

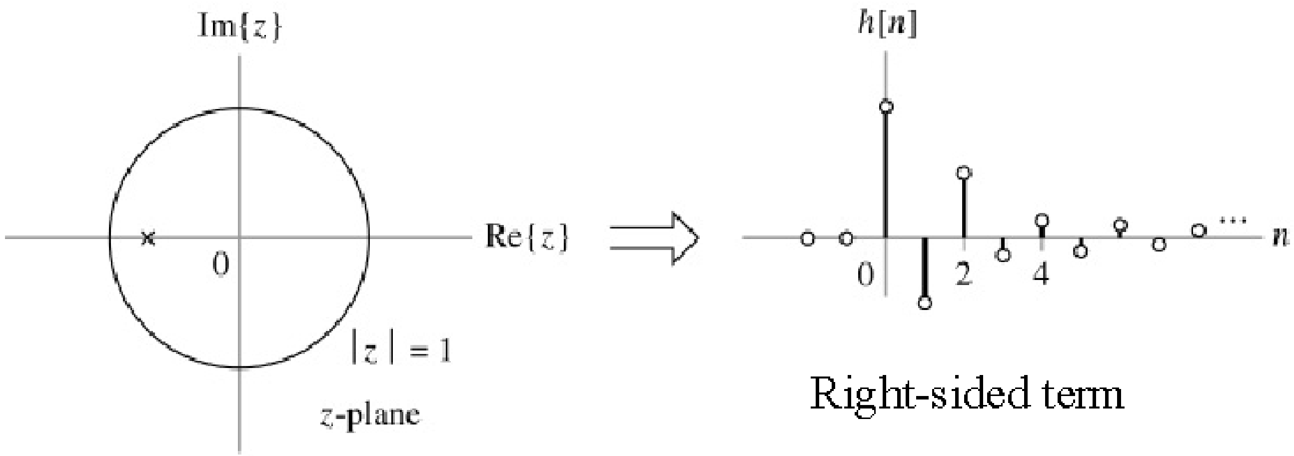
- Conclusion: If a system is causal and stable, then *all poles of* $H(z)$ *are inside the unit circle*.

System causality and stability (cont.)

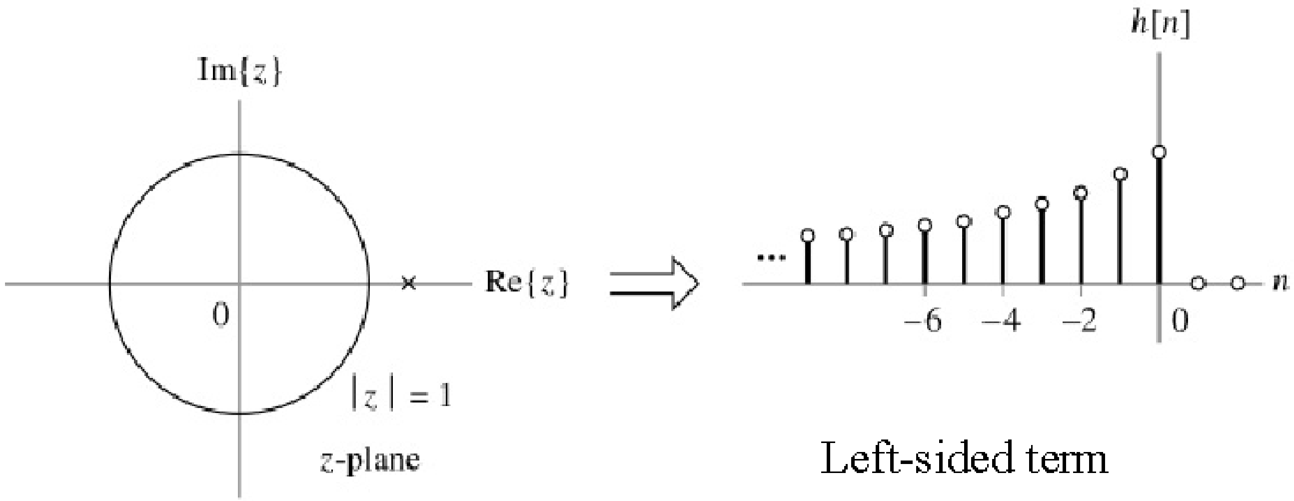


For causal systems

System causality and stability (cont.)



(a)



(b)

For stable systems

System causality and stability - Examples

- Read Example 7.16, p_{584}
- Read Example 7.17, p_{585}

Problem 7.10, p_{585}: A stable and causal LTI system is described by the difference equation

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = -2x[n] + \frac{5}{4}x[n-1].$$

Find the system impulse response.

System causality and stability - Examples (cont.)

Taking z-transform of both sides of the difference equation, we obtain

$$
\begin{aligned}
Y(z) + \frac{1}{4}Y(z)z^{-1} - \frac{1}{8}Y(z)Z^{-2} &= -2X(z) + \frac{5}{4}X(z)z^{-1} \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{-2 + \frac{5}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \\
&= \frac{-2 + \frac{5}{4}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + \frac{1}{2}z^{-1})} \\
&= \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}}
\end{aligned}
$$

System causality and stability - Examples (cont.)

Poles at $z = 1/4$ and $z = -1/2$, both are inside the unit circle. Because system is causal, both poles corresponding to right-sided terms.

$$\begin{aligned}
A &= \left.\frac{-2 + \frac{5}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right|_{z^{-1}=4} = 1 \\
B &= \left.\frac{-2 + \frac{5}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\right|_{z^{-1}=-2} = -3 \\
H(z) &= \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{-3}{1 + \frac{1}{2}z^{-1}} \\
h[n] &= \left(\frac{1}{4}\right)^n u[n] - 3\left(-\frac{1}{2}\right)^n u[n]
\end{aligned}$$

Freq. response from poles and zeros

- If ROC of an LTI system transfer function includes the unit circle, frequency response can be obtained as $H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}}$.

- For a rational transfer function assuming a p^{th}-order pole at $z = 0$ and an l^{th}-order zero at $z = 0$ expressed as

$$H(z) = \frac{\tilde{b} z^{-p} \prod_{k=1}^{M-p}(1 - c_k z^{-1})}{z^{-l} \prod_{k=1}^{N-l}(1 - d_k z^{-1})}$$

where $\tilde{b} = b_p / a_l$, the frequency response is obtained by substituting $z = e^{j\Omega}$:

$$\begin{aligned} H(e^{j\Omega}) &= \frac{\tilde{b} e^{-jp\Omega} \prod_{k=1}^{M-p}(1 - c_k e^{-j\Omega})}{e^{-jl\Omega} \prod_{k=1}^{N-l}(1 - d_k e^{-j\Omega})} \\ &= \frac{\tilde{b} e^{-j(N-M)\Omega} \prod_{k=1}^{M-p}(e^{j\Omega} - c_k)}{\prod_{k=1}^{N-l}(e^{j\Omega} - d_k)} \end{aligned}$$

Freq. response from poles and zeros (cont.)

For a particular frequency Ω_0, the overall magnitude is evaluated in terms of the magnitude associated with each pole and zero as

$$|H(e^{j\Omega_0})| = \frac{|\tilde{b}| \prod_{k=1}^{M-p}(e^{j\Omega_0} - c_k)}{\prod_{k=1}^{N-l}(e^{j\Omega_0} - d_k)}$$

and the overall phase is evaluated in terms of the phase associated with each pole and zero as

$$arg\{H(e^{j\Omega_0})\} = \quad arg\{\tilde{b}\} + (N-M)\Omega_0 + \sum_{k=1}^{M-p} arg\{e^{j\Omega_0} - c_k\} - \sum_{k=1}^{N-l} arg\{e^{j\Omega_0} - d_k\}$$

Applications to Filters and Equalizers

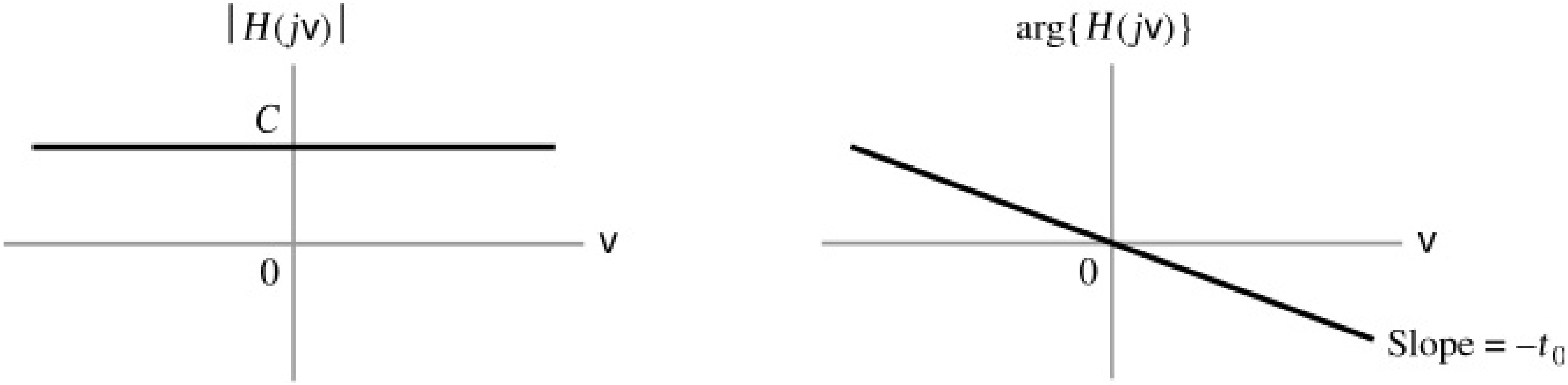
- Distortionless transmission:
  - ⋆ A scaling of magnitude
  - ⋆ A constant time delay

  Let $x(t)$ be the input to an LTI system. If the system is distortionless, output must be $y(t) = Cx(t - t_0)$, where C is a constant and t_0 is the transmission delay.

  - ⋆ The impulse response of the system is: $h(t) = C\delta(t - t_0)$.
  - ⋆ Fourier transform of $y(t)$: $Y(j\omega) = CX(j\omega)e^{-j\omega t_0}$.

Distortionless transmission

- The system transfer function: $H(j\omega) = Ce^{-j\omega t_0}$
  - Magnitude response: $|H(j\omega)| = C$
  - Phase response: $arg\{H(j\omega)\} = -\omega t_0$



(a) (b)

Ideal low-pass filters

Consider

$$H(j\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

where

- the constant is set to $C = 1$.
- a finite delay t_0 is chosen.
- ω_c is the cutoff frequency.

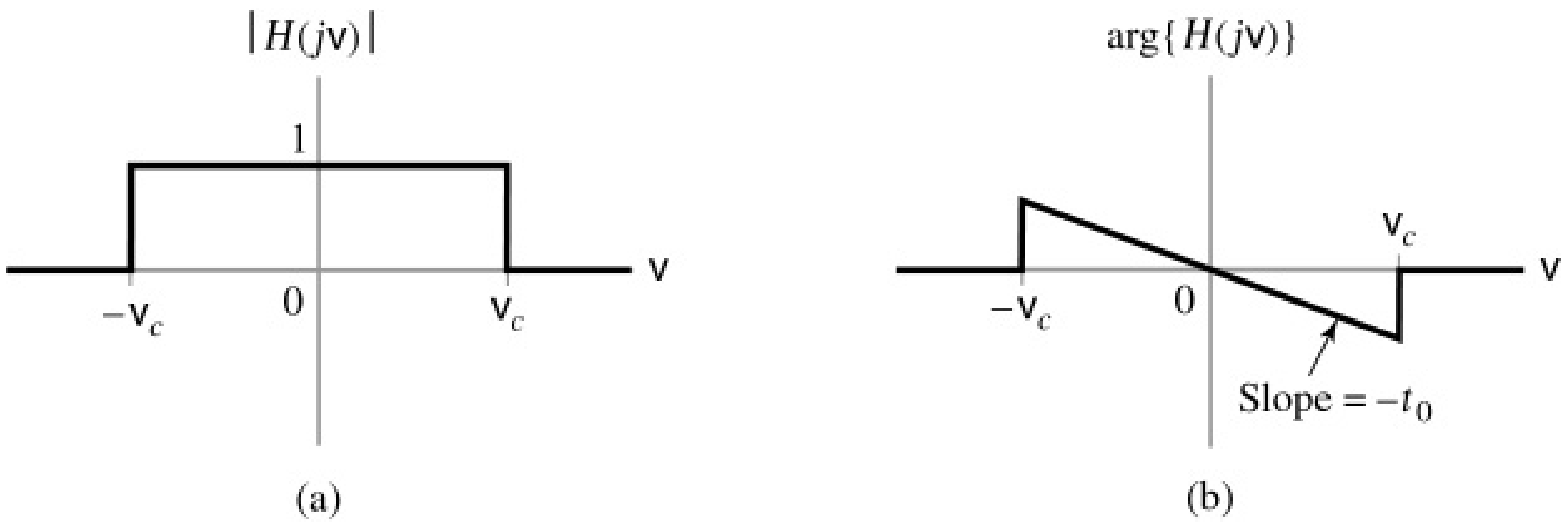
Ideal low-pass filters (cont.)

To evaluate the filter impulse response $h(t)$, we take the inverse Fourier transform:

$$\begin{aligned} H(j\omega) \overset{FT}{\longleftrightarrow} h(t) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)} d\omega \\ &= \frac{1}{2\pi}\frac{e^{j\omega(t-t_0)}}{j(t-t_0)}\Bigg|_{-\omega_c}^{-\omega_c} \\ &= \frac{sin(\omega_c(t-t_0))}{\pi(t-t_0))} = \frac{\omega_c}{\pi} sinc\left(\frac{\omega_c}{\pi}(t-t_0)\right) \end{aligned}$$

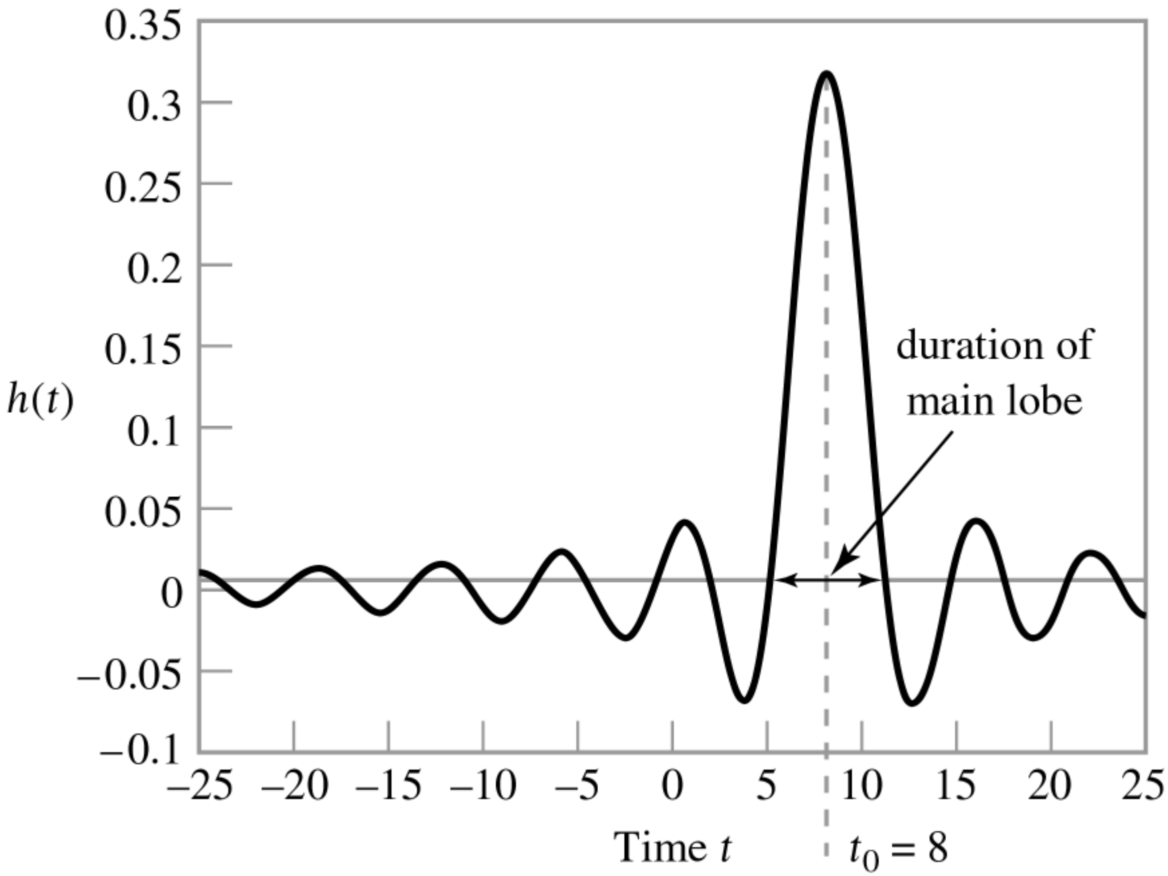
where the definition of $sinc(\omega t) = \frac{sin(\pi\omega t)}{\pi\omega t}$ is applied.

Ideal low-pass filters (cont.)



Frequency response of ideal low-pass filters. (a) Magnitude response. (b) Phase response.

Ideal low-pass filters (cont.)



Time-shifted form of the impulse response of an ideal, noncausal, low-pass filter for $\omega_c = 1$ and $t_0 = 8$.

Design of filters

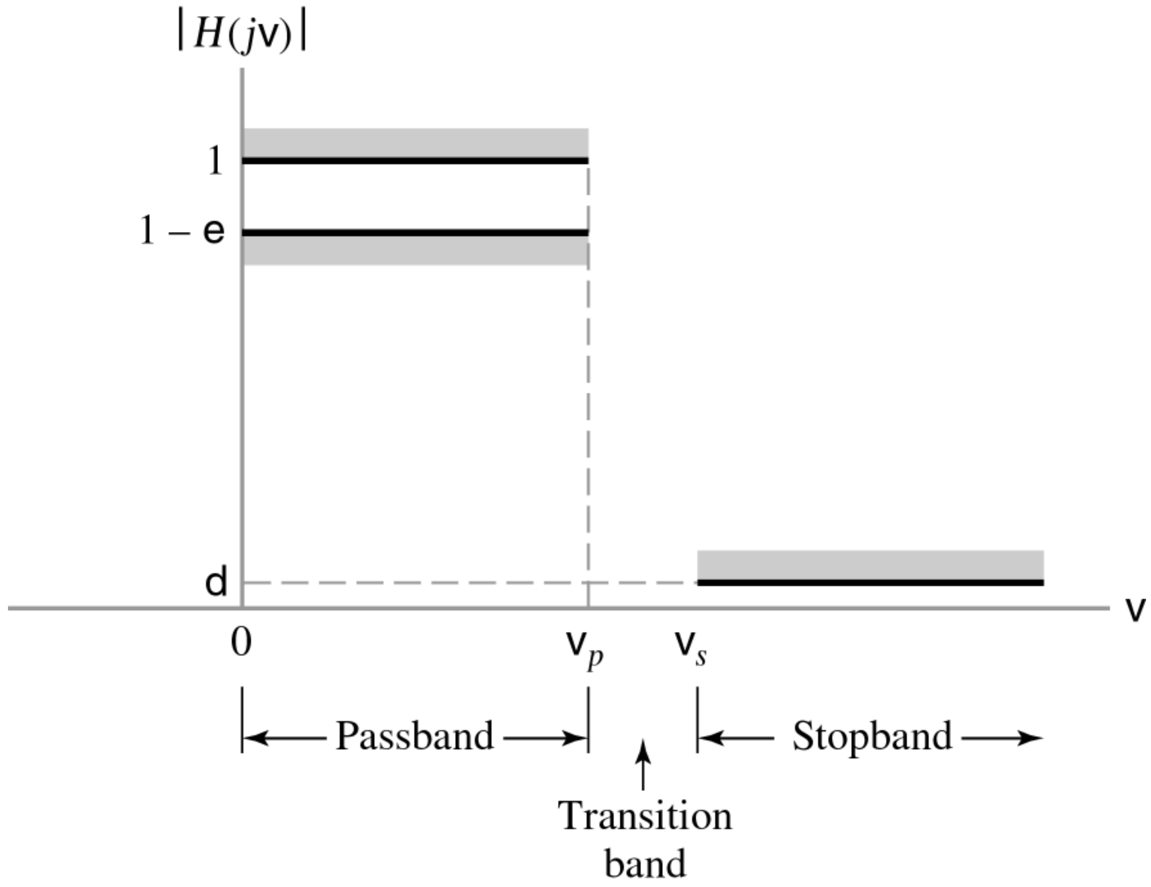
Impulse responses of ideal filters are noncausal and infinite length. These filters are nonimplementable. Practical filters allow

- Passband ripple: $1 - \epsilon \leq |H(j\omega)| \leq 1$ for $0 \leq |\omega| \leq \omega_p$, where ω_p is the passband cutoff frequency and ϵ is a tolerance parameter.

- Stop band ripple: $|H(j\omega)| \leq \delta$ for $|\omega| \geq \omega_s$, where ω_s is the stopband cutoff frequency and δ is the tolerance parameter.

- Transition band: $\omega_s - \omega_p$, a finite width.

Ideal low-pass filters (cont.)



Tolerance diagram of a practical low-pass filter. The passband, transition band, and stopband are shown for positive frequencies.

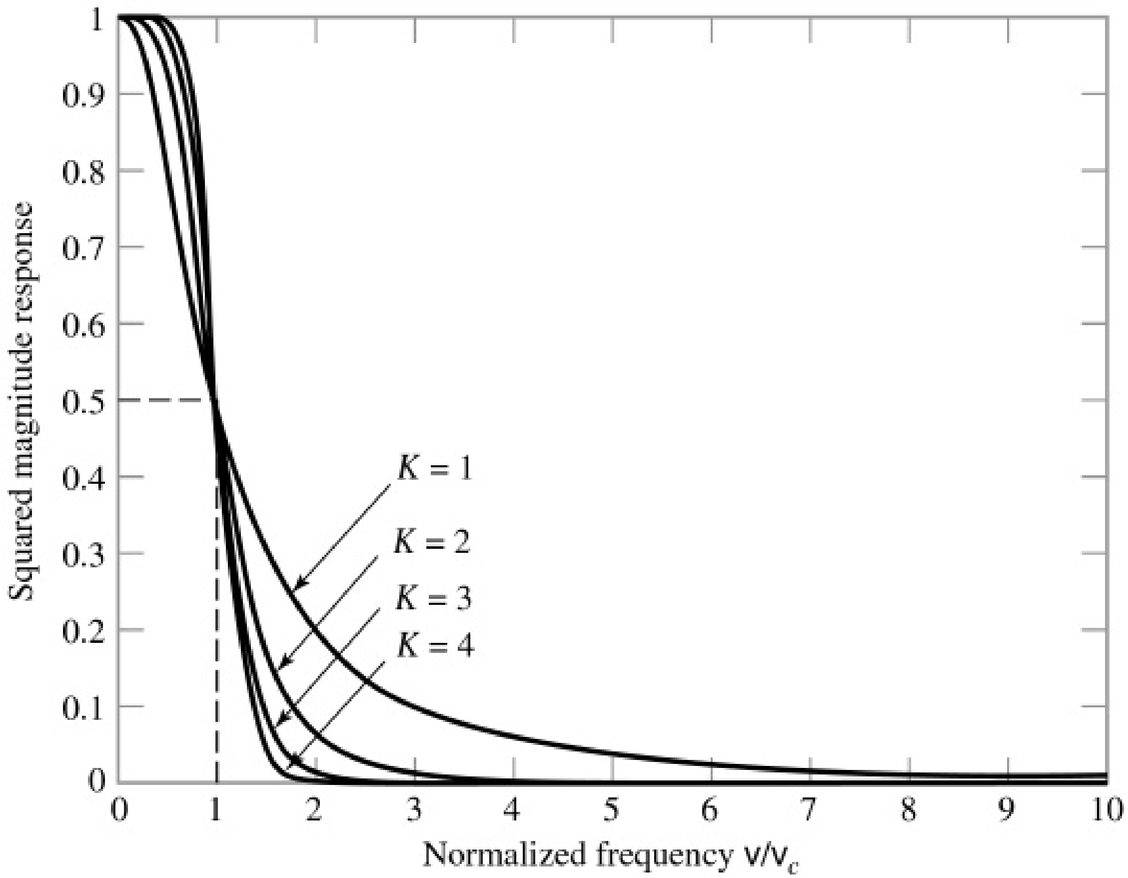
Approximating functions – Butterworth prototype

Butterworth function of order K:

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2K}}, \; K = 1, 2, 3, \cdots,$$

- ω_c: cutoff frequency.
- K: filter order.
- For prescribed values of tolerance parameters ϵ and δ, the passband and stopband frequencies are:
  - ⋆ $\omega_p = \omega_c \left(\frac{\epsilon}{1-\epsilon}\right)^{1/(2K)}$
  - ⋆ $\omega_s = \omega_c \left(\frac{1-\delta}{\delta}\right)^{1/(2K)}$

Approximating functions – Butterworth prototype (cont.)



Magnitude response of Butterworth filter for varying orders.

Approximating functions – Butterworth prototype (cont.)

- Butterworth filters are *maximally flat* at $\omega = 0$ (i.e., the first $2K - 1$ derivatives of $|H(j\omega)|^2$ at $\omega = 0$ are equal to zero).

- For any given set of specifications $(\omega_p, \omega_s, \epsilon, \delta)$, K and ω_c can be calculated, and hence $|H(j\omega)|^2$ can be determined.

Given Butterworth function $|H(j\omega)|^2$, the transfer function $H(s)$ maybe obtained by using the following procedures:

- Find the $2K$ pole locations of $H(s)H(-s)|_{s=j\omega} = |H(j\omega)|^2$: $s = \omega_c e^{j\pi(2k+1)/(2K)}$, for $k = 0, 1, \cdots, 2K - 1$.

- For a stable system, the K poles on the left half plane belong to $H(s)$.

Approximating functions – Butterworth prototype (cont.)

- Read example 8.3, p_{627}.

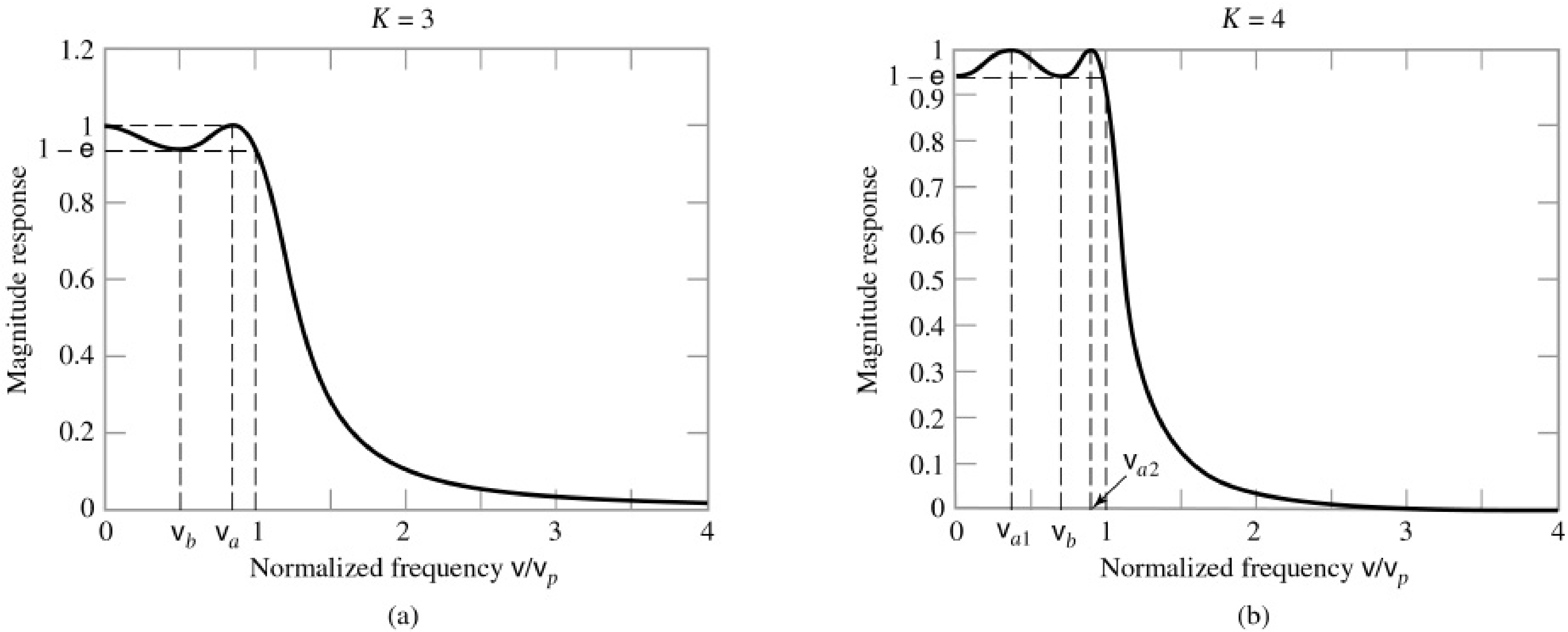
- Problem 8.3, p_{628}: Find the transfer function of a Butterworth filter with cutoff frequency $\omega_c = 1$ and filter order $K = 2$.

  ⋆ The $2K = 4$ poles of $H(s)H(-s)$ are determined to be ($s = \omega_c e^{j(2k+1)/(2K)}, k = 0, 1, 2, 3$): $s_{1,2} = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$, $s_{3,4} = -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$.

  ⋆ Poles of $H(s)$ are: $s_{3,4} = -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$. Thus,

$$
\begin{aligned}
H(s) &= \frac{1}{(s + \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2})(s + \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})} \\
&= \frac{1}{s^2 + \sqrt{2}s + 1}
\end{aligned}
$$

Approximating functions – Chebyshev prototype



Magnitude response of Chebyshev filter for order (a) $K = 3$ and (b) $K = 4$ and passband ripple = 0.5 dB. The frequencies ω_b and ω_a in case (a) and the frequencies ω_{a1} and ω_b, and ω_{a2} in case (b) are defined in accordance with the optimality criteria for equiripple amplitude response.

Approximating functions – Chebyshev prototype (cont.)

- Equiripple in the passband.
- Monotonic in the stopband.
- Approximation functions with an equiripple magnitude response are known collectively as *Chebyshev functions.*
- A filter designed on this basis is called a *Chebyshev filter.*

Frequency transformations

- So far, have considered only low-pass filters.

- High-pass, band-pass, and band-stop filters can be designed by an appropriate transformation of the independent variable.

  ⋆ Low-pass to high-pass transformation: $s \longrightarrow \frac{\omega_c}{s}$, where ω_c is the desired cutoff frequency of the high-pass filter.

  ⋆ Low-pass to band-pass transformation: $s \longrightarrow \frac{s^2+\omega_0^2}{Bs}$, where B is the bandwidth of the band-pass filter and ω_0 is the midband frequency of the band-pass filter, both measured in radians per second.