ECE352
Spring 07

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Homework 1
Due 04/13/07 at the beginning of the class

3.75. Evaluate the following quantities.

(a)

$$\frac{2}{1-\frac{1}{3}e^{-j\Omega}} \xleftrightarrow{DTFT} 2(\frac{1}{3})u[n]$$

$$\begin{aligned}
\int_{-\pi}^{\pi} |\frac{2}{1-\frac{1}{3}e^{-j\Omega}}|^2 d\Omega &= 2\pi \sum_{n=-\infty}^{\infty} |2(\frac{1}{3})^n u[n]|^2 \\
&= 8\pi \sum_{n=0}^{\infty} |(\frac{1}{9})^n| \\
&= \frac{8\pi}{1-\frac{1}{9}} \\
&= 9\pi
\end{aligned}$$

(b)

$$X[k] = \frac{\sin(k\frac{\pi}{8})}{\pi k} \xleftrightarrow{FS;\,\pi} x(t) = \begin{cases} 1 & |t| \le \frac{\pi}{8\omega_o} \\ 0, & \frac{\pi}{8\omega_o} < |t| \le \frac{2\pi}{\omega_o} \end{cases}$$

$$\begin{aligned}
\pi^2 \sum_{k=-\infty}^{\infty} \frac{\sin^2(k\pi/8)}{\pi^2 k^2} &= \frac{\pi^2}{T} \int_{-0.5T}^{0.5T} |x(t)|^2\, dt \\
&= \frac{\pi\omega_o}{2} \int_{-\frac{\pi}{8\omega_o}}^{\frac{\pi}{8\omega_o}} |1|^2\, dt \\
&= \frac{\omega_o 2\pi^2}{2(8)\omega_o} \\
&= \frac{\pi^2}{8}
\end{aligned}$$

(c)

$$
\begin{aligned}
X(j\omega) = \frac{2(2)}{\omega^2+2^2} \quad &\overset{FT}{\longleftrightarrow} \quad x(t) = e^{-2|t|} \\
\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{4}{\omega^2+2^2}\right)^2 d\omega \quad &= \quad \pi\int_{-\infty}^{\infty}|x(t)|^2\,dt \\
&= \quad 2\pi\int_0^{\infty} e^{-4t}\,dt \\
&= \quad \frac{\pi}{2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
x(t) = \frac{\sin(\pi t)}{\pi t} \quad &\overset{FT}{\longleftrightarrow} \quad X(j\omega) = \begin{cases} 1 & |\omega| \le \pi \\ 0, & \text{otherwise} \end{cases} \\
\pi\int_{-\infty}^{\infty}\left(\frac{\sin(\pi t)}{\pi t}\right)^2 dt \quad &= \quad \frac{1}{2}\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega \\
&= \quad \frac{1}{2}\int_{-\pi}^{\pi} 1\,d\omega \\
&= \quad \pi
\end{aligned}
$$

3.76. Use the duality property to evaluate

(a)

$$
\begin{aligned}
x(t) \quad &\overset{FT}{\longleftrightarrow} \quad e^{-2\omega}u(j\omega) \\
e^{2t}u(-t) \quad &\overset{FT}{\longleftrightarrow} \quad \frac{1}{2-j\omega} \\
\text{thus:} & \\
\frac{1}{2-jt} \quad &\overset{FT}{\longleftrightarrow} \quad 2\pi e^{-2\omega}u(\omega) \\
x(t) \quad &= \quad \frac{1}{2\pi}\frac{1}{2-jt}
\end{aligned}
$$

(b)

$$
X(j\omega) \quad \overset{FT}{\longleftrightarrow} \quad \frac{1}{(2+jt)^2}
$$

$$
\begin{aligned}
te^{-2t}u(t) &\overset{FT}{\longleftrightarrow} \frac{1}{(2+j\omega)^2} \\
&\text{thus:} \\
\frac{1}{(2+jt)^2} &\overset{FT}{\longleftrightarrow} -2\pi\omega e^{2\omega}u(-\omega) \\
X(j\omega) &= -2\pi\omega e^{2\omega}u(-\omega)
\end{aligned}
$$

(c)

$$
\begin{aligned}
x[n] = \frac{\sin\left(\frac{11\pi}{20}n\right)}{\sin\left(\frac{\pi}{20}n\right)} &\overset{DTFS;\,\frac{\pi}{10}}{\longleftrightarrow} X[k] \\
\omega_o = \frac{\pi}{10} &\quad \text{implies} \quad N = 20 \\
\begin{cases} 1 & |n| \le 5 \\ 0, & 5 < |n| \le 10 \end{cases} &\overset{DTFS;\,\frac{\pi}{10}}{\longleftrightarrow} \frac{\sin(\frac{11\pi}{20}k)}{\sin(\frac{\pi}{20}k)} \\
&\text{implies} \\
\frac{\sin(\frac{11\pi}{20}k)}{\sin(\frac{\pi}{20}k)} &\overset{DTFS;\,\frac{\pi}{10}}{\longleftrightarrow} \frac{1}{20}\begin{cases} 1 & |k| \le 5 \\ 0, & 5 < |k| \le 10 \end{cases} \\
X[k] &= \begin{cases} \frac{1}{20} & |k| \le 5 \\ 0, & 5 < |k| \le 10 \end{cases} \\
X[k+iN] = X[k+i20] &= X[k] \text{ where } k, i \text{ are integers.}
\end{aligned}
$$

3.81. In this problem we show that Gaussian pulses acheive the lower bound in the time-bandwidth product. *Hint:* Use the definite integrals in Appendix A.4.

(a) Let $x(t) = e^{-\frac{t^2}{2}}, X(j\omega) = e^{-\frac{\omega^2}{2}}$. Find the effective duration, T_d, bandwidth, B_w, and evaluate the time-bandwidth product.

$$
\begin{aligned}
T_d &= \left\{ \frac{\int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2}\,dt} \right\}^{\frac{1}{2}} \\
&= \left\{ \frac{(\frac{1}{\sqrt{2}})^3\sqrt{2\pi}}{(\frac{1}{\sqrt{2}})\sqrt{2\pi}} \right\}^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \\
B_w &= \left\{ \frac{\int_{-\infty}^{\infty} \omega^2 e^{-\omega^2}\,d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2}\,d\omega} \right\}^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \\
T_d B_w &= \frac{1}{2}
\end{aligned}
$$

(b) Let $x(t) = e^{-\frac{t^2}{2a^2}}$. Find the effective duration, T_d, bandwidth, B_w, and evaluate the time-bandwidth product. What happens to T_d, B_w, and T_dB_w as a increases?

$$
\begin{aligned}
f(\frac{t}{a}) &\overset{FT}{\longleftrightarrow} aF(j\omega a) \\
\text{so } X(j\omega) &= ae^{-\frac{\omega^2 a^2}{2}}
\end{aligned}
$$

4.16. Find the FT representations for the following periodic signals: Sketch the magnitude and phas spectra.

(a) $x(t) = 2\cos(\pi t) + \sin(2\pi t)$

$$\begin{aligned} x(t) &= e^{j\pi t} + e^{-j\pi t} + \frac{1}{2j}e^{j2\pi t} - \frac{1}{2j}e^{-j2\pi t} \\ \omega_o &= lcm(\pi, 2\pi) = \pi \\ x[1] &= x[-1] = 1 \\ x[2] &= -x[-2] = \frac{1}{2j} \\ X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o) \\ X(j\omega) &= 2\left[\pi\delta(\omega-\pi) + \pi\delta(\omega+\pi)\right] + \frac{1}{j}\left[\pi\delta(\omega-2\pi) - \pi\delta(\omega+2\pi)\right] \end{aligned}$$

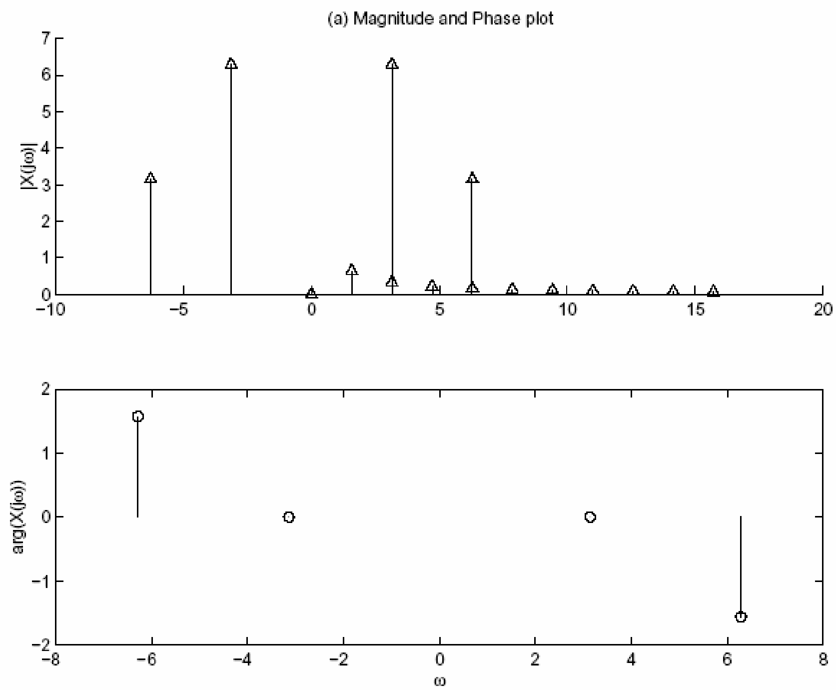

Figure P4.16. (a) Magnitude and Phase plot

(b) $x(t) = \sum_{k=0}^{4} \frac{(-1)^k}{k+1}\cos((2k+1)\pi t)$

$$x(t) = \frac{1}{2}\sum_{k=0}^{4} \frac{(-1)^k}{k+1}\left[e^{j(2k+1)\pi t} + e^{-j(2k+1)\pi t}\right]$$

$$X(j\omega) = \pi \sum_{k=0}^{4} \frac{(-1)^k}{k+1} \left[\delta(\omega - (2k+1)\pi) + \delta(\omega + (2k+1)\pi)\right]$$

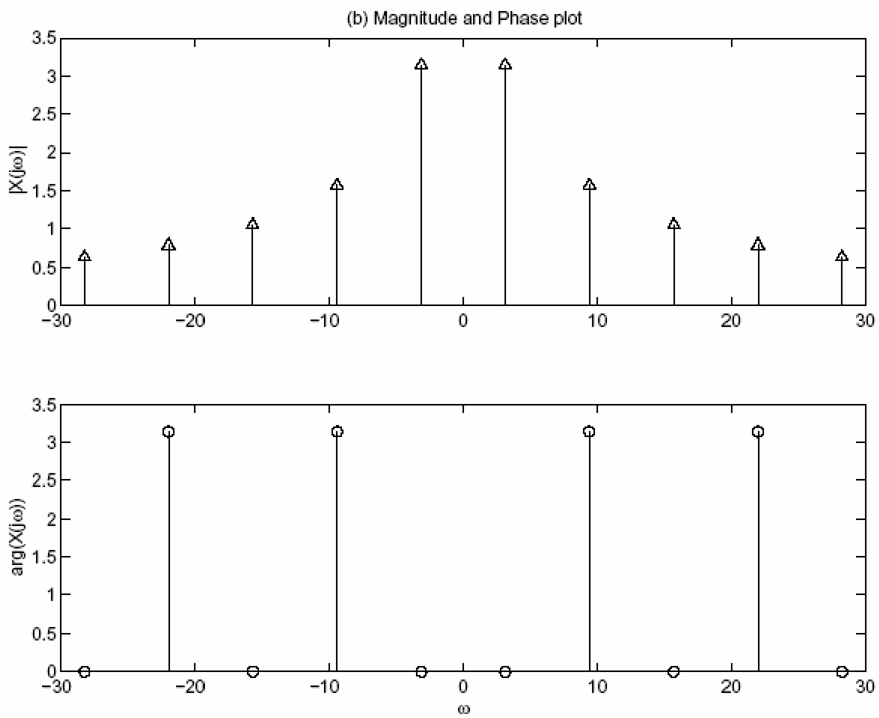

Figure P4.16. (b) Magnitude and Phase plot

(c) $x(t)$ as depicted in Fig. P4.16 (a).

$$x(t) = \begin{cases} 1 & |t| \le 1 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 2 & |t| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \left[\frac{2\sin(k\frac{2\pi}{3})}{k} + \frac{4\sin(k\frac{\pi}{3})}{k}\right] \delta(\omega - k\frac{2\pi}{3})$$

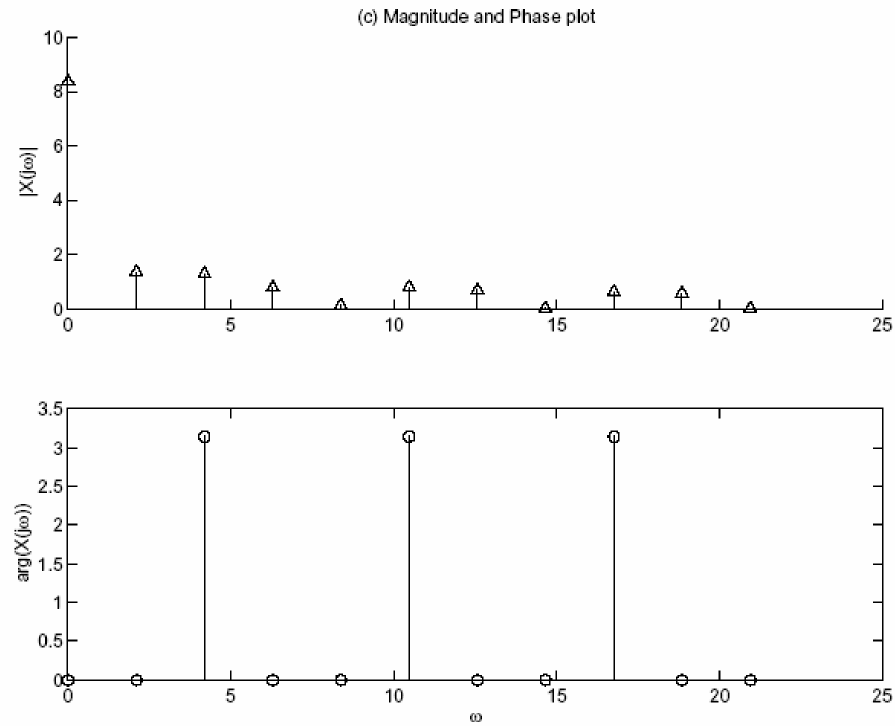

Figure P4.16. (c) Magnitude and Phase plot

(d) $x(t)$ as depicted in Fig. P4.16 (b).

$$T = 4 \qquad \omega_o = \frac{\pi}{2}$$

$$\begin{aligned} X[k] &= \frac{1}{4}\int_{-2}^{2} 2te^{-j\frac{\pi}{2}kt}dt \\ &= \begin{cases} 0 & k = 0 \\ \frac{2j\cos(\pi k)}{\pi k} & k \neq 0 \end{cases} \\ X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - \frac{\pi}{2}k) \end{aligned}$$

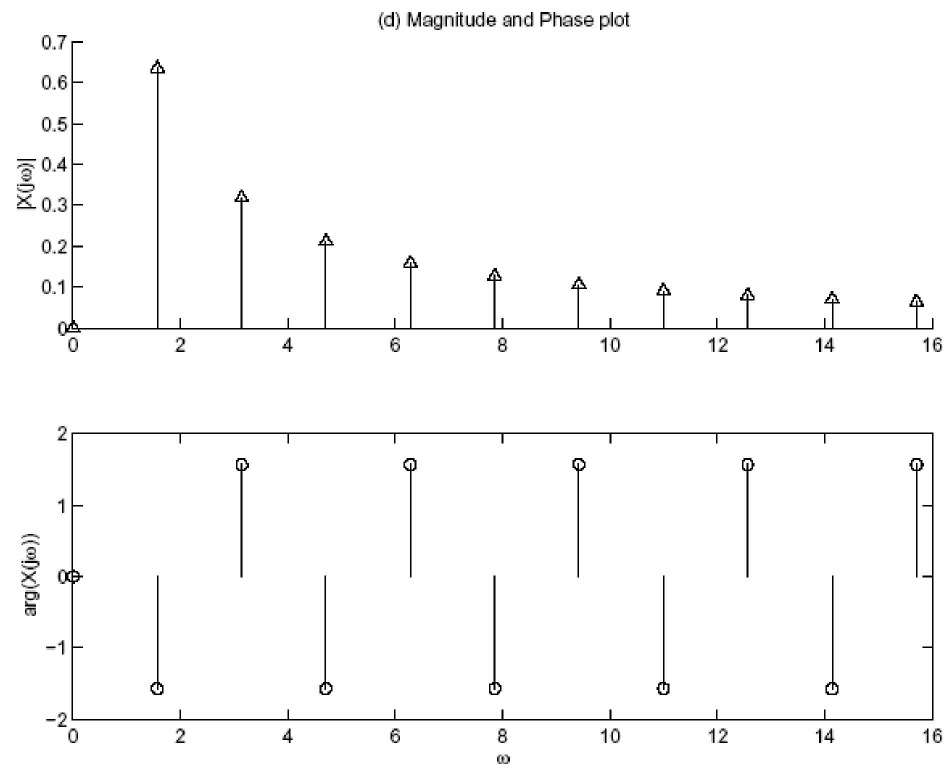

Figure P4.16. (d) Magnitude and Phase plot

4.17. Find the DTFT representations for the following periodic signals: Sketch the magnitude and phase spectra.

(a) $x[n] = \cos(\frac{\pi}{8}n) + \sin(\frac{\pi}{5}n)$

$$\begin{aligned}
x[n] &= \frac{1}{2}\left[e^{j\frac{\pi}{8}n} + e^{-j\frac{\pi}{8}n}\right] + \frac{1}{2j}\left[e^{j\frac{\pi}{5}n} - e^{-j\frac{\pi}{5}n}\right] \\
\Omega_o &= lcm(\frac{\pi}{8}, \frac{\pi}{5}) = \frac{\pi}{40} \\
X[5] &= X[-5] = \frac{1}{2} \\
X[8] &= -X[-8] = \frac{1}{2j} \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o) \\
X(e^{j\Omega}) &= \pi\left[\delta(\Omega - \frac{\pi}{8}) + \delta(\Omega + \frac{\pi}{8})\right] + \frac{\pi}{j}\left[\delta(\Omega - \frac{\pi}{5}) - \delta(\Omega + \frac{\pi}{5})\right]
\end{aligned}$$

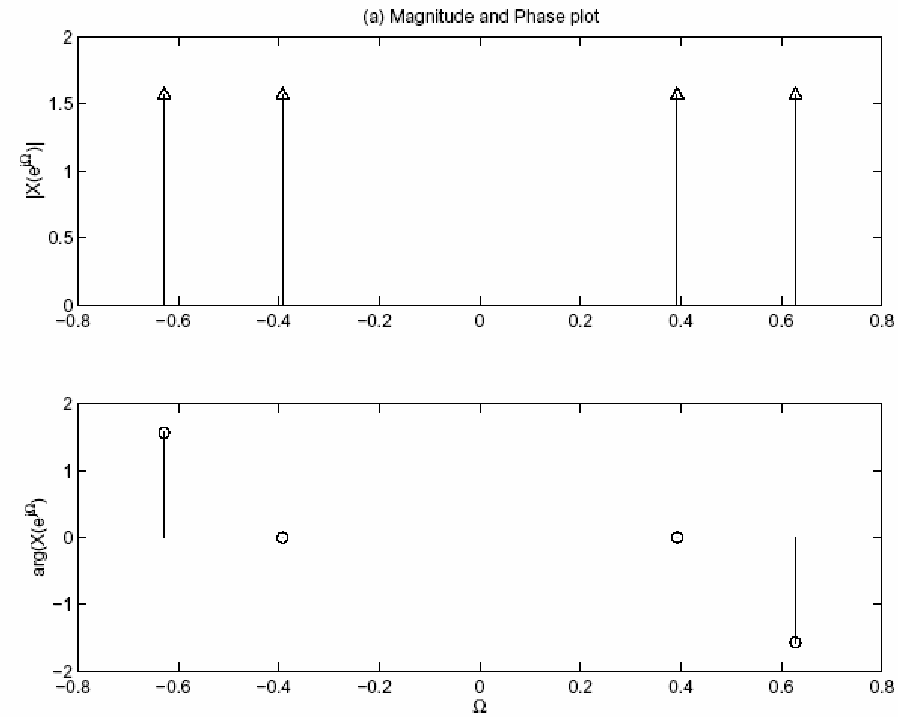

Figure P4.17. (a) Magnitude and Phase response

(b) $x[n] = 1 + \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{4}m)\delta[n-m]$

$$N = 8 \qquad \Omega_o = \frac{\pi}{4}$$

$$\begin{aligned}
x[n] &= 1 + \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{4}m)\delta[n-m] \\
&= 1 + \cos(\frac{\pi}{4}n) \\
X[k] &= \frac{1}{8}\sum_{n=-4}^{3} x[n]e^{-jk\frac{\pi}{4}n} \\
& \text{For one period of } X[k], k \in [-4,3] \\
X[-4] &= 0 \\
X[-3] &= \frac{1-2^{-0.5}}{8}e^{jk\frac{3\pi}{4}} \\
X[-2] &= \frac{1}{8}e^{jk\frac{2\pi}{4}} \\
X[-1] &= \frac{1+2^{-0.5}}{8}e^{jk\frac{\pi}{4}} \\
X[0] &= \frac{2}{8} \\
X[1] &= \frac{1+2^{-0.5}}{8}e^{-jk\frac{\pi}{4}} \\
X[2] &= \frac{1}{8}e^{-jk\frac{2\pi}{4}}
\end{aligned}$$

$$
\begin{aligned}
X[3] &= \frac{1-2^{-0.5}}{8} e^{-jk\frac{3\pi}{4}} \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o) \\
&= \pi\left[\frac{(1-2^{-0.5})}{4}\delta(\Omega+\frac{3\pi}{4}) + \frac{1}{4}\delta(\Omega+\frac{\pi}{2}) + \frac{(1+2^{-0.5})}{4}\delta(\Omega+\frac{\pi}{4}) + \frac{1}{4}\delta(2\Omega)\right] + \\
&\quad \pi\left[\frac{(1+2^{-0.5})}{4}\delta(\Omega-\frac{\pi}{4}) + \frac{1}{4}\delta(\Omega-\frac{\pi}{2}) + \frac{(1-2^{-0.5})}{4}\delta(\Omega-\frac{3\pi}{4})\right]
\end{aligned}
$$

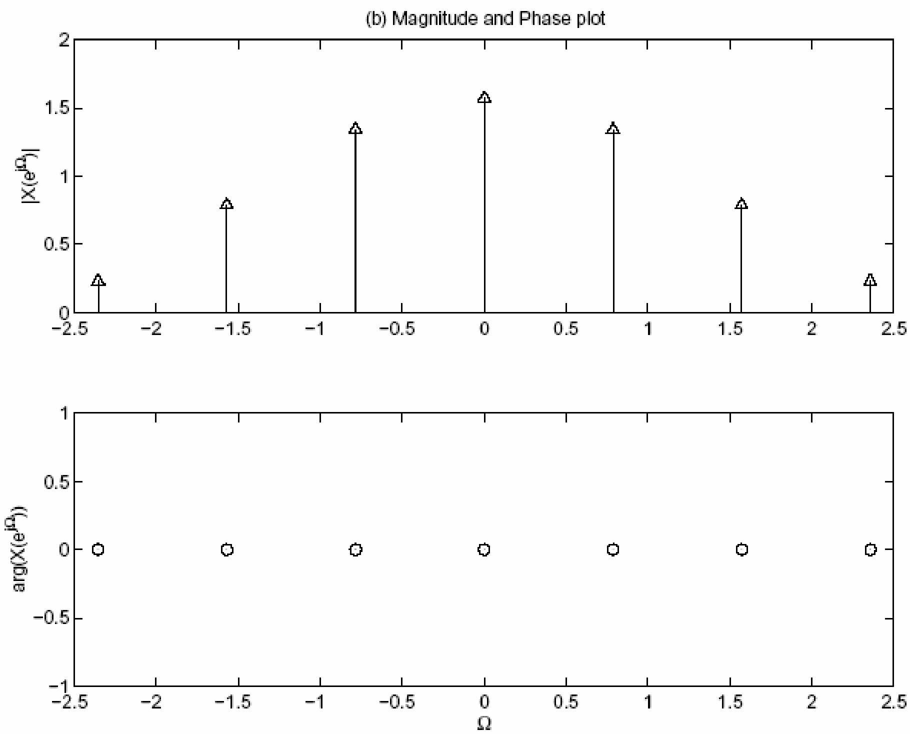

Figure P4.17. (b) Magnitude and Phase response

(c) $x[n]$ as depicted in Fig. P4.17 (a).

$$
N = 8 \qquad \Omega_o = \frac{\pi}{4}
$$

$$
\begin{aligned}
X[k] &= \frac{\sin(k\frac{5\pi}{8})}{8\sin(\frac{\pi}{8}k)} \\
X[0] &= \frac{5}{8} \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\frac{\pi}{4})
\end{aligned}
$$

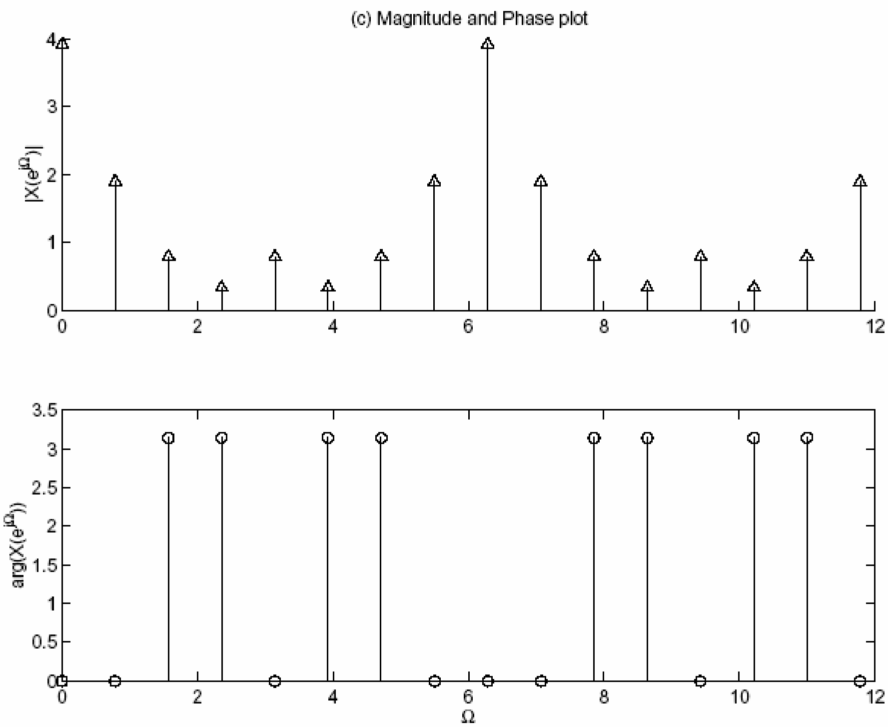

Figure P4.17. (c) Magnitude and Phase response

(d) $x[n]$ as depicted in Fig. P4.17 (b).

$$N = 7 \qquad \Omega_o = \frac{2\pi}{7}$$

$$\begin{aligned} X[k] &= \frac{1}{7}\left(1 - e^{jk\frac{2\pi}{7}} - e^{-jk\frac{2\pi}{7}}\right) \\ &= \frac{1}{7}\left(1 - 2\cos(k\frac{2\pi}{7})\right) \\ X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\frac{2\pi}{7}) \end{aligned}$$

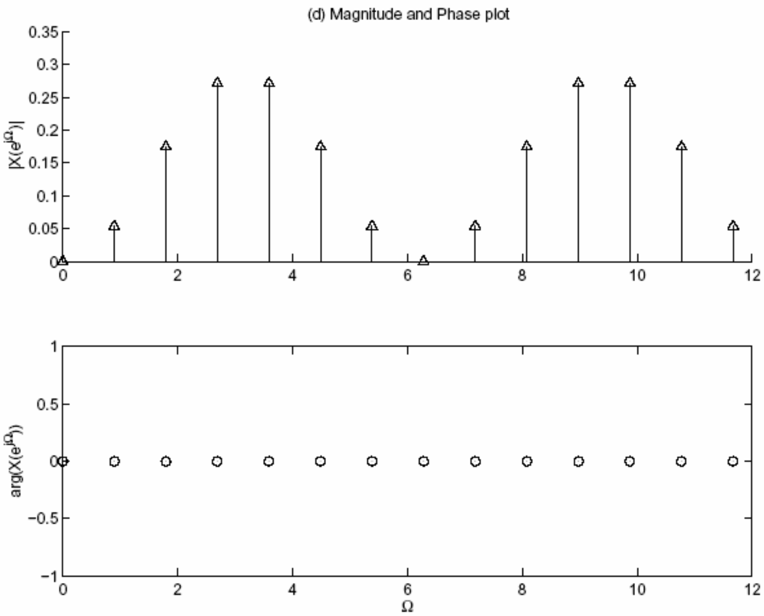

Figure P4.17. (d) Magnitude and Phase response

(e) $x[n]$ as depicted in Fig. P4.17 (c).

$$N = 4 \qquad \Omega_o = \frac{\pi}{2}$$

$$\begin{aligned}
X[k] &= \frac{1}{4}\left(1 + e^{-jk\frac{\pi}{2}} - e^{-jk\pi} - e^{-jk\frac{3\pi}{2}}\right) \\
&= \frac{1}{4}(1-(-1)^k) - \frac{j}{2}\sin(k\frac{\pi}{2}) \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\frac{\pi}{2})
\end{aligned}$$

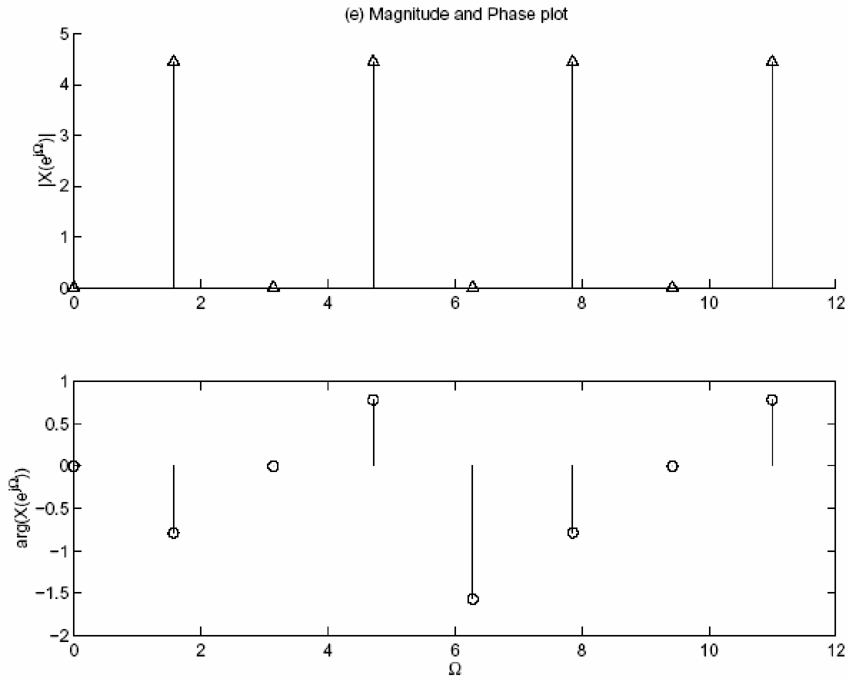

Figure P4.17. (e) Magnitude and Phase response

4.20. Consider the system depicted in Fig. P4.20 (a). The FT of the input signal is depicted in Fig. 4.20 (b). Let $z(t) \stackrel{FT}{\longleftrightarrow} Z(j\omega)$ and $y(t) \stackrel{FT}{\longleftrightarrow} Y(j\omega)$. Sketch $Z(j\omega)$ and $Y(j\omega)$ for the following cases.
(a) $w(t) = \cos(5\pi t)$ and $h(t) = \frac{\sin(6\pi t)}{\pi t}$

$$\begin{aligned}
Z(j\omega) &= \frac{1}{2\pi} X(j\omega) * W(j\omega) \\
W(j\omega) &= \pi\left(\delta(\omega - 5\pi) + \delta(\omega + 5\pi)\right) \\
Z(j\omega) &= \frac{1}{2}\left(X(j(\omega - 5\pi)) + X(j(\omega + 5\pi))\right) \\
H(j\omega) &= \begin{cases} 1 & |\omega| < 6\pi \\ 0 & \text{otherwise} \end{cases} \\
C(j\omega) &= H(j\omega)Z(j\omega) = Z(j\omega) \\
Y(j\omega) &= \frac{1}{2\pi} C(j\omega) * \left[\pi\left(\delta(\omega - 5\pi) + \delta(\omega + 5\pi)\right)\right] \\
Y(j\omega) &= \frac{1}{2}\left[Z(j(\omega - 5\pi)) + Z(j(\omega + 5\pi))\right]
\end{aligned}$$

$$= \frac{1}{4}\left[X(j(\omega - 10\pi)) + 2X(j\omega) + X(j(\omega + 10\pi))\right]$$

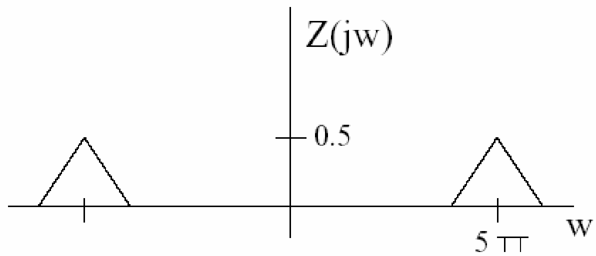

Figure P4.20. (a) Sketch of $Z(j\omega)$

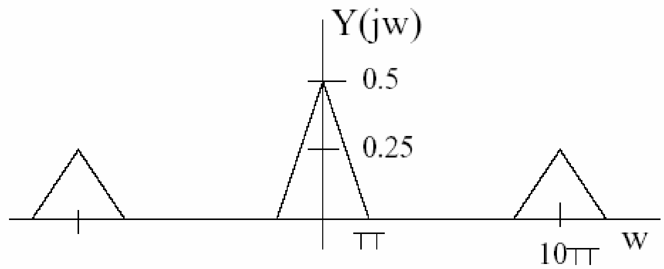

Figure P4.20. (a) Sketch of $Y(j\omega)$

(b) $w(t) = \cos(5\pi t)$ and $h(t) = \frac{\sin(5\pi t)}{\pi t}$

$$\begin{aligned}
Z(j\omega) &= \frac{1}{2}\left[X(j(\omega - 5\pi)) + X(j(\omega + 5\pi))\right] \\
H(j\omega) &= \begin{cases} 1 & |\omega| < 5\pi \\ 0 & \text{otherwise} \end{cases} \\
C(j\omega) &= H(j\omega)Z(j\omega) \\
Y(j\omega) &= \frac{1}{2\pi}C(j\omega) * \left[\pi\left(\delta(\omega - 5\pi) + \delta(\omega + 5\pi)\right)\right] \\
Y(j\omega) &= \frac{1}{2}\left[C(j(\omega - 5\pi)) + C(j(\omega + 5\pi))\right]
\end{aligned}$$

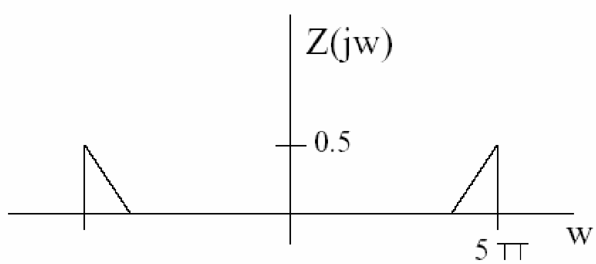

Figure P4.20. (b) Sketch of $Z(j\omega)$

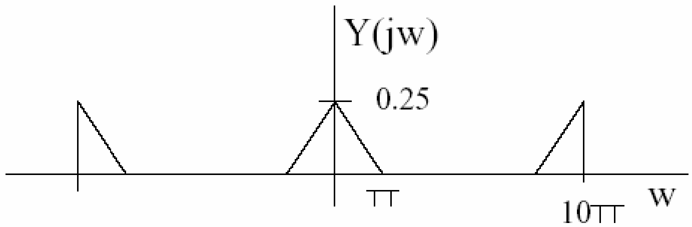

Figure P4.20. (b) Sketch of $Y(j\omega)$

(c) $w(t)$ depicted in Fig. P4.20 (c) and $h(t) = \frac{\sin(2\pi t)}{\pi t}\cos(5\pi t)$
$T = 2,\ \omega_o = \pi,\ T_o = \frac{1}{2}$

$$
\begin{aligned}
W(j\omega) &= \sum_{k=-\infty}^{\infty} \frac{2\sin(k\frac{\pi}{2})}{k}\delta(\omega - k\pi) \\
Z(j\omega) &= \frac{1}{2\pi}X(j\omega) * W(j\omega) \\
&= \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{2})}{k\pi}X(j(\omega - k\pi)) \\
C(j\omega) &= H(j\omega)Z(j\omega) \\
&= \sum_{k=3}^{7} \frac{\sin(k\frac{\pi}{2})}{k\pi}X(j(\omega - k\pi)) + \sum_{k=-3}^{-7} \frac{\sin(k\frac{\pi}{2})}{k\pi}X(j(\omega - k\pi)) \\
Y(j\omega) &= \frac{1}{2}\left[C(j(\omega - 5\pi)) + C(j(\omega + 5\pi))\right]
\end{aligned}
$$

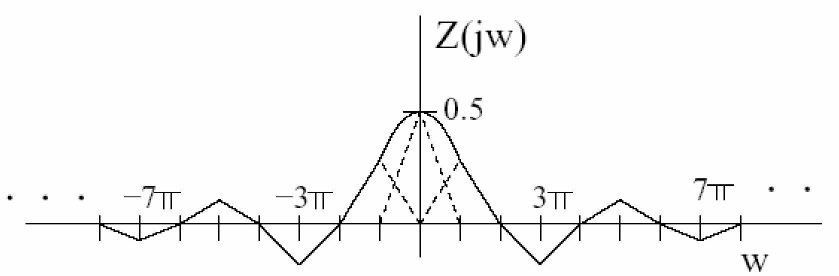

Figure P4.20. (c) Sketch of $Z(j\omega)$

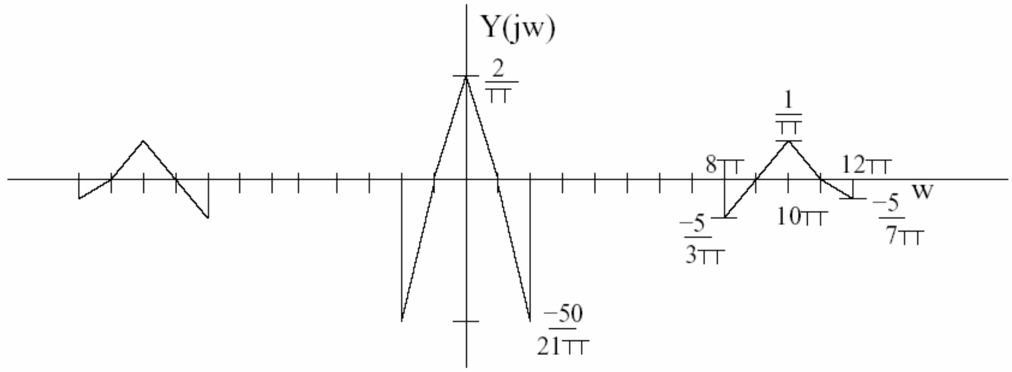

Figure P4.20. (c) Sketch of $Y(j\omega)$