Introduction to Wavelet

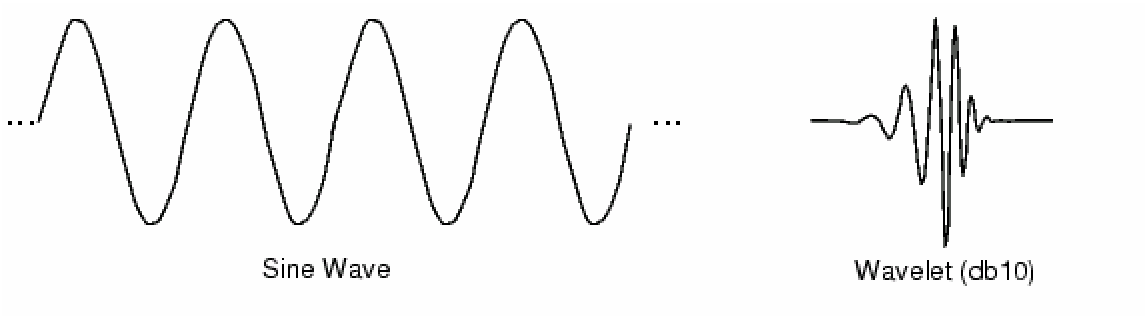
Based on A. Mukherjee's lecture notes

Contents

Wavelet Definition

"The wavelet transform is a tool that cuts up data, functions or operators into different frequency components, and then studies each component with a resolution matched to its scale"

---- Dr. Ingrid Daubechies, Lucent, Princeton U



Wavelet Coding Methods

- EZW - Shapiro, 1993
  - Embedded Zerotree coding.
- SPIHT - Said and Pearlman, 1996
  - Set Partitioning in Hierarchical Trees coding. Also uses “zerotrees”.
- ECECOW - Wu, 1997
  - Uses arithmetic coding with context.
- EBCOT – Taubman, 2000
  - Uses arithmetic coding with different context.
- JPEG 2000 – new standard based largely on EBCOT

Comparison of Wavelet Based JPEG 2000 and DCT Based JPEG

- JPEG2000 image shows almost no quality loss from current JPEG, even at 158:1 compression.

Introduction to Wavelets

- *"... the new computational paradigm [wavelets] eventually may swallow Fourier transform methods..."*

- *" ...a new approach to data crunching that, if successful, could spawn new computer architectures and bring about commercial realization of such difficult data-compression tasks as sending images over telephone lines. "*

---- from "New-wave number crunching" C. Brown, Electronic Engineering Times, 11/5/90.

Timeline

Wavelets have had an unusual scientific history, marked by many independent discoveries and rediscoveries. The most rapid progress has come since the early 1980s, when a coherent mathematical theory of wavelets finally emerged.

1807
Jean Baptiste Joseph Fourier claims that any periodic function, or wave, can be expressed as an infinite sum of sine and cosine waves of various frequencies. Because of serious doubts over the correctness of his arguments, his paper is not published until 15 years later.

1909
Alfred Haar, a Hungarian mathematician, discovers a "basis" of functions that are now recognized as the first wavelets. They consist of a short positive pulse followed by a short negative pulse.

1930
John Littlewood and R.A.E.C. Paley, of Cambridge University, show that local information about a wave, such as the timing of a pulse of energy, can be retrieved by grouping the terms of its Fourier series into "octaves."

1946
Dennis (Denes) Gabor, a British-Hungarian physicist who invented holography, decomposes signals into "time-frequency packets" or "Gabor chirps."

1976
IBM physicists Claude Galand and Daniel Esteban discover subband coding, a way of encoding digital transmissions for the telephone.

1981
Petroleum engineer Jean Morlet of Elf-Aquitaine finds a way to decompose seismic signals into what he calls "wavelets of constant shape." He turns to quantum physicist Alex Grossmann for help in proving that the method works.

1984
Joint paper by Morlet and Grossmann brings the word "wavelet" into the mathematical lexicon for the first time.

1985
Yves Meyer of the University of Paris discovers the first smooth orthogonal wavelets.

1986
Stéphane Mallat, then at the University of Pennsylvania, shows that the Haar basis, the Littlewood-Paley octaves, the Gabor chirps, and the subband filters of Galand and Esteban are all related to wavelet-based algorithms.

1987
Ingrid Daubechies constructs the first smooth orthogonal wavelets with compact support. Her wavelets turn the theory into a practical tool that can be easily programmed and used by any scientist with a minimum of mathematical training.

1990
David Donoho and Iain Johnstone, at Stanford University, use wavelets to "denoise" images, making them even sharper than the originals.

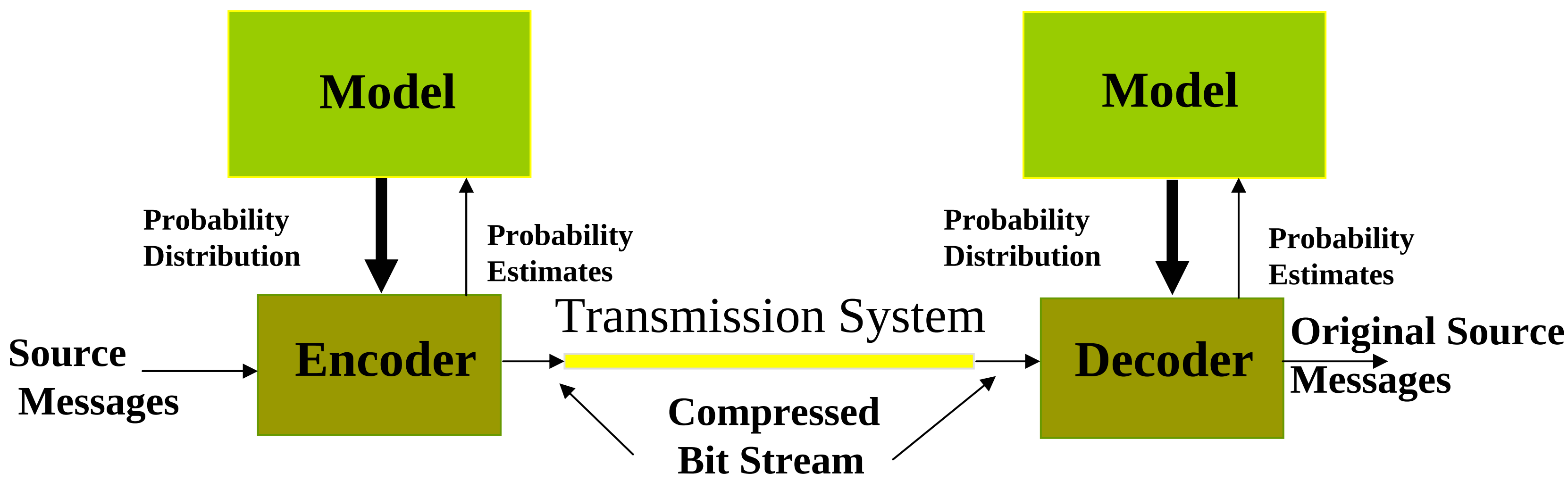
1992
The FBI chooses a wavelet method developed by Tom Hopper of the FBI's Criminal Justice Information Services Division and Jonathan Bradley and Chris Brislawn from Los Alamos National Laboratory, to compress its enormous database of fingerprints.

1995
Pixar Studios releases the movie *Toy Story*, the first fully computer-animated cartoon. In the sequel, *Toy Story 2*, some shapes are rendered by subdivision surfaces, a technique mathematically related to wavelets.

1999
The International Standards Organization approves a new standard for digital picture compression, called JPEG-2000. The new standard uses wavelets to compress image files by 1:200 ratios with no visible loss in image quality. Web browsers are expected to support the new standard by 2001.

Model and Prediction

- Compression is **PREDICTION**.
- There are many **decomposition** approaches to modeling the signal.
  - Every signal is a function.
  - Modeling is function representation/approximation.



Methods of Function Approximation

- Sequence of samples
  - Time domain
- Pyramid (hierarchical)
- Polynomial
- Piecewise polynomials
  - Finite element method
- Fourier Transform
  - Frequency domain
  - Sinusoids of various frequencies
- Wavelet Transform
  - Time/frequency domain

The Fourier Transform

- Analysis, forward transform:

$$F(u) = \int f(t) e^{-j2\pi ut} dt$$

- Synthesis, inverse transform:

$$f(t) = \int F(u) e^{j2\pi ut} du$$

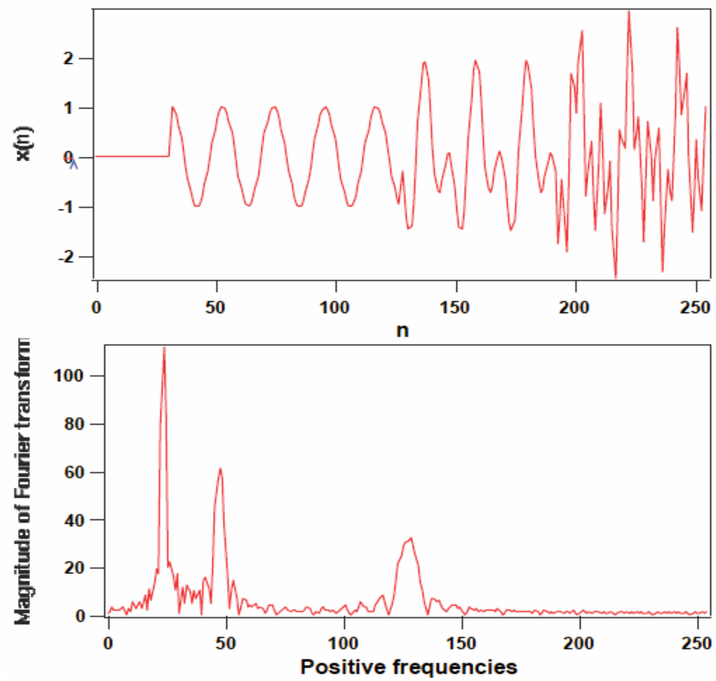
- Forward transform decomposes f(t) into sinusoids.
  - F(u) represents how much of the sinusoid with frequency u is in f(t).
- Inverse transform synthesizes f(t) from sinusoids, weighted by F(u).

The Fourier Transform Properties

- Linear Transform.
- Analysis (decomposition) of signals into sines and cosines has physical significance
  - tones, vibrations.
- Fast algorithms exist.
  - The fast Fourier transform requires O(nlogn) computations.

Problems With the Fourier Transform

- Fourier transform well-suited for stationary signals – statistics of the signals do not vary with time. This model does not fit real signals well.
- For time-varying signals or signals with abrupt transitions, the Fourier transform does not provide information on when transitions occur.



Problems With the Fourier Transform

- Fourier transform is a “global” analysis. A small perturbation of the function at any one point on the time-axis influences all points on the frequency-axis and vise versa.

- A qualitative explanation of why Fourier transform fails to capture time information is the fact that the set of basis functions (sines and cosines) are infinitely long and the transform picks up the frequencies regardless of where it appears in the signal.

- Need a better way to represent functions that are localized in both time and frequency.

Uncertainty Principle
--- Preliminaries for the STFT

- The time and frequency domains are complimentary.
  - If one is local, the other is global.
  - For an impulse signal, which assumes a constant value for a very brief period of time, the frequency spectrum is infinite.
  - If a sinusoidal signal extends over infinite time, its frequency spectrum is a single vertical line at the given frequency.
- We can always localize a signal or a frequency but we cannot do that simultaneously.
  - If the signal has a short duration, its band of frequency is wide and vice versa.

Uncertainty Principle

- Heisenberg’s uncertainty principle was enunciated in the context of quantum physics which stated that the position and the momentum of a particle cannot be precisely determined simultaneously.

- This principle is also applicable to signal processing.

Uncertainty Principle -- In Signal Processing

Let $g(t)$ be a function with the property .

Then $$\int_{-\infty}^{\infty} \left| g(t)^2 \right| dt = 1$$

$$\left(\int_{-\infty}^{\infty} (t - t_m)^2 \left| g(t) \right|^2 dt\right)\left(\int_{-\infty}^{\infty} (f - f_m)^2 \left| G(f) \right|^2 dt\right) \geq \frac{1}{16\Pi^2}$$

where t_m, f_m denote average values of t and f ,and $G(f)$ is the Fourier transform of $g(t)$.

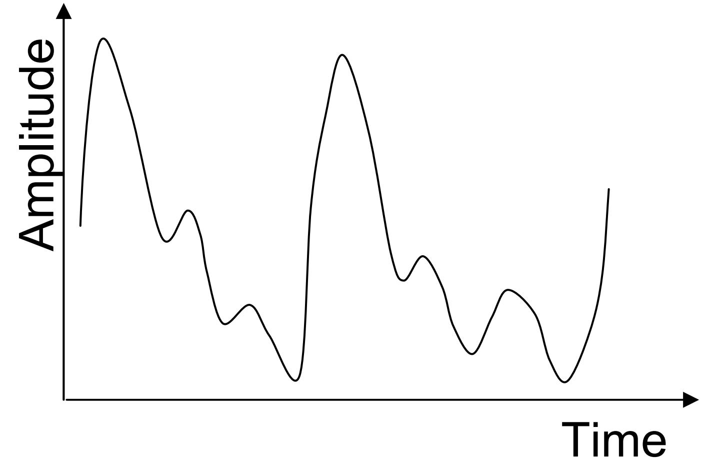
$$t_m = \int t \mid g(t) \mid^2 dt$$

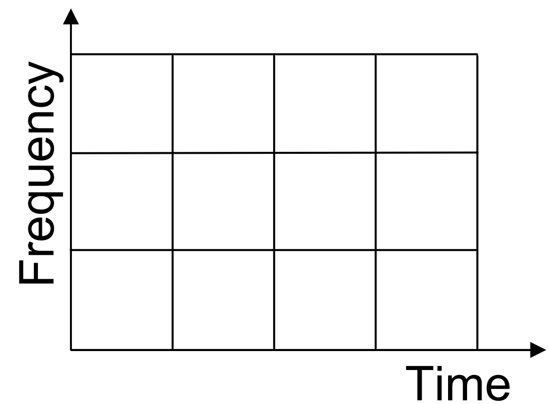
$$f_m = \int f \mid G(t) \mid^2 dt$$

Gabor's Proposal
Short-time Fourier Transform.

- The STFT is an attempt to alleviate the problems with FT.

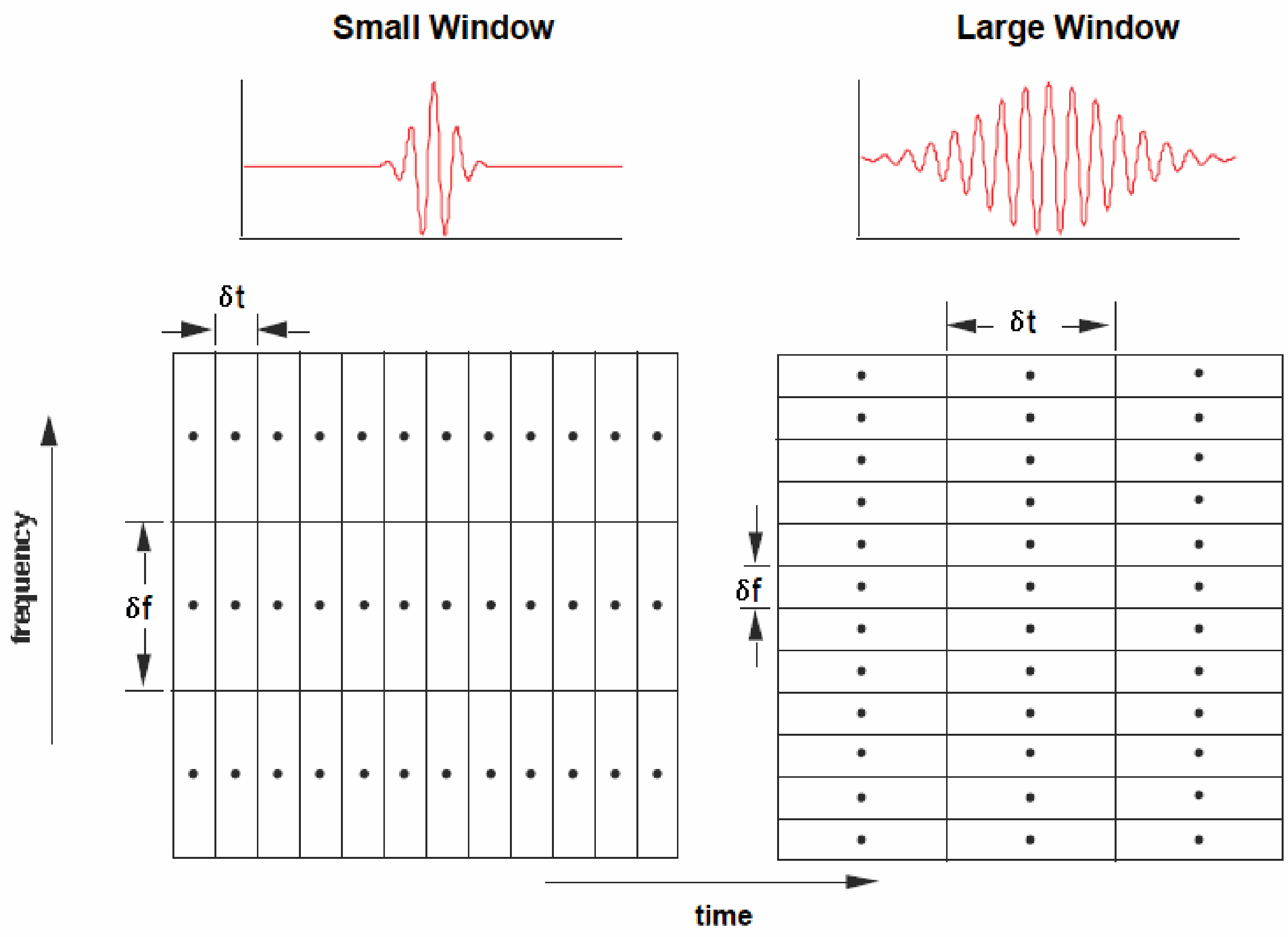
- It takes a non-stationary signal and breaks it down into "windows" of signals for a specified short period of time and does Fourier transform on the window by considering the signal to consist of repeated windows over time.





STFT

The Short-time Fourier Transform Time-frequency Resolution



The Short-time Fourier Transform

- Analysis:

$$STFT(\tau,u)=\int f(t)w^*(t-\tau)e^{-j2\pi ut}dt$$

- Synthesis:

$$f(t)=\int STFT(\tau,u)w(t-\tau)e^{j2\pi ut}d\tau du$$

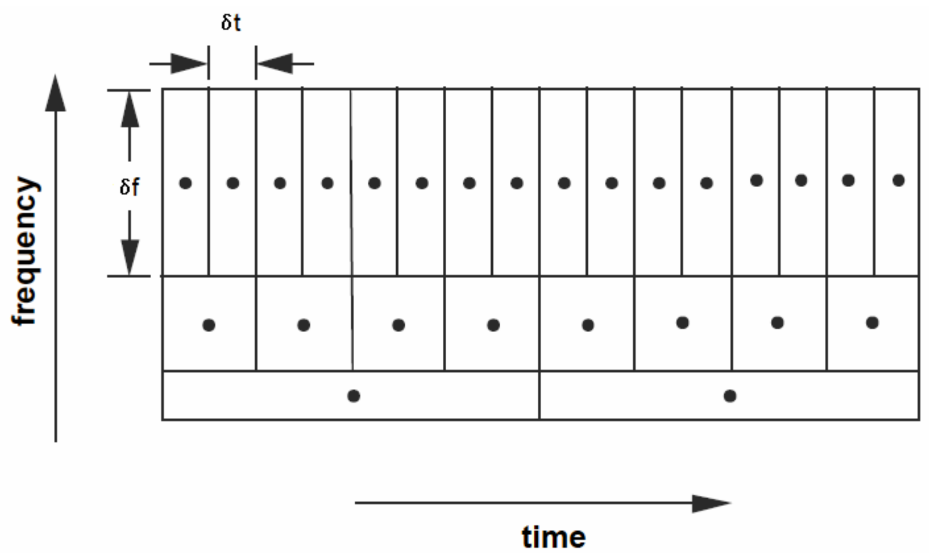
- where w(t) is a window function localized in time and frequency.
- Analysis functions are sinusoids windowed by w(t).
- Common window functions
  - Gaussian (Gabor), Hamming, Hanning, Kaiser.

The Short-time Fourier Transform Properties

- Linear transform.
- Time resolution (Δt) and frequency resolution (Δu) are determined by w(t), and remain fixed regardless of τor u.
- Biggest disadvantage:
  - since Δt and Δu are fixed by choice of w(t), need to know a priori what w(t) will work for the desired application.

Basic Idea of the Wavelet Transform -- Time-frequency Resolution

- Basic idea:
  - Δt, Δu vary as a function of scale (scale = 1/frequency).



Wavelet Transformation

- Analysis

$$WT(s,\tau)=\langle \psi_{\tau,s}, f(t)\rangle = \frac{1}{\sqrt{s}}\int f(t)\psi\left(\frac{t-\tau}{s}\right)dt$$

s : scale

τ : position

WT: coefficient

- Synthesis

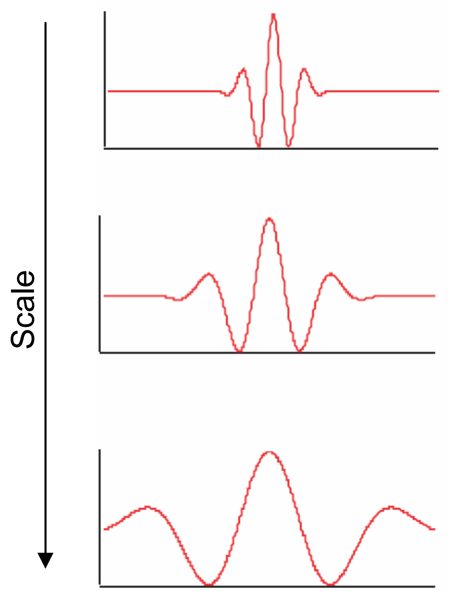
$$x(t)=\frac{1}{C_\psi\sqrt{s}}\iint_{s>0} WT(s,\tau)\psi\left(\frac{t-\tau}{s}\right)ds\,d\tau + c\Phi(t)$$

  - where $\psi(t)$ is the mother wavelet.
  - admissibility condition
  -

$$C_\psi = \int_0^\infty \frac{\left|\Psi(f)\right|^2}{f}df < \infty \qquad \int_{-\infty}^{\infty}\psi_{s,\tau}(t)=0$$

Scaling

- Scaling = frequency band
- Small scale
  - Rapidly changing details,
  - Like high frequency
- Large scale
  - Slowly changing details
  - Like low frequency



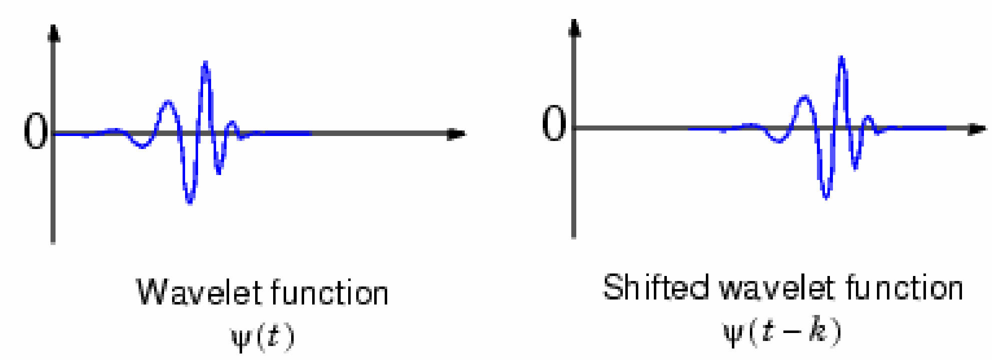
Wavelet Basis functions at 3 different scales

More on Scale

- It lets you either narrow down the frequency band of interest, or determine the frequency content in a narrower time interval
- Good for non-stationary data
- Low scale→ *a* Compressed wavelet→ Rapidly changing details→ High frequency.
- High scale →*a* Stretched wavelet → Slowly changing, coarse features → Low frequency.

Shifting

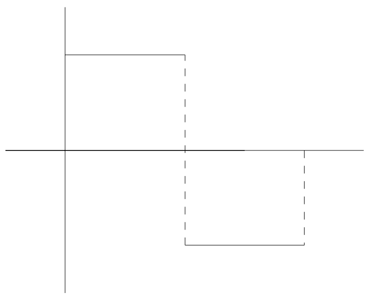
- Shifting a wavelet simply means delaying (or hastening) its onset.
- Mathematically, shifting a function f(t) by *k* is represented by f(t-k).



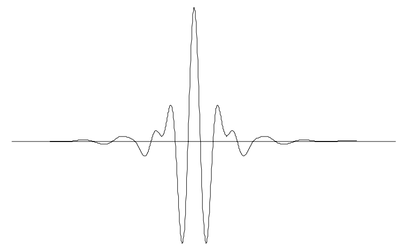
The Wavelet Transform Properties

- Linear transform.
- All analysis functions $\psi_{s,\tau} = \psi(\frac{t-\tau}{s})$ are shifts and dilations of the mother wavelet $\psi(t)$
- Time resolution and frequency resolution vary as a function of scale.
- Continuous wavelet transform (CWT)
  - s and τ are continuous.
- Discrete wavelet transform (DWT)
  - s and τ are discrete.

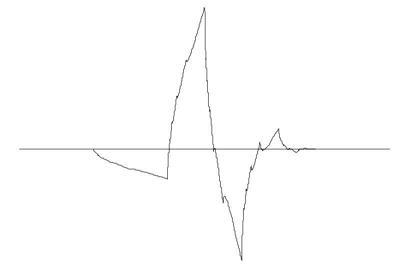
Different Types of Mother Wavelets



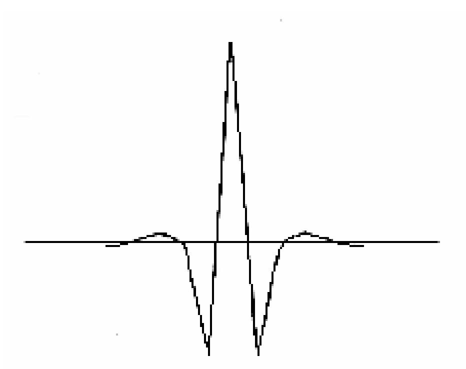
Haar



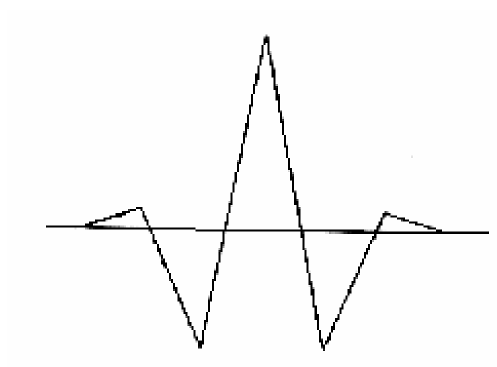
Meyer



Daubechies



Battle-Lemarie



Chui-Wang

Calculate the CWT Coefficients

- The result of the CWT are many *wavelet coefficients* *WT*

$$WT(s,\tau) = \langle \psi_{\tau,s}, f(t) \rangle = \frac{1}{\sqrt{s}} \int f(t)\psi(\frac{t-\tau}{s})dt$$

  - Function of scale and position.
- How to calculate the coefficient?

```
for each SCALE s
    for each POSITION t
        WT (s, t) =   Signal  x Wavelet (s, t)
    end
end
```

Calculate the CWT Coefficients

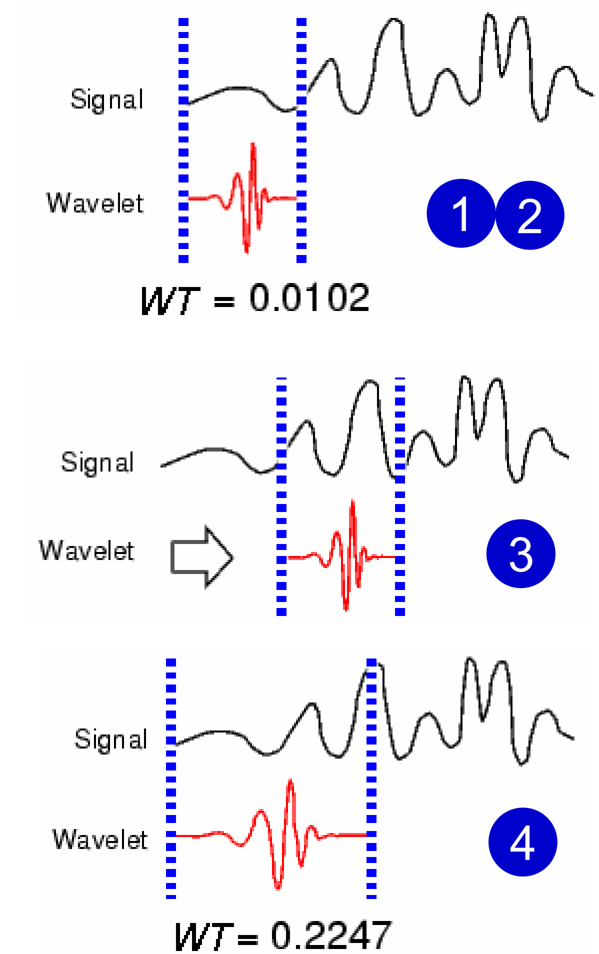
1. Take a wavelet and compare it to a section at the start of the original signal.

2. Calculate a correlation coefficient *WT*

3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.

4. Scale (stretch) the wavelet and repeat steps 1 through 3.

5. Repeat steps 1 through 4 for all scales.



Discrete Wavelet Transform

- Calculating wavelet coefficients at every possible scale is a fair amount of work, and it generates an awful lot of data.
- What if we choose only a subset of scales and positions at which to make our calculations?
- It turns out, rather remarkably, that if we choose scales and positions based on powers of two --- so-called *dyadic* scales and positions --- then our analysis will be much more efficient and just as accurate.

Discrete Wavelet Transform

- If (s, τ) take discrete value in R^2, we get DWT.
- A popular approach to select (s, τ) is

$$s = \frac{1}{s_0^m} \qquad s_0 = 2 \quad \rightarrow \quad s = \frac{1}{2^m} = <1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots>, \qquad \text{m: integer}$$

$$\tau = \frac{n\tau_0}{s_0^m} \qquad s_0 = 2\,, \quad \tau_0 = 1, \qquad \tau = \frac{n}{2^m} \qquad \text{n, m: integer}$$

- So,

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right) = 2^{m/2}\psi\left(\frac{t-\frac{n}{2^m}}{\frac{1}{2^m}}\right) = \psi_{m,n}(t) = 2^{m/2}\psi\left(2^m t - n\right)$$

Discrete Wavelet Transform

- Wavelet Transform:

$$DWT_{m,n} = < f(t), \psi_{m,n}(t) > = 2^{m/2} \int f(t) \psi(2^m t - n) dt$$

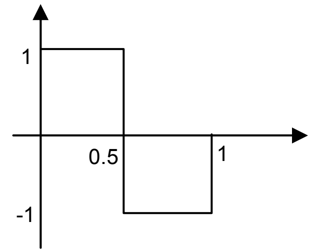
- Inverse Wavelet Transform

$$f(t) = \sum_{m} \sum_{n} DWT_{m,n} \psi_{m,n}(t) + c\Phi(t)$$

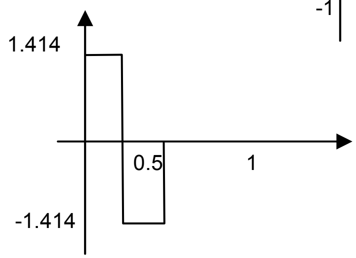
- If *f(t)* is continuous while *(s, τ)* consists of discrete values, the series is called the Discrete Time Wavelet Transform (DTWT).

- If *f(t)* is sampled (that is, discrete in time, as well as *(s, τ)* are discrete, we get Discrete Wavelet Transform (DWT).

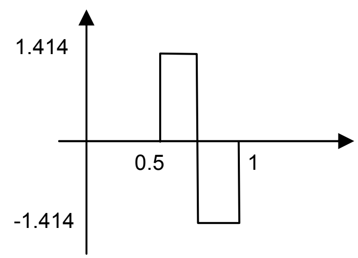
$$\psi_{0,0}(t) = \begin{cases} 1 & 0 \leq \mathrm{t} < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \end{cases}$$



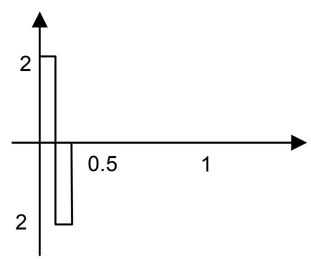
$$\psi_{1,0}(t) = \sqrt{2}\psi(2t)$$



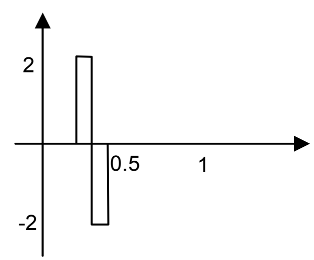
$$\psi_{1,1}(t) = \sqrt{2}\psi(2t-1)$$



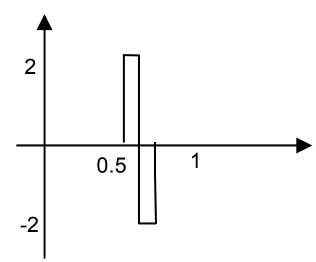
$$\psi_{2,0}(t) = 2\psi(4t)$$



$$\psi_{2,1}(t) = 2\psi(4t-1)$$



$$\psi_{2,2}(t) = 2\psi(4t-2)$$

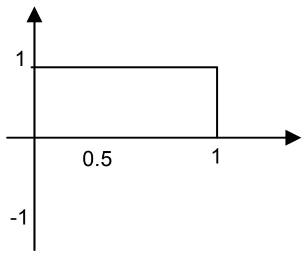


Haar Scaling function

- The Haar transform uses a scaling function and wavelet functions.

- Scaling function
  - Calculate scaling function

$$c = < f(t), \Phi(t) > = \int f(t)\Phi(t)dt$$



  - Synthesis the original signal

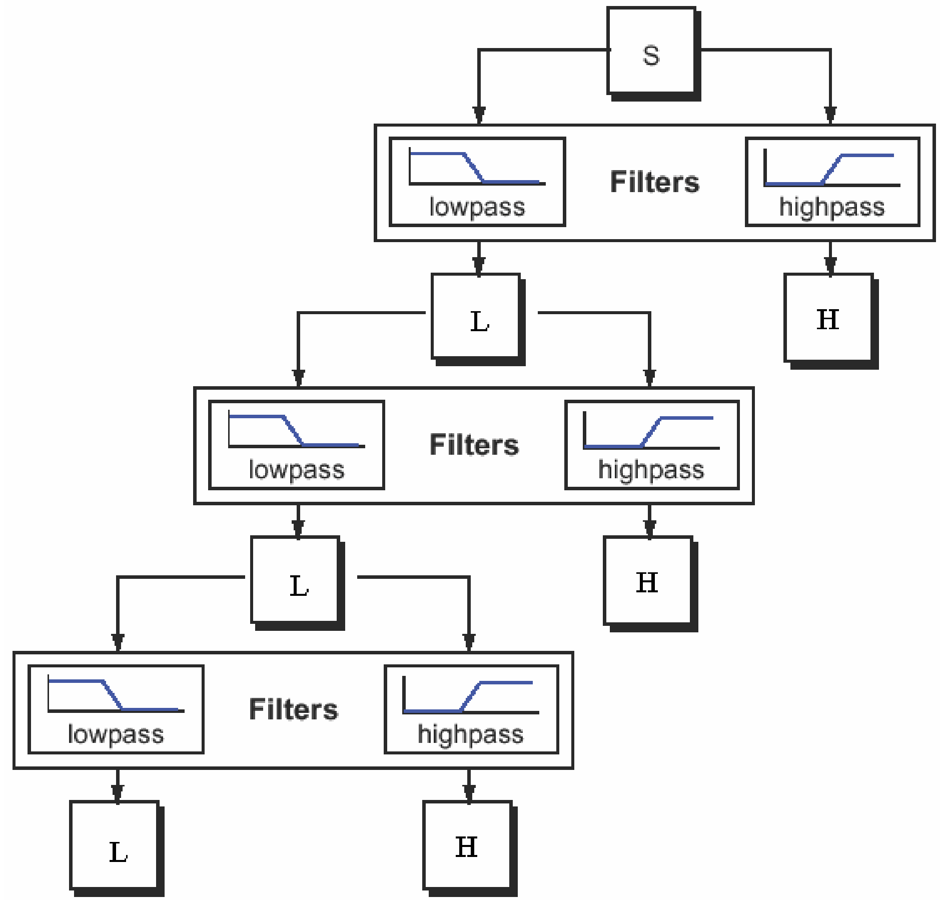
$$f(t) = \sum_m \sum_n DWT_{m,n}\psi_{m,n}(t) + c\Phi(t)$$

Example of Fast DWT, Haar Wavelet

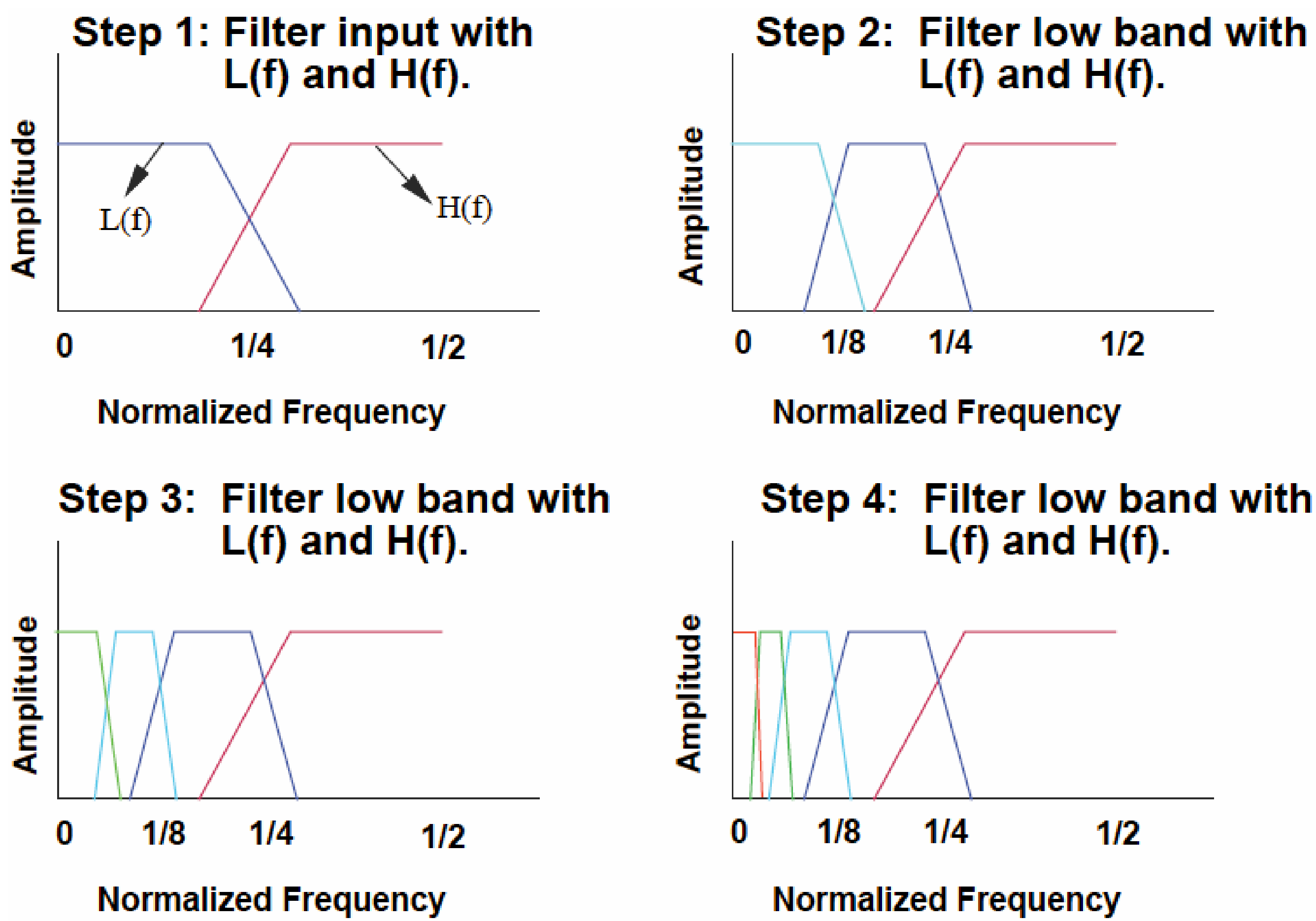
- Given input value {1, 2, 3, 4, 5, 6, 7, 8}
- Step #1
  - Output Low Frequency {1.5, 3.5, 5.5, 7.5}
  - Output High Frequency {-0.5, -0.5, -0.5, -0.5}
- Step #2
  - Refine Low frequency output in Step #1
    - L: {2.5, 6.5}
    - H: {-1, -1}
- Step #3
  - Refine Low frequency output in Step #2
    - L: {4.5}
    - H: {-2}

Fast Discrete Wavelet Transform

- Behaves like a filter bank
  - signal in
  - coefficients out
- Recursive application of a two-band filter bank to the lowpass band of the previous stage.



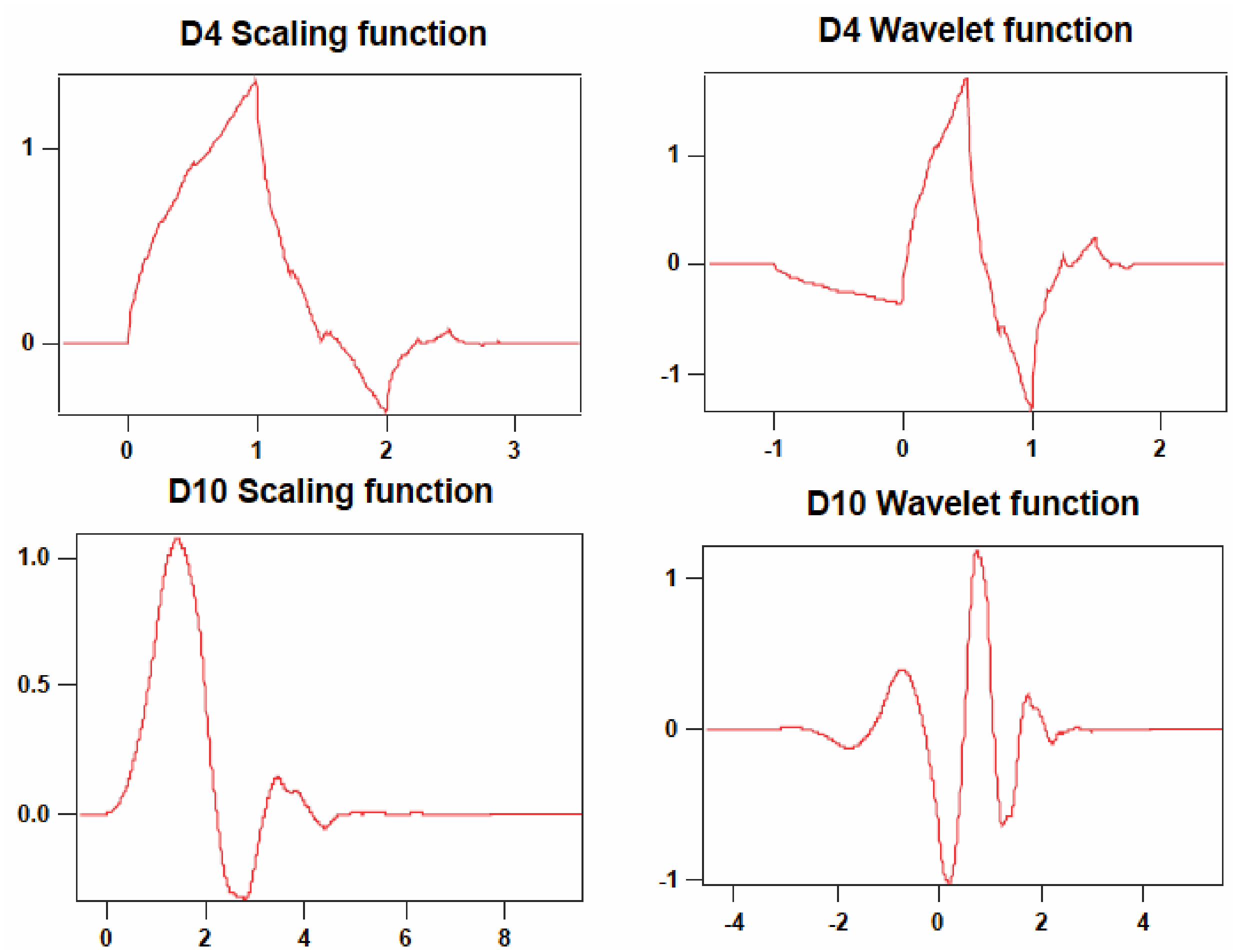
Fast Discrete Wavelet Transform



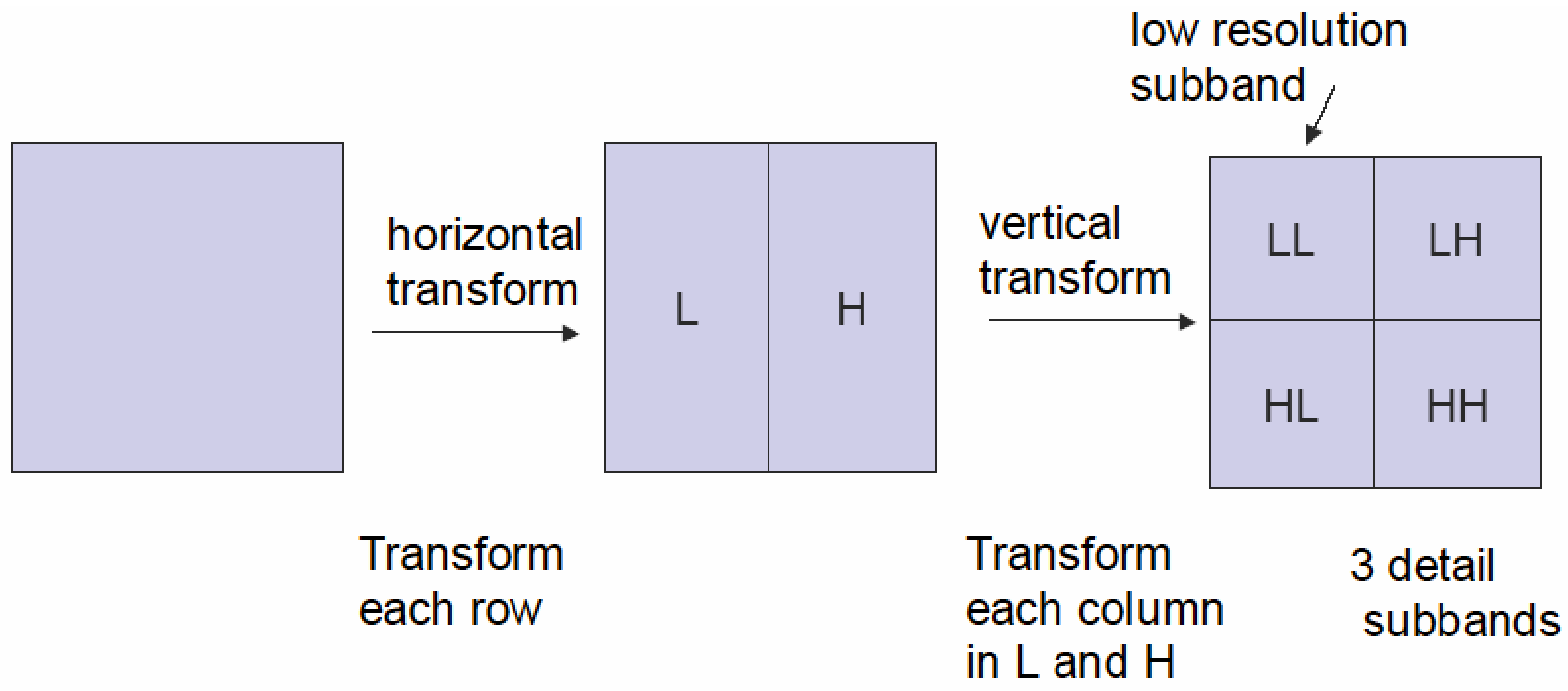
Fast Wavelet Transform Properties

- Algorithm is very fast
  - O(n) operations.
- Discrete wavelet transform is not shift invariant.
  - A deficiency.
- Key to the algorithm is the design of h(n) and l(n).
  - Can h(n) and l(n) be designed so that wavelets and scaling function form an orthonormal basis?
  - Can filters of finite length be found?
  - YES.
    - Daubechies Family of Wavelets
    - Haar basis is a special case of Daubechies wavelet.

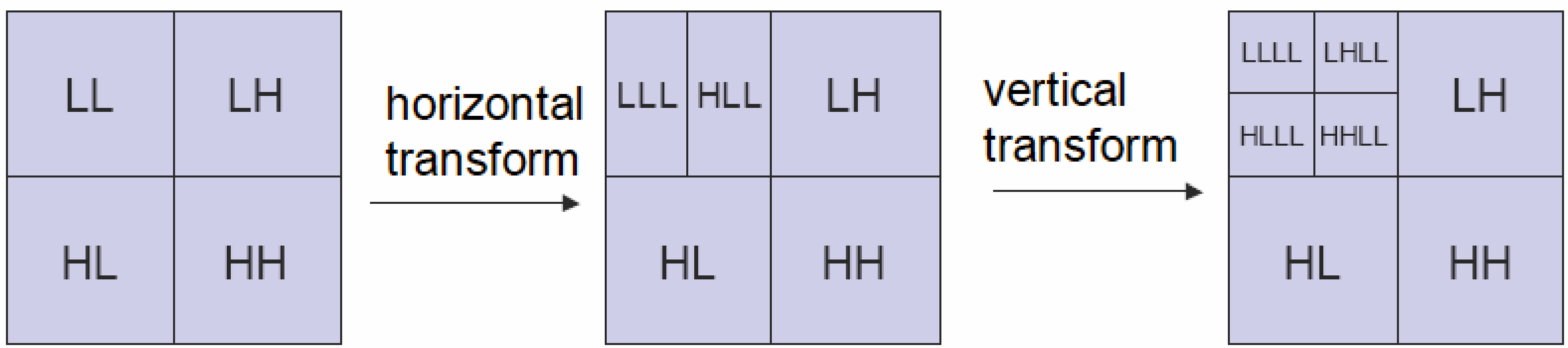
Daubechies Family of Wavelets Examples



Two Dimensional Transform



Two Dimensional Transform (Continued)



Transform each row in LL

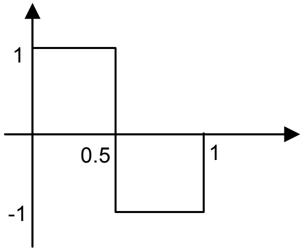
Transform each column in LLL and HLL

2 levels of transform gives 7 subbands.
k levels of transform gives 3k + 1 subbands.

1D Haar Wavelet

- Mother Wavelet

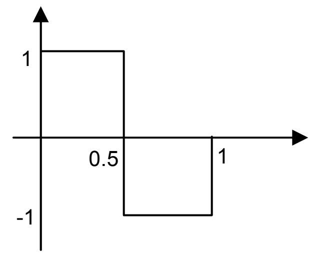
$$\psi_{0,0}(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$



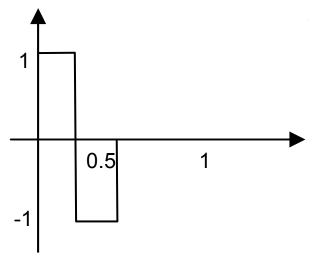
- Wavelet basis function

$$\psi_{j,i}(t) = \psi(2^j t - i)$$

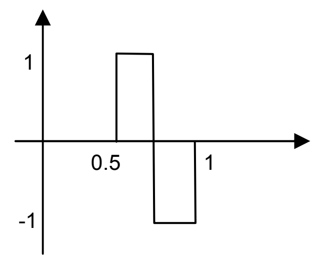
$$\psi_{0,0}(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$



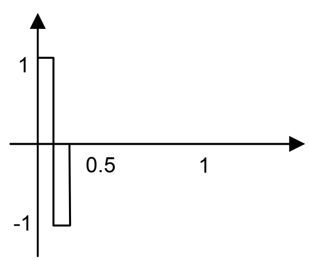
$$\psi_{1,0}(t) = \psi(2t)$$



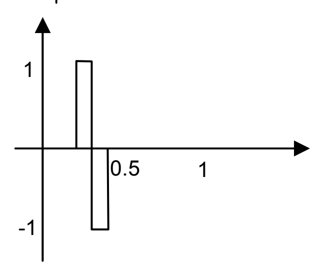
$$\psi_{1,1}(t) = \psi(2t-1)$$



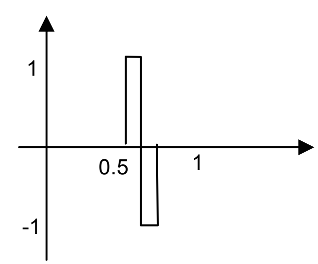
$$\psi_{2,0}(t) = \psi(4t)$$



$$\psi_{2,1}(t) = \psi(4t-1)$$



$$\psi_{2,2}(t) = \psi(4t-2)$$



Wavelet Transformed Image

- Three levels of Wavelet transform.
  - One low resolution subband.
  - Nine detail subband.