Z_min and Z_max of Standing Waves

Sometimes we can determine the distance from the load to the first minimum (Z_min) or the distance to the first maximum (Z_max) of the standing wave ratio pattern.

Then we could find the phase angle of the reflection coefficient and be able to determine what the load "looks" like.

The instrument to do this is called a slotted line; not often used except in demonstrations or in modest applications.

The slotted line is simply a calibrated length of coaxial line with a movable probe that may be slid up or down the line. The probe slides in a longitudinal line and is close to, but does not touch, the inner conductor.

The probe is terminated in a simple diode detector that measures the voltage amplitude of the center conductor against the outer shield.
Earlier we saw that:

\[ V(z') = V_0^+ e^{j\beta z'} + V_0^- \prod \left[ e^{j\theta} - e^{-j\beta z'} \right] \]

And since \( \prod = |\prod| e^{j\theta} \),

\[ = V_0^+ e^{j\beta z'} + V_0^- |\prod| e^{j\theta} e^{-j\beta z'} \]

because \( e^a e^b = e^{a+b} \)

\[ = V_0^+ e^{j\beta z'} \left[ 1 + |\prod| e^{j(\theta - 2\beta z')} \right] \]

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This could also be considered as \( \prod(z') \), a generalized voltage reflection coefficient that can be found at any point \( z' \) on the line. Important points:

1) Magnitude of the reflection coefficient is unchanged across a lossless line.

2) The phase of the reflection coefficient changes at a rate of \( 2\pi \) the electrical distance from the load.

For current, we can likewise say:

\[ I(z') = \frac{V_0^+}{Z_0} e^{j\beta z'} \left[ 1 - |\prod| e^{j(\theta - 2\beta z')} \right] \]

We are now interested in how \( |V(z')| \) varies with distance \( z' \) from the load, especially looking for minimums or maximums.

\( |V(z')| + V(z') \) still. \( |V(z')| \) gives us a mono-polar value, \( V(z') \) a p-p value.

\[ V(z') \text{ or } \left< \begin{array}{c} \text{peak-to-peak} \end{array} \right> \]

\[ |V(z')| \text{ or } \left< \begin{array}{c} \text{mono-polar} \end{array} \right> \]

\[ \uparrow \text{ easier to find min & max!} \]
Z_{\text{min}} + Z_{\text{max}} \text{ of Standing Waves}

From before:
\[ V(z') = V_0 e^{j \beta z'} \left[ 1 + |\Gamma|^2 e^{j(\theta_L - 2\beta z')} \right] \]

- We would like to have the standing wave plot \(|V(z')|\) and find its maximum.
- The magnitude of \(V(z')\), \(|V(z')|\), is \(|V(z')| = \left| V(z') V^*(z') \right|^{1/2} \)
  where \(V^*(z')\) is the complex conjugate of \(V(z')\). We find that: \((\text{See page} 407)\)
- \(|V(z')| = |V_0|^2 \left[ (1 + |\Gamma|^2)^2 \cos^2 \left( \beta z' - \frac{\theta_L}{2} \right) + (1 - |\Gamma|^2)^2 \sin^2 \left( \beta z' - \frac{\theta_L}{2} \right) \right]\)

Maximize this term to find the maximum of \(|V(z')|\).

The maximum will be found where the argument of the \(\cos^2\) function is equal to \(n\pi\).
Since \(\cos(x)\) is at a maximum at \(n\pi\) where \(n = 0, 1, 2, ...\), so will \(\cos^2\).

- So, we can say the maximum standing wave voltages are found from the condition:
  \[ \beta z'_{\text{max}} - \frac{\theta_L}{2} = n\pi \quad n = 0, 1, 2, 3, ... \]
  thus, \(z'_{\text{max}} = \frac{n\pi + \theta_L}{\beta} = \frac{2n\pi + \theta_L}{2\beta}\); also \(\theta_L = z'_{\text{max}} 2\beta - 2n\pi\)

- The voltage minimums will be found away or when:
  \[ \beta z'_{\text{min}} - \frac{\theta_L}{2} = \frac{\pi}{2} + n\pi \quad n = 0, 1, 2 \]
Z_{min} + Z_{max} of Standing Waves

Being able to find a voltage max or min, including the first one measured from the load,

1. The wavelength of the signal \( \lambda \) on the line since points of \( V_{max}/V_{min} \) occur at \( \frac{\lambda}{2} \) intervals. This yields \( \beta \) then as \( \beta = \frac{2\pi}{\lambda} \)

2. If we know \( \lambda \), and thus \( \beta \), and have a relationship such as

\[ Z'_{max} = \frac{\Theta_L + 2n\pi}{2\beta} \]

we can find \( \Theta_L \), the phase angle of \( \Pi \), where \( \Theta_L = Z'_{max} - 2n\pi \)

A use \( \Theta_L \), again.

Then, if we know the SWR, which is related to \( |\Pi| \), we can fully characterize a transmission line terminated in an arbitrary load impedance.

\[ \text{SWR} = \frac{V_{max}}{V_{min}} = \frac{1 + |\Pi|}{1 - |\Pi|} \Rightarrow \text{gives } |\Pi| \]

\[ Z'_{max} = \frac{\Theta_L + 2n\pi}{2\beta} \]

\[ \Rightarrow \text{gives } e^{i\Theta} \]

\[ Z'_{min} = \frac{\Theta_L + (2n+1)\pi}{2\beta} \]

\[ \Rightarrow |\Pi| e^{i\Theta} = \Pi \]
Calculate $\Gamma_L$, $|\Gamma_L|$, $V_{max}$, $V_{min}$, SWR, $\theta_L$

Sketch $|V(z')|$ as a function of $\frac{Z}{\lambda}$

\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{12.5 - 50}{12.5 + 50} = -0.6, \text{ also expressed as } 0.6 e^{j\pi} \Rightarrow \Gamma = -0.6, \quad |\Gamma| = 0.6, \quad \theta_L = 180^\circ
\]

\[V_{max} = |V'| (1 + |\Gamma_L|) = 1.6 (1 + 0.6) = 2.56 V\]
\[V_{min} = |V'| (1 - |\Gamma_L|) = 1 (1 - 0.6) = 0.4 V\]

\[\text{SWR} = \frac{|V_{max}|}{|V_{min}|} = \frac{1.6}{0.4} = 4\] (A pretty bad match; most transmitters require SWR < 2)

To sketch $|V(z')|$, determine where first maximum is. This is where $\beta Z'_{max} = \frac{\theta_L}{2} = \frac{n \pi}{n}$ ($n = 0, 1, 2, ...$)

Closest location is where $n=0$.

Since $\beta = \frac{2 \pi}{\lambda}$ and $\theta_L = \pi$, and $n=0$, $\beta Z'_{max} = \frac{\pi}{2} = 0$; $\lambda$

\[Z'_{max} = \frac{\lambda}{\beta} = \frac{\lambda}{\frac{2 \pi}{\lambda}} = \frac{\lambda}{\frac{2 \pi}{\lambda}} = \frac{\lambda}{2 \pi} = \frac{\lambda}{2} \lambda\]

Note:
- Resistive loads will show this type of waveform
  - If $R_L < Z_0$, $|V(z')|$ will be a minimum
    - $R_L = Z_0$; maximum
  - Adjacent min or max values are $\lambda/4$ away
  - Pattern repeats every $\lambda/2$
  - If $R_L = 0$ min would be at load + $\frac{\lambda}{4}$ away Peak would occur, this what we saw before.
A more interesting case:

An SWR measurement on a line reveals an \( \text{SWR} = \frac{|V_{\text{max}}|}{|V_{\text{min}}|} = 5 \) (this would typically be done with an SWR meter).

Distance between voltage minima is 20 cm.
Distance from termination to nearest voltage minima is 4 cm.

What is the load impedance \( Z_L \)?

Since minima are 20 cm apart, \( \gamma_2 = 20 \text{ cm} + \frac{\lambda}{2} = 40 \text{ cm} \)

Thus the distance from the load to the minima (as fraction of \( \lambda \)) is \( \frac{1}{40} = 0.1 \lambda \)

So the distance \( Z_{\text{max}} \) from the load to the first maxima is 14 cm or \( 0.35 \lambda \), \( 0.1 \lambda \)

At the first maxima (\( n=0 \)), \( \beta Z_{\text{max}} = \frac{\theta_L}{Z_0} \) or

\[ \theta_L = 2 \beta Z_{\text{max}} = \frac{(2)2\pi (35\lambda)}{1.414} \]

From above \( \text{SWR} = \frac{|V_{\text{max}}|}{|V_{\text{min}}|} = 5 \) so

\[ 1 = \frac{\text{SWR}_1}{\text{SWR}_2} = \frac{5-1}{5+1} = \frac{4}{6} = 0.666 \quad \rightarrow \quad \text{So } \Gamma_L = 0.666 e^{125.2^\circ} \]

\[ -0.206 + j0.1034 \quad (\text{the reflection coefficient has both magnitude and phase, unlike a resistive load}) \]

So, finally to find what \( Z_L \) is:

\[ Z_L = Z_o \left( \frac{1 + \Gamma_L}{1 - \Gamma_L} \right) = Z_o \left( \frac{0.7944 - j0.1034}{1.206 + j0.1034} \right) = Z_o \left( \frac{1.0169 e^{-j(28.42^\circ)}}{1.362 e^{j(27.73^\circ)}} \right) = Z_o \left( 0.795 e^{-j(66.33^\circ)} \right) \]

This is not to imply that \( Z_L \) changes \( Z_o \), \( Z_L \) is fixed! If \( Z_o \) changes, SWR would change, changing \( \Gamma_L \) and will give an unchanged \( Z_L \).
Standing Wave Patterns for Open, Short and Resistive Terminations
Standing wave patterns for inductive ($+jx$) and capacitive ($-jx$) loads

* hence why: $Z_{max} = \frac{\theta_L + 2\pi n}{2\beta}$ and since $\beta = \frac{\pi}{\lambda}$, at the closest point ($n=0$);

$Z_{max} = \frac{\theta_L}{2\beta} \implies 2\beta Z_{max} = \theta_L$; substitute in for $\beta$

$\frac{2\pi \theta_L}{\lambda} = \frac{\theta_L}{2}$

let us express the fraction as a function of $2\pi$ radians