First Order Circuits

- Usually contain one inductor or capacitor

\[ C \quad R \quad \text{OR} \quad L \quad \frac{1}{R} \]  

(2 caps or 2 inductors would make it a 2nd order circuit)

(can have a source although one is not shown)

- Can still be a 1st order circuit if we can reduce the # of L's or C's to one element.

- The behavior of this circuit is described by a first-order differential equation.

- The solution of this equation is known most often as the "natural response." (Also transients response)

- The natural response is the behavior of the circuit with no external sources of excitation.

- These types of circuits are very important.
  - Coupling networks
  - Compensation networks in control systems

- Since these circuits have no independent sources, they are called "source-free" circuits.

- When we introduce sources again, we will have a "complete response" comprised of a natural response (has characteristics only of the network) + forced response (has characteristics of the source in it) + complete response
**Source-Free RC + RL circuits**

- There are no independent sources, we depend on some initial conditions (stored energy)
  - For RC circuit → A charged capacitor
  - For RL circuit → An initial inductor current

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**Source-Free RC circuit**

- We assume the capacitor was charged prior to analysis to $V_0$.

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**I C L C at top node:**

\[
I_C + I_R = 0 \\
C \frac{dv}{dt} + \frac{v}{R} = 0 \\
\frac{dv}{v} + \frac{1}{RC} \, dt = 0 \\
\int_{V_0}^{V(t)} \frac{dv}{v} = -\frac{1}{RC} \int_0^t dt
\]

\[
\ln \frac{V}{V_0} = -\frac{1}{RC} t \\
\ln V = -\frac{1}{RC} t + \ln V_0 \\
V(t) = V_0 e^{-\frac{t}{RC}}
\]

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The voltage response of the RC circuit is an exponential decay of the initial voltage.
Response of an RC (source-free) circuit.

\[ V(t) = V_0 e^{-\frac{t}{\tau}} \]
where \( \tau = RC \)

At \( t = \tau \),
\[ V(t) = V_0 e^{-1} = 0.368V_0 \]

\( \tau \) is known as the time constant and is equal to \( (R \cdot C) \).

note that \( RC \) has units of seconds

\[ R \cdot C = \frac{\text{Volt} \cdot \text{Coulomb}}{\text{Volt} \cdot \text{Sec}} = \frac{\text{Coulomb}}{\text{Sec}} = \text{Sec} \]
Source-Free RL Circuits

- We assume that an initial inductor current is flowing at time zero: \( i(0) = I_0 \).

- We are looking for the circuit response as a function of inductor current \( i(t) \).

\[ V_L + V_r = 0 \]
\[ L \frac{di}{dt} + iR = 0 \]
\[ \frac{di}{dt} + \frac{1}{\tau} i = 0 \]

where \( \tau = \frac{L}{R} \)

let's find \( i(t) \) (current through inductor)

- KVL around loop:

\[ \int_{i(0)}^{i(t)} \frac{1}{i} \, di = -\frac{R}{L} \int_{0}^{t} \, dt \]

\[ \ln i - \ln i_0 = -\frac{R}{L} t \]

\[ e(\ln i) = e(-\frac{R}{L} t + \ln I_0) \]

\[ i(t) = I_0 e^{-\frac{R}{L} t} \]

- The current response for an RL circuit is an exponential decay of the initial current.

- As before if we let \( \tau = \frac{L}{R} \), we can write

\[ i(t) = I_0 e^{-\frac{1}{\tau} t} \]
Current response of the RL circuit

\[ i(t) \]

\[ .368 I_0 \]

where \( \tau = \frac{L}{R} \)

\( \frac{L}{R} \) has units of seconds

\[ V_L = L \frac{di}{dt} \implies L = \frac{\text{volt} \cdot \text{sec}}{\text{ampere}} \]

\[ \frac{L}{R} = \frac{\text{volt} \cdot \text{sec}}{\text{ampere}} \cdot \frac{\text{ampere}}{\text{volt}} = \frac{\text{volts} \cdot \text{sec}}{\text{volts}} = \text{sec} \]
Note the similarity between the RC and RL circuits:

\[ \text{RC: } V(t) = V_0 e^{-\frac{t}{\tau}} \quad \tau = RC \]
\[ \text{RL: } I(t) = I_0 e^{-\frac{t}{\tau}} \quad \tau = \frac{L}{R} \]

- Both circuits are described by 1st order diff. eq.
- Both circuits solution in the form:
  \[ X(t) = X(0)e^{-\frac{t}{\tau}} \]
  - \( \tau \): time constant, equal to
  - Initial voltage across cap or initial current through inductor
  - \( R_{TH} \) for RC ckt
  - \( \frac{L_{EQ}}{R_{TH}} \) for RL ckt

where \( R_{TH} \) = Thevenin resistance
\( L_{EQ} \) = Equivalent L
\( C_{EQ} \) = Equivalent C
Let's use these ideas:

- Switch has been in position 1 for a long time
- At t=0 it moves to position 2

Find initial voltage across cap
- Cap is open circuit to DC
- Circuit looks like:

At time=0 circuit changes into:

- What is the Req then?

We know \( V_c(0) \), \( R_C \) so solve,

\[
V_c(t) = V_c(0) e^{-t/R_C} \quad \text{where} \quad R_C = (2 \times 10^3)(100 \times 10^{-6}) = 0.2
\]

- \( V_c(t) = 4e^{-t/0.2} \) or
- \( V_c(t) = 4e^{-5t} \)

\( i(t) \) is the current through the 3k resistor which is in parallel with the 100\( \mu \)F cap. So...

\[
i(t) = \frac{4e^{-5t}}{3000} \quad \text{Check: at } t=0, \quad i(0) = 1.33 \text{ mA} \checkmark
\]

at \( t=5 \) \( i(t) = 9 \mu\text{A} \) (rounded to two) \( \checkmark \)
Another example:

\[ \text{Circuit transforms to } \Rightarrow 20 \, \text{V}^+ \rightarrow 4 \, \mu F \rightarrow 250 \, \Omega \]

\[ V_c(t) = 20 \, e^{-t/\left(4 \times 10^{-6}\right) \left(250 \times 10^3\right)} \]

\[ V_c(t) = 20 \, e^{-t} \]

\[ i(t) = \frac{20 \, e^{-t}}{250 \, 000} = \frac{8 \times 10^{-5}}{e^{-t}} \, A \]

\[ = 80 \, e^{-t} \, \text{mA} \]

\[ C_{eq} = \frac{(5)(20)}{s + 20} = 4 \, \mu F \]
now, lets try an inductor

\[ \text{find } i(t) \]

looking to get this form

\[ \text{Inductor is short ckt to DC so the circuit (} t < 0 \text{) looked like:} \]

\[ \begin{align*}
\text{IT} &= \frac{3.6}{2 + \frac{2}{8}} = 6 \text{A} \\
V_{2R} &= 12 \text{V}, \ V_{6R} = 24 \text{V} \\
\therefore I_{6R} &= 4 \text{A} \text{ so} \\
i_L(0) &= 4 \text{A} \\
\end{align*} \]

when switch opens what does ckt look like?

\[ \begin{align*}
\text{so } t &= \frac{L}{R} = \frac{2H}{18} = 0.111\text{sec} \\
\text{thus} \\
i_L(t) &= 4e^{-\frac{t}{0.111}} \\
i_L(t) &= 4e^{-9t} \text{A} \\
\end{align*} \]

so \( i(t) \) (through 8Ω resistor) = \( -i_L(t) \)

\[
\begin{align*}
\begin{array}{c}
i(0) = -4e^{-9t} \text{A} \quad t > 0 \\
i(t) = 2A \quad t < 0 \\
\end{array}
\end{align*}
\]

\[ \text{note that when switch opens } t = 0, \text{ the current change in the resistors is instantaneous (discontinuous), but remains continuous thru inductor!} \]
Another:

Find \( V_c(t) \) if \( V_c(0^-) = 10 \text{V} \)

- Need to find \( \frac{\Delta V_a}{\Delta t} \) at \( a,b \)
  - Independent sec present, drive with ext. SRC.

\[ 0.75V_x + \frac{V_x}{4} + 1 = 0 \]

\[ 3V_x - V_x + 4 = 0 \]

\[ V_x = -2 \]

\[ R_{th} = \frac{-2}{4} = -0.5 \text{ \Omega} \]

(negative resistance)

- So \( T = (-2)(0.25) \)
  \[ = -0.5 \text{ sec} \]

- Thus \( V_c(t) = 10e^{2t} \text{ volts} \)
  (An unstable ckt, but can be useful)