Singularity Functions (switching functions)

- discontinuous or have discontinuous derivatives
- looking at 3 functions
  - unit step function
  - unit impulse function
  - unit ramp function

- Unit step function
  - \( u(t) \) zero for \( t < 0 \)
  - one for \( t > 0 \)
  - undefined at \( t = 0 \)
  - dimensionless

\[
u(t) = \begin{cases} 
0, & t < 0 \\
1, & t > 0 
\end{cases}
\]

- Unit step (delayed)

- Unit step (advanced)

* Used to represent an abrupt change in voltage or current
Unit impulse function (delta function)

$$\delta(t) = \frac{d}{dt} \eta(t)$$

\[
\begin{align*}
\delta(t) &= \begin{cases} 
0, & t < 0 \\
\text{undefined}, & t = 0 \\
0, & t > 0
\end{cases} \\
\delta(t-0) &= \begin{cases} 
0, & t < 0 \\
\text{undefined}, & t = 0 \\
0, & t > 0
\end{cases} \\
\delta(t+0) &= \begin{cases} 
0, & t < 0 \\
\text{undefined}, & t = 0 \\
0, & t > 0
\end{cases}
\end{align*}
\]

- Not physically realizable
- Think of as an applied shock, a very short pulse of unit area
- $$\int_0^\infty \delta(t) \, dt = 1$$

![CMOS Inverter](image1.png)

![Switch Equipment](image2.png)

![Nearly a delta function](image3.png)
Unit ramp function

- Obtained by integrating the unit step function

\[ r(t) = \left\{ \begin{array}{ll}
0 & t \leq 0 \\
t & t > 0
\end{array} \right. \]

\[ r(t) = \int_{-\infty}^{t} u(t') dt' = t u(t) \]

\[ r(t) = \left\{ \begin{array}{ll}
0 & t \leq 0 \\
t & t > 0
\end{array} \right. \]
**Step Response - RC Circuit**

- Sudden application of DC source \( v(t) \) (or removal)

- Step response of a circuit
- Its behavior when the excitation source is a step function
- Let's look at it...

\[
\begin{align*}
V_s(t) & \quad (A) \\
R & \quad \text{Assuming an initial voltage of } V_0 \text{ on cap.} \\
\frac{V}{R} & \quad v(0^-) = V_0 \\
\text{since voltage on a cap cannot change immediately,} & \quad v(0^+) = v(0^+) = V_0 \\
C & \quad \frac{C}{RC} = -\frac{t}{RC} \quad \text{let } RC = \tau \\
\frac{V(t) - V_s}{V_0 - V_s} & \quad \frac{C}{RC} = \frac{V(t) - V_s}{V_0 - V_s} = e^{-\frac{t}{\tau}} \\
V(t) & \quad \text{the "complete response"} \\
V(t) & \quad \text{IF cap was not initially charged, } (V_0 = 0) \\
v(t) & \quad V_0 (1 - e^{-\frac{t}{\tau}})
\end{align*}
\]
Back to the RC circuit:

\[ i_C = i_R \]

\[ C \frac{dv}{dt} + V(s) - v = \frac{V_s(t)}{R} \]

\[ \frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} V_s(t) \]

More general case of 1st order, non-homogeneous linear differential equation with constant coefficients in the form of...

\[ \frac{dx(t)}{dt} + \frac{x(t)}{T} = f(t) \]

where:
- \( t \) is the independent variable
- \( x(t) \) is the dependent variable
  - typically \( i(t) \) or \( v(t) \)
- \( f(t) \) is the forcing function that provides stimulus to the circuit

For the step function case:

\[ \frac{dx(t)}{dt} + \frac{x(t)}{T} = \mu(t) \]

Referring to

the solution to (this differential equation) is given by:

\[ x(t) = x(\infty) + \left[ x(t_0^+) - x(\infty) \right] e^{-(t-t_0)/\tau} \]

\( t_0^+ \): time at which switching occurs
\( x(\infty) \): value of \( x(t) \) at steady state

\( \tau \): time constant

\( \tau \) = RC for RC circuits

\( \tau \) = LR for RL circuits

(R, L, C are equivalent quantities)
Step response - RC circuit

Back to the previous result

\[ V_n = V_o e^{-t/\tau} \]

\[ V_f = V_s (1 - e^{-t/\tau}) \]

natural response
decays in time
dies out naturally
stored energy is released

forced response
response to external force
2 components: transient +
steady-state independent source causes

Either way, the complete response may be written as:

\[ V(t) = V(\infty) + (V(0) - V(\infty)) e^{-t/\tau} \]

Find an steady-state
value initial voltage
at \( t = 0^+ \)

To find the step response of an RC circuit, we need:

- initial cap voltage \( V(0) \) \( \Rightarrow \) find when \( t < 0 \)
- final cap voltage \( V(\infty) \) \( \Rightarrow \) find when \( t > 0 \)
- time constant \( \tau \)

This technique also works for RL circuits, but we look for:

- initial inductor current \( i(0) \)
- final inductor current \( i(\infty) \)
- time constant \( \tau \)
Find: $i(t)$

- Initial charge on cap (do by superposition method) to find $i(0^+)$

$$V_c(0^-) = 36 \left( \frac{10}{12} \right) + 12 \left( \frac{2}{12} \right) = 32 V$$

now need to find $i(0^+)$

$$i(0^+) = \frac{32}{6000} = 5.3 mA$$

- now find $i(\infty)$

$$i(\infty) = \frac{36}{2000 + 6000} = 4.5 mA$$

- Find $R_{T1}$ for time constant: (What res. does the cap see across its terminals?)

$$R_{T1} = \frac{2k \cdot 6k}{2000 \cdot 6000} = 1500 \Omega = 1.5 k\Omega$$

- $T = RC = (1500)(100 \times 10^{-6}) = 0.15 s$

- Solution: $i(t) = i(\infty) + \left[ i(0^+) - i(\infty) \right] e^{-t / T}$

$$= 4.5 mA + \left[ 5.3 mA - 4.5 mA \right] e^{-t / 0.15}$$

$$= 4.5 mA + 0.83 mA e^{-6.67t}$$
Alternative solution (more straightforward)

- We know \( V_c(0^-) = 32V \)
- Now find \( V_c(\infty) \):

\[
\begin{align*}
V_c(\infty) &= 36 \left( \frac{6}{8} \right) = 27V
\end{align*}
\]

- Find \( \tau \) from \( \text{Req} \):

\[
\text{Req} = 2k||6k = \frac{(2000)(6000)}{8000} = 1500
\]

\[
\therefore \tau = (100 \times 10^{-6})(1500) = 0.15s
\]

- So \( V_c(t) = V_c(\infty) + \left[ V(0^-) - V(\infty) \right] e^{-\frac{t}{\tau}} \)

\[
\begin{align*}
&= 27 + \left[ 32 - 27 \right] e^{-\frac{t}{0.15}} \\
&= 27 + 5 e^{-6.67t}
\end{align*}
\]

- \( i(t) = \frac{V(t)}{6000} = 4.5 + 0.833e^{-6.67t} \text{ mA} \)