Denotational Semantics
and
Domain Theory
Outline

Denotational Semantics

Basic Domain Theory
  - Introduction and history
  - Primitive and lifted domains
  - Sum and product domains
  - Function domains

Meaning of Recursive Definitions
  - Compositionality and well-definedness
  - Least fixed-point construction
  - Internal structure of domains
A denotational semantics relates each term to a denotation

- an abstract syntax tree
- a value in some semantic domain

Valuation function

\[
[\cdot] : \text{abstract syntax} \rightarrow \text{semantic domain}
\]

Valuation function in Haskell

\[
eval :: \text{Term} \rightarrow \text{Value}
\]
**Semantic domain**: captures the set of possible meanings of a program/term

*what is a meaning? — it depends on the language!*

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Defining a language with denotational semantics

Example encoding in Haskell:

1. Define the **abstract syntax**, $T$
   the set of abstract syntax trees
   
   data Term = ...

2. Identify or define the **semantic domain**, $V$
   the representation of semantic values
   
   type Value = ...

3. Define the **valuation function**, $\llbracket \cdot \rrbracket : T \rightarrow V$
   the mapping from ASTs to semantic values
   a.k.a. the “semantic function”

   sem :: Term -> Value
Example: simple arithmetic expressions

1. Define abstract syntax

\[ n \in \text{Nat} \ ::= \ 0 \mid 1 \mid 2 \mid \ldots \]
\[ e \in \text{Exp} \ ::= \ \text{add} \ e \ e \mid \text{mul} \ e \ e \mid \text{neg} \ e \mid n \]

2. Define semantic domain

Use the set of all integers, \( \text{Int} \)

Comes with some operations:
\[ +, \times, -, \text{toInt} : \text{Nat} \to \text{Int}, \ldots \]

3. Define the valuation function

\[ [\text{Exp}] : \text{Int} \]
\[ [\text{add} \ e_1 \ e_2] = [e_1] + [e_2] \]
\[ [\text{mul} \ e_1 \ e_2] = [e_1] \times [e_2] \]
\[ [\text{neg} \ e] = -[e] \]
\[ [n] = \text{toInt}(n) \]
Encoding denotational semantics in Haskell

1. **abstract syntax**: define a new **data type**, as usual
2. **semantic domain**: identify and/or define a new **type**, as needed
3. **valuation function**: define a **function** from ASTs to semantic domain

Valuation function in Haskell

```haskell
sem :: Exp -> Int
sem (Add l r) = sem l + sem r
sem (Mul l r) = sem l * sem r
sem (Neg e) = negate (sem e)
sem (Lit n) = n
```
Desirable properties of a denotational semantics

**Compositionality**: a program’s denotation is built from the denotations of its parts
- supports modular reasoning, extensibility
- supports proof by structural induction

**Completeness**: every value in the semantics domain is denoted by some program
- ensures that semantics domain and language align
- if not, language has expressiveness gaps, or semantics domain is too general

**Soundness**: two programs are “equivalent” iff they have the same denotation
- equivalence: same w.r.t. to some other definition, e.g. operational semantics
- ensures that the denotational semantics is correct
More on compositionality

**Compositionality**: a program’s denotation is built from the denotations of its parts. An AST, sub-ASTs.

Example: What is the meaning of $\text{op } e_1 \ e_2 \ e_3$?

1. Determine the meaning of $e_1, e_2, e_3$
2. Combine these submeanings in some way specific to $\text{op}$

Implications:

- The valuation function is probably **recursive**
- We need different valuation functions for each syntactic category
Example: move language

A language describing movements on a 2D plane

- a **step** is an $n$-unit horizontal or vertical movement
- a **move** is described by a sequence of steps

Abstract syntax

\[

g o \ N \ 3; \ g o \ E \ 4; \ g o \ S \ 1;
\]

\[
\begin{align*}
n &\in \text{Nat} &::= &\ 0 \mid 1 \mid 2 \mid \ldots \\
d &\in \text{Dir} &::= &\ N \mid S \mid E \mid W \\
s &\in \text{Step} &::= &\text{go} \ d \ n \\
m &\in \text{Move} &::= &\epsilon \mid s \ ; \ m
\end{align*}
\]
1. Abstract syntax

\[ n \in \text{Nat} ::= 0 \mid 1 \mid 2 \mid \ldots \]

\[ d \in \text{Dir} ::= \text{N} \mid \text{S} \mid \text{E} \mid \text{W} \]

\[ s \in \text{Step} ::= \text{go} \ d \ n \]

\[ m \in \text{Move} ::= \epsilon \mid s ; m \]

2. Semantic domain

\[ \text{Pos} = \text{Int} \times \text{Int} \]

Domain: \( \text{Pos} \rightarrow \text{Pos} \)

3. Valuation function (Step)

\[ S[\text{Step}] : \text{Pos} \rightarrow \text{Pos} \]

\[ S[\text{go N} \ k] = \lambda(x, y). (x, y + k) \]

\[ S[\text{go S} \ k] = \lambda(x, y). (x, y - k) \]

\[ S[\text{go E} \ k] = \lambda(x, y). (x + k, y) \]

\[ S[\text{go W} \ k] = \lambda(x, y). (x - k, y) \]

3. Valuation function (Move)

\[ M[\text{Move}] : \text{Pos} \rightarrow \text{Pos} \]

\[ M[\epsilon] = \lambda p. p \]

\[ M[s ; m] = M[m] \circ S[s] \]
Alternative semantics

Often multiple interpretations (semantics) of the same language

Example: Database schema

One declarative spec, used to:
- initialize the database
- generate APIs
- validate queries
- normalize layout
- ...

Distance traveled

\[
S_D[\text{Step}] : \text{Int} \\
S_D[\text{go } d \; k] = k \\
M_D[\text{Move}] : \text{Int} \\
M_D[\epsilon] = 0 \\
M_D[s ; m] = S_D[s] + M_D[m]
\]

Combined trip information

\[
M_C[\text{Move}] : \text{Int} \times (\text{Pos} \rightarrow \text{Pos}) \\
M_C[m] = (M_D[m], M[m])
\]
Picking the right semantic domain

Simple semantics domains can be combined in two ways:

- **product**: contains a value from both domains
  - e.g. combined trip information for move language
  - use Haskell \((a,b)\) or define a new data type

- **sum**: contains a value from one domain or the other
  - e.g. IntBool language can evaluate to **Int** or **Bool**
  - use Haskell **Either** \(a\ b\) or define a new data type

Can errors occur?

- use Haskell **Maybe** \(a\) or define a new data type

Does the language manipulate state or use naming?

- use a **function type**
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  Least fixed-point construction
  Internal structure of domains
What is domain theory?

**Domain theory**: a mathematical framework for constructing **semantic domains**

Recall …

A denotational semantics relates each **term** to a **denotation**

- an abstract syntax tree
- a value in some **semantic domain**

**Semantic domain**: captures the set of possible meanings of a program/term
Origins of domain theory:

- **Christopher Strachey**, 1964
  - early work on denotational semantics
  - used *lambda calculus* for denotations

- **Dana Scott**, 1975
  - goal: denotational semantics for lambda calculus itself
  - created domain theory for meaning of recursive functions

More on Dana Scott:

- Turing award in 1976 for nondeterminism in automata theory
- PhD advisor: **Alonzo Church**, 20 years after **Alan Turing**
Two views of denotational semantics

View #1: Translation from one formal system to another
  • e.g. translate object language into lambda calculus

View #2: “True meaning” of a program as a mathematical object
  • e.g. map programs to elements of a semantic domain
  • need domain theory to describe set of meanings
Domains as semantic algebras

A **semantic domain** can be viewed as an **algebraic structure**

- a set of **values** the meanings of the programs
- a set of **operations** on the values used to compose meanings of parts

Domains also have internal structure: **complete partial ordering** (later)
- like a “half-lattice” – any two elements must have a **join**
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Primitive domains

Values are **atomic**
- often correspond to **built-in types** in Haskell
- **nullary operations** for naming values explicitly

**Domain: \textit{Bool}**
- \textit{true} : \textit{Bool}
- \textit{false} : \textit{Bool}
- \textit{not} : \textit{Bool} → \textit{Bool}
- \textit{and} : \textit{Bool} × \textit{Bool} → \textit{Bool}
- \textit{or} : \textit{Bool} × \textit{Bool} → \textit{Bool}

**Domain: \textit{Int}**
- \textit{0}, \textit{1}, \textit{2}, \ldots : \textit{Int}
- \textit{negate} : \textit{Int} → \textit{Int}
- \textit{plus} : \textit{Int} × \textit{Int} → \textit{Int}
- \textit{times} : \textit{Int} × \textit{Int} → \textit{Int}

**Domain: \textit{Unit}**
- () : \textit{Unit}

Also: \textit{Nat}, \textit{Name}, \textit{Addr}, …
Lifted domains

Construction: add \( \bot \) (bottom) to an existing domain

\[
A_{\bot} = A \cup \{ \bot \}
\]

New operations

\[
\bot : A_{\bot}
\]

\[
map : (A \to B) \times A_{\bot} \to B_{\bot}
\]

\[
maybe : B \times (A \to B) \times A_{\bot} \to B_{\bot}
\]
Encoding lifted domains in Haskell

**Option #1: Maybe**

```haskell
data Maybe a = Nothing
  | Just a

fmap :: (a -> b) -> Maybe a -> Maybe b
maybe :: b -> (a -> b) -> Maybe a -> Maybe b
```

Can also use pattern matching!

**Option #2: new data type with nullary constructor**

```haskell
data Value = Success Int | Error
```

Best when combined with other constructions
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Sum domains

**Construction:** the disjoint union of two existing domains
- contains a value from either one domain or the other

\[ A \oplus B = A \cup B \]

**New operations**

\[
\begin{align*}
\text{inL} &: A \to A \oplus B \\
\text{inR} &: B \to A \oplus B \\
\text{case} &: (A \to C) \times (B \to D) \times (A \oplus B) \to C \oplus D
\end{align*}
\]
Encoding sum domains in Haskell

Option #1: Either

```haskell
data Either a b = Left a
               | Right b
either :: (a -> c) -> (b -> d) -> Either a b -> Either c d
```

Can also use pattern matching!

Option #2: new data type with multiple constructors

```haskell
data Value = I Int | B Bool
```

Best when combined with other constructions, or more than two options
Example: a language with multiple types

\[
\begin{align*}
  b \in \text{Bool} & \quad ::= \quad \text{true} \mid \text{false} \\
  n \in \text{Nat} & \quad ::= \quad 0 \mid 1 \mid 2 \mid \ldots \\
  e \in \text{Exp} & \quad ::= \quad \text{add} \quad e \quad e \\
  & \quad \mid \quad \text{neg} \quad e \\
  & \quad \mid \quad \text{equal} \quad e \quad e \\
  & \quad \mid \quad \text{cond} \quad e \quad e \quad e \\
  & \quad \mid \quad n \\
  & \quad \mid \quad b
\end{align*}
\]

Design a denotational semantics for \(\text{Exp}\)

1. How should we define our semantics domain?
2. Define a valuation semantics function

- **neg** – negates either a numeric or boolean value
- **equal** – compares two values of the same type for equality
- **cond** – equivalent to **if-then-else**
Solution

\[
\begin{align*}
[Exp] &: (Int \oplus \text{Bool}) \perp \\
[\text{add } e_1 \ e_2] &= \begin{cases} 
[e_1] + [e_2] & [e_1] \in Int, [e_2] \in Int \\
\perp & \text{otherwise}
\end{cases} \\
[\text{neg } e] &= \begin{cases} 
-\lbrack e\rbrack & [e] \in Int \\
\lbrack e \rbrack & [e] \in \text{Bool} \\
\perp & \text{otherwise}
\end{cases} \\
[\text{equal } e_1 \ e_2] &= \begin{cases} 
[e_1] =_{\text{Int}} [e_2] & [e_1] \in \text{Int}, [e_2] \in \text{Int} \\
[e_1] =_{\text{Bool}} [e_2] & [e_1] \in \text{Bool}, [e_2] \in \text{Bool} \\
\perp & \text{otherwise}
\end{cases} \\
[\text{cond } e_1 \ e_2 \ e_3] &= \begin{cases} 
[e_2] & [e_1] = \text{true} \\
[e_3] & [e_1] = \text{false} \\
\perp & \text{otherwise}
\end{cases} \\
[n] &= n \\
[b] &= b
\end{align*}
\]
Product domains

**Construction:** the **cartesian product** of two existing domains
- contains a value from both domains

\[
A \otimes B = \{(a, b) \mid a \in A, b \in B\}
\]

**New operations**
- pair : \( A \times B \rightarrow A \otimes B \)
- \( \text{fst} : A \otimes B \rightarrow A \)
- \( \text{snd} : A \otimes B \rightarrow B \)
Encoding product domains in Haskell

Option #1: Tuples

type Value a b = (a,b)

fst :: (a,b) -> a
snd :: (a,b) -> b

Can also use pattern matching!

Option #2: new data type with multiple arguments

data Value = V Int Bool

Best when combined with other constructions, or more than two
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Function space domains

Construction: the set of functions from one domain to another

- **Create a function:** $A \to B$
  - Lambda notation: $\lambda x. y$
    - where $\Gamma, x : A \vdash y : B$

- **Eliminate a function**
  - $apply : (A \to B) \times A \to B$
Denotational semantics of naming

**Environment**: a function associating names with things

\[ Env = Name \rightarrow Thing \]

**Naming concepts**

- **declaration**: `add` a new name to the environment
- **binding**: `set` the thing associated with a name
- **reference**: `get` the thing associated with a name

**Example semantic domains for expressions with …**

- **immutable** variables (Haskell) \( Env \rightarrow Val \)
- **mutable** variables (C/Java/Python) \( Env \rightarrow Env \otimes Val \)
Example: Denotational semantics of let language

1. Abstract syntax

\[ i \in \text{Int} ::= \text{(any integer)} \]
\[ v \in \text{Var} ::= \text{(any variable name)} \]
\[ e \in \text{Exp} ::= i \]
\[ \quad \mid \text{add} \ e \ e \]
\[ \quad \mid \text{let} \ v \ e \ e \]
\[ \quad \mid v \]

2. Identify semantic domain

i. Result of evaluation: \( \text{Int}_\bot \)
ii. Environment: \( \text{Env} = \text{Var} \rightarrow \text{Int}_\bot \)
iii. Semantic domain: \( \text{Env} \rightarrow \text{Int}_\bot \)

3. Define a valuation function

\[ [\text{Exp}] : (\text{Var} \rightarrow \text{Int}_\bot) \rightarrow \text{Int}_\bot \]

\[ [i] = \lambda m. i \]
\[ [\text{add} \ e_1 \ e_2] = \lambda m. \[e_1\](m) + \bot \[e_2\](m) \]
\[ [\text{let} \ v \ e_1 \ e_2] = \lambda m. \[e_2\](\lambda w. \text{if } w = v \text{ then } [e_1](m) \text{ else } m(w)) \]
\[ [v] = \lambda m. m(v) \]

\[ i +_\bot j = \begin{cases} i + j & i \in \text{Int}, j \in \text{Int} \\ \bot & \text{otherwise} \end{cases} \]
What is mutable state?

**Mutable state**: stored information that a program can **read** and **write**

Typical semantic domains with state domain $S$:

- $S \rightarrow S$  
  state mutation as **main effect**

- $S \rightarrow S \otimes Val$  
  state mutation as **side effect**

Often: lifted codomain if mutation can fail

**Examples**

- the memory cell in a calculator  
  $S = Int$

- the stack in a stack language  
  $S = Stack$

- the store in many programming languages  
  $S = Name \rightarrow Val$
Example: Single register calculator language

1. Abstract syntax

\[ i \in \text{Int} ::= \text{(any integer)} \]
\[ e \in \text{Exp} ::= i \]
\[ \quad | \quad e + e \]
\[ \quad | \quad \text{save } e \]
\[ \quad | \quad \text{load} \]

Examples:

- \text{save } (3+2) + \text{load} \implies 10
- \text{save } 1 +
  \quad (\text{save } 10 + \text{load}) + \text{load} \implies 31

2. Identify semantic domain

i. State (side effect): \text{Int}
ii. Result (main effect): \text{Int}
iii. Semantic domain: \text{Int} \rightarrow \text{Int} \otimes \text{Int}
Example: Single register calculator language

1. Abstract syntax

\[ i \in \text{Int} ::= (\text{any integer}) \]
\[ e \in \text{Exp} ::= i \]
\[ e + e \]
\[ \text{save } e \]
\[ \text{load} \]

Examples:

- \text{save} \ (3+2) + \text{load} \rightarrow 10
- \text{save} \ 1 + \\
  (\text{save} \ 10 + \text{load}) + \text{load} \rightarrow 31

3. Define valuation function

\[ [\text{Exp}] : \text{Int} \rightarrow \text{Int} \otimes \text{Int} \]

\[ [i] = \lambda s. (s, i) \]
\[ [e_1 + e_2] = \lambda s. \text{let} \ ((s_1, i_1) := [e_1](s)) \]
\[ (s_2, i_2) := [e_2](s_1) \]
\[ \text{in} \ (s_2, i_1 + i_2) \]

\[ [\text{save } e] = \lambda s. \text{let} \ ((s', i) := [e](s)) \text{ in} \ (i, i) \]

\[ [\text{load } e] = \lambda s. (s, s) \]
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Compositionality and well-definedness

Recall: a **denotational semantics** must be **compositional**
- a term’s denotation is built from the denotations of its parts

Example: integer expressions

\[
\begin{align*}
i \in \text{Int} &::= (\text{any integer}) \\
e \in \text{Exp} &::= i \mid \text{add } e \ e \mid \text{mul } e \ e
\end{align*}
\]

\[
\begin{align*}
\mathbb{J}[\text{Exp}] &::= \text{Int} \\
\mathbb{J}[i] &= i \\
\mathbb{J}[\text{add } e_1 \ e_2] &= \mathbb{J}[e_1] + \mathbb{J}[e_2] \\
\mathbb{J}[\text{mul } e_1 \ e_2] &= \mathbb{J}[e_1] \times \mathbb{J}[e_2]
\end{align*}
\]

Compositionality ensures the semantics is **well-defined** by **structural induction**

Each AST has **exactly one** meaning.
A non-compositional (and ill-defined) semantics

Anti-example: while statement

\[ t \in \text{Test} ::= \ldots \]
\[ c \in \text{Cmd} ::= \ldots \mid \text{while } t\ c \]

\[ T[\text{Test}] : S \to \text{Bool} \]
\[ C[\text{Cmd}] : S \to S \]
\[ C[\text{while } t\ c] = \lambda s. \text{if } T[t](s) \text{ then } C[\text{while } t\ c](C[c](s)) \text{ else } s \]

Meaning of \textbf{while } t\ c in state \textit{s}:

1. evaluate \textit{t} in state \textit{s}
2. if true:
   a. run \textit{c} to get updated state \textit{s}'
   b. re-evaluate \textbf{while} in state \textit{s}'
      (not compositional)
3. otherwise return \textit{s} unchanged

A \textbf{while} statement may have \textit{infinitely many} meanings! –or–

Translational view: meaning is an \textit{infinite} lambda expression!
Mathematical function
Defined **extensionally**:
- a relation between inputs and outputs

Computational function  (e.g. Haskell)
Usually defined **operationally**:
- compute output by sequence of reductions

Example (intensional definition)
\[
 fac(n) = \begin{cases} 
 1 & n = 0 \\
 n \cdot fac(n - 1) & \text{otherwise} 
\end{cases}
\]

Extensional meaning
\{..., (2, 2), (3, 6), (4, 24), ...\}

Operational meaning
\[
\begin{align*}
 fac(3) & \rightsquigarrow 3 \cdot fac(2) \\
 & \rightsquigarrow 3 \cdot 2 \cdot fac(1) \\
 & \rightsquigarrow 3 \cdot 2 \cdot 1 \cdot fac(0) \\
 & \rightsquigarrow 3 \cdot 2 \cdot 1 \cdot 1 \\
 & \rightsquigarrow 6
\end{align*}
\]
Extensional meaning of recursive functions

\[ \text{grow}(n) = \begin{cases} 
1 & n = 0 \\
\text{grow}(n + 1) - 2 & \text{otherwise}
\end{cases} \]

Best extension (use \( \bot \) if undefined):

- \( \{(0, 1), (1, \bot), (2, \bot), (3, \bot), (4, \bot), \ldots\} \)

Other valid extensions:

- \( \{(0, 1), (1, 2), (2, 4), (3, 6), (4, 8), \ldots\} \)

- \( \{(0, 1), (1, 5), (2, 7), (3, 9), (4, 11), \ldots\} \)

- \( \ldots \)

Goal: best extension = \textbf{only} extension
Connection back to denotational semantics

A **function space domain** is a set of **mathematical functions**

**Anti-example: while statement**

\[
\begin{align*}
    t & \in \text{Test} \quad ::= \quad \ldots \\
    c & \in \text{Cmd} \quad ::= \quad \ldots \quad | \quad \textbf{while } t \ c
\end{align*}
\]

\[
\begin{align*}
    T[\text{Test}] & : S \rightarrow \text{Bool} \\
    C[\text{Cmd}] & : S \rightarrow S \\
    C[\textbf{while } t \ c] & = \lambda s. \text{if } T[t](s) \text{ then } C[c](s) \text{ else } s
\end{align*}
\]

**Ideal semantics of Cmd:**
- semantic domain: \( S \rightarrow S_\bot \)
- contains \((s, s')\) if \(c\) terminates
- contains \((s, \bot)\) if \(c\) diverges
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Least fixed points

Basic idea:

1. A recursive function defines a set of non-recursive, finite subfunctions.
2. Its meaning is the "union" of the meanings of its subfunctions.

Iteratively grow the extension until we reach a fixed point:
- Essentially encodes computational functions as mathematical functions.
Example: unfolding a recursive definition

### Recursive definition

\[
fac(n) = \begin{cases} 
  1 & \text{if } n = 0 \\
  n \cdot fac(n - 1) & \text{otherwise}
\end{cases}
\]

### Non-recursive, finite subfunctions

\[
\begin{align*}
fac_0(n) &= \bot \\
fac_1(n) &= \begin{cases} 
  1 & \text{if } n = 0 \\
  n \cdot fac_0(n - 1) & \text{otherwise}
\end{cases} \\
fac_2(n) &= \begin{cases} 
  1 & \text{if } n = 0 \\
  n \cdot fac_1(n - 1) & \text{otherwise}
\end{cases} \\
&\vspace{5pt} \vdots
\end{align*}
\]

\[
\begin{align*}
fac_0 &= \{\} \\
fac_1 &= \{(0, 1)\} \\
fac_2 &= \{(0, 1), (1, 1)\} \\
fac_3 &= \{(0, 1), (1, 1), (2, 2)\} \\
&\vspace{5pt} \vdots
\end{align*}
\]

\[
fac = \bigcup_{i=0}^{\infty} fac_i
\]

Fine print:
- each \( fac_i \) maps all other values to \( \bot \)
- \( \bigcup \) operation prefers non-\( \bot \) mappings
Computing the fixed point

In general

\[ \text{fac}_0(n) = \bot \]

\[ \text{fac}_i(n) = \begin{cases} 1 & n = 0 \\ n \cdot \text{fac}_{i-1}(n-1) & \text{otherwise} \end{cases} \]

A template to represent all \( \text{fac}_i \) functions:

\[ F = \lambda f. \lambda n. \begin{cases} 1 & n = 0 \\ n \cdot f(n-1) & \text{otherwise} \end{cases} \]

Fixpoint operator

\[ \text{fix} : (A \rightarrow A) \rightarrow A \]

\[ \text{fix}(g) = \text{let } x := g(x) \text{ in } x \]

\[ \text{fix}(h) = h(h(h(h(\ldots)))) \]

Factorial as a fixed point

\[ \text{fac} = \text{fix}(F) \]

\[ \text{takes fac}_{i-1} \text{ as input} \]
Outline

Denotational Semantics

Basic Domain Theory
  Introduction and history
  Primitive and lifted domains
  Sum and product domains
  Function domains

Meaning of Recursive Definitions
  Compositionality and well-definedness
  Least fixed-point construction
  Internal structure of domains

Meaning of Recursive Definitions 47 / 51
Why domains are not flat sets

**Internal structure** of domains supports the least fixed-point construction

Recall fine print from factorial example:

- each $fac_i$ maps all other values to $\bot$
- $\cup$ operation prefers non-$\bot$ mappings

How can we *generalize* and *formalize* this idea?
Partial orderings and joins

Partial ordering: \( \sqsubseteq : D \times D \to \mathbb{B} \)
- reflexive: \( \forall x \in D. \ x \sqsubseteq x \)
- antisymmetric: \( \forall x, y \in D. \ x \sqsubseteq y \land y \sqsubseteq x \implies x = y \)
- transitive: \( \forall x, y, z \in D. \ x \sqsubseteq y \land y \sqsubseteq z \implies x \sqsubseteq z \)

Join: \( \sqcup : D \times D \to D \)
\( \forall a, b \in D, \text{ the element } c = a \sqcup b \in D, \text{ if it exists, } \)
\( \text{is the smallest element that is larger than both } a \text{ and } b \)

i.e. \( a \sqsubseteq c \) and \( b \sqsubseteq c \), and there is no \( d = a \sqcup b \in D \) where \( d \sqsubseteq c \)
Directed-complete partial orderings

A directed subset of domain is a subset of elements related by the ordering relation $\sqsubseteq$

Every domain is a directed-complete partial ordering (dcpo):

- every directed subset $D$ has a least element

A function is Scott-continuous if it preserves the least element

Finally, the meaning of a Scott-continuous recursive function $f$ is:

$$\bigcup_{i=0}^{\infty} f_i$$

where $f_i$ are the finite approximations of $f$
Well-defined semantics for the while statement

Syntax

\[ t \in \text{Test} ::= \ldots \]
\[ c \in \text{Cmd} ::= \ldots \mid \text{while } t \ c \]

Semantics

\[
T[	ext{Test}] : S \rightarrow \text{Bool}
\]
\[
C[	ext{Cmd}] : S \rightarrow S
\]
\[
C[\text{while } t \ c] = \text{fix}(\lambda f. \lambda s. \text{if } T[t](s) \text{ then } f(C[c](s)) \text{ else } s)
\]