SECTION 3: RESISTIVE CIRCUIT ANALYSIS II

ENGR 201 – Electrical Fundamentals I

Resistive Network Analysis

Circuit Analysis Methods

- Circuit analysis objective is to determine all:
  - ***Node voltages***
  - ***Branch currents***
- Circuit analysis tools:
  - Ohm's law
  - Kirchhoff's laws – KVL, KCL
- Circuit analysis methods:
  - ***Nodal analysis***
    - Systematic application of KCL
  - ***Mesh/loop analysis***
    - Systematic application of KVL

Nodal Analysis

Nodal Analysis

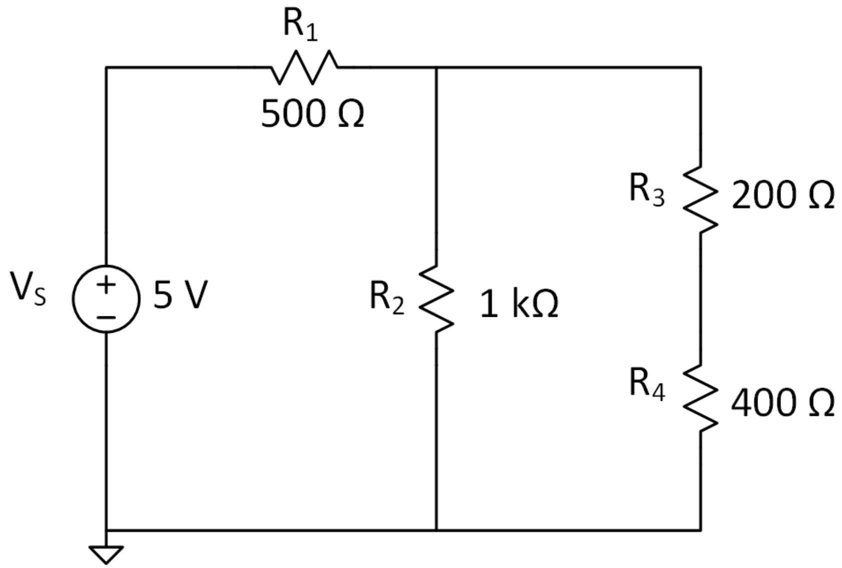
- ***Nodal analysis***
  - Systematic application of ***KCL***
  - Generate a system of equations
    - Node voltages are the unknown variables
    - Number of equations equals number of unknown node voltages
  - Solve equations to determine node voltages
  - Apply Ohm's law to determine branch currents

Nodal Analysis – Step-by-Step Procedure

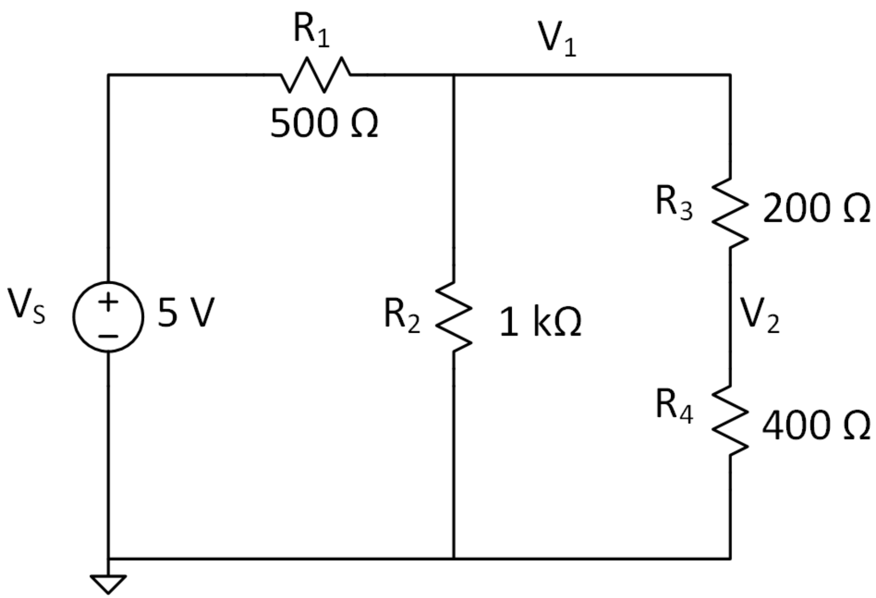
1) Identify and label all nodes in the circuit – distinguish known from unknown node voltages
2) Assign and label polarities of currents through all branches
3) Apply KCL at each node, using Ohm's Law to express branch currents in terms of node voltages
4) Solve the resulting simultaneous system of equations using substitution, calculator, Cramer's Rule, etc.
5) Use Ohm's Law and node voltages to determine branch currents

Nodal Analysis – Example

- Apply nodal analysis to determine all node voltages and branch currents in the following circuit

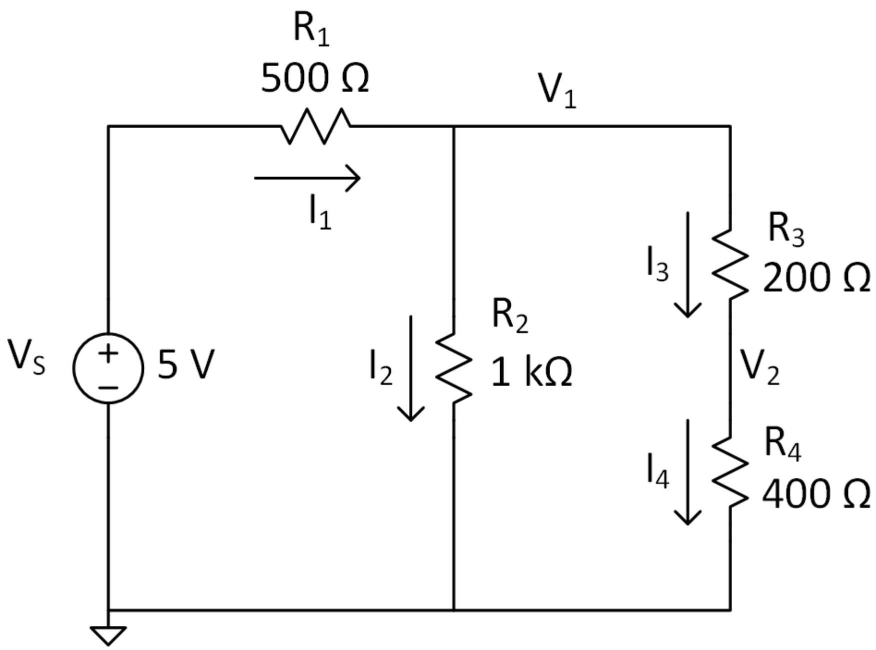


Nodal Analysis – Step 1



- **<u>Step 1</u>:** Identify and label all nodes in the circuit – distinguish known from unknown node voltages
  - V_s is a known node voltage (5 V)
  - V_1 and V_2 are unknown

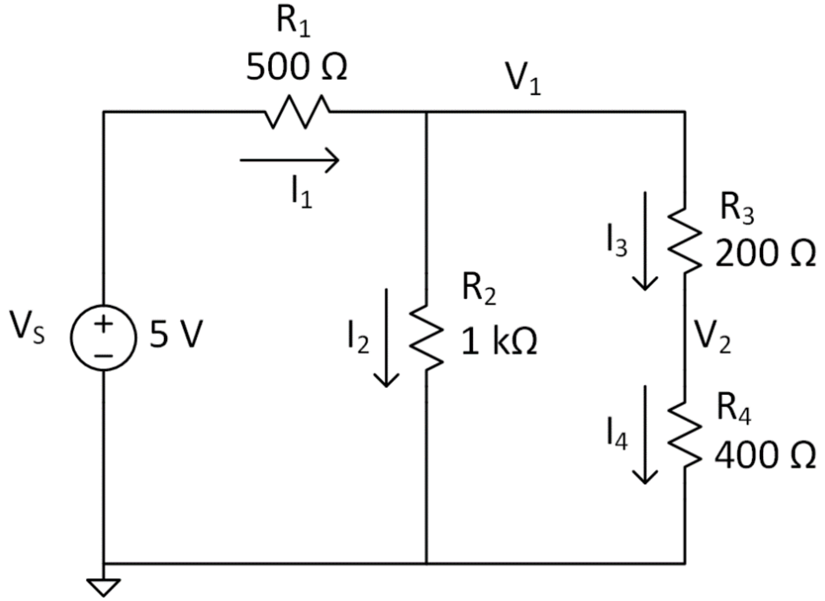
Nodal Analysis – Step 2



- **<u>Step 2</u>:** Assign and label polarities of currents through all branches
  - Assumed polarities needn't be correct
    - Correct polarity given by the sign of the determined quantity

Nodal Analysis – Step 3

- **Step 3**: Apply KCL at each node, using Ohm's Law to express branch currents in terms of node voltages



KCL at node 1

$$I_1 - I_2 - I_3 = 0$$

$$\frac{5V - V_1}{R_1} - \frac{V_1}{R_2} - \frac{V_1 - V_2}{R_3} = 0$$

KCL at node 2

$$I_3 - I_4 = 0$$

$$\frac{V_1 - V_2}{R_3} - \frac{V_2}{R_4} = 0$$

Nodal Analysis – Step 4

- **<u>Step 4</u>:** Solve the resulting system of equations
  - First, organize the equations

$$\frac{5V - V_1}{R_1} - \frac{V_1}{R_2} - \frac{V_1 - V_2}{R_3} = 0$$

$$\frac{V_1 - V_2}{R_3} - \frac{V_2}{R_4} = 0$$

$$\Rightarrow$$

$$V_1\left(-\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3}\right) + V_2\left(\frac{1}{R_3}\right) = -\frac{5V}{R_1}$$

$$V_1\left(\frac{1}{R_3}\right) + V_2\left(-\frac{1}{R_3} - \frac{1}{R_4}\right) = 0$$

- Solve using Gaussian elimination, Cramer's rule, or using calculator or computer
  - Put into matrix form for solution in calculator or MATLAB:

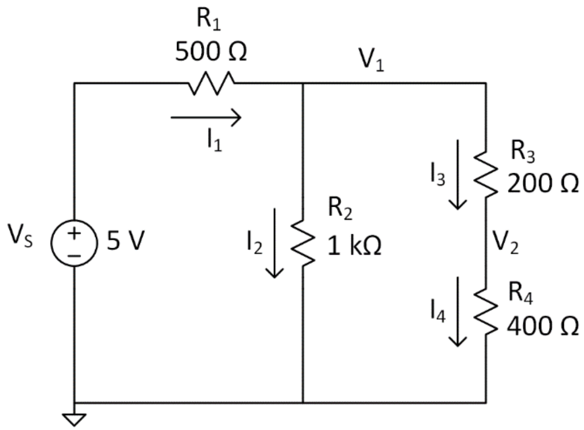
$$\begin{bmatrix} -\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} & \frac{1}{R_3} \\ \frac{1}{R_3} & -\frac{1}{R_3} - \frac{1}{R_4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\frac{5V}{R_1} \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -8mS & 5mS \\ 5mS & -7.5mS \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -10mA \\ 0 \end{bmatrix}$$

$$\mathbf{GV} = \mathbf{I}$$

Nodal Analysis – Step 5

- **Step 5:** Use Ohm's Law and node voltages to determine branch currents
  - Solution to system of equations yields node voltages:

$$V_1 = \mathbf{2.14\ V}$$
$$V_2 = \mathbf{1.43\ V}$$



  - Branch currents are

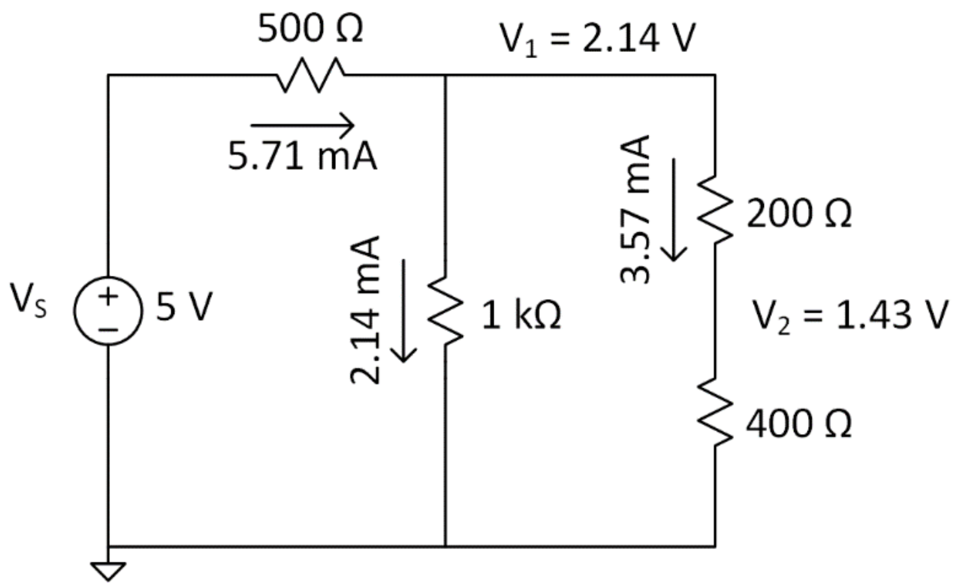
$$I_1 = \frac{5V - V_1}{R_1} = \frac{5\ V - 2.14\ V}{500\ \Omega} = \mathbf{5.71\ mA}$$

$$I_2 = \frac{V_1}{R_2} = \frac{2.14\ V}{1\ k\Omega} = \mathbf{2.14\ mA}$$

$$I_3 = I_4 = \frac{V_2}{R_4} = \frac{1.43\ V}{400\ \Omega} = \mathbf{3.57\ mA}$$

Nodal Analysis

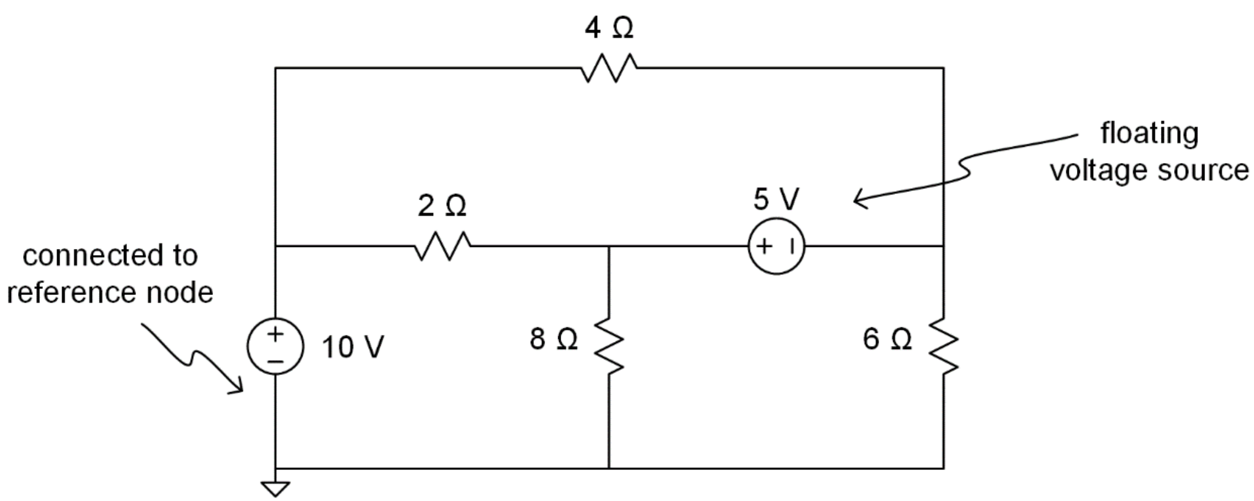
- Nodal analysis yields all node voltages and branch currents



Supernodes

Nodal Analysis – Floating Voltage Sources

- When performing nodal analysis on circuits with voltage sources, there are two possible scenarios:
  - Voltage source connected to the reference node
    - As in the last example
  - Voltage source is ***floating***
    - Both terminals connected to non-reference nodes

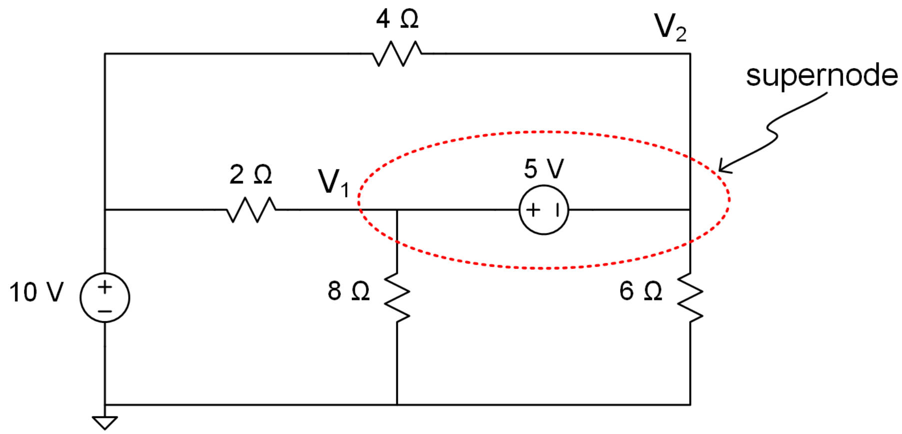


Nodal Analysis - Supernodes

- Floating voltage sources pose a problem
  - Cannot use Ohm's law to represent the current through the source
    - Ohm's law applies only to resistors
- Solution:
  - Form a ***supernode*** enclosing the source
    - Formed by two non-reference nodes
  - Apply KCL to the supernode
    - *One* equation for the *two* unknown nodes
  - Apply KVL to relate the voltages of the nodes forming the supernode
    - Providing the required additional equation

Supernode – Example

- Nodes V_1 and V_2 form a supernode, enclosing the floating voltage source
- Circuit has two unknown node voltages, V_1 and V_2
  - System of two equations is required
- KCL will be applied at the supernode
  - Only one equation will result



- Additional required equation obtained by applying KVL to relate V_1 to V_2

Nodal Analysis with Supernodes – Step-by-Step

1) Identify and label all nodes in the circuit – distinguish known from unknown node voltages

2) Assign and label polarities of currents through all branches

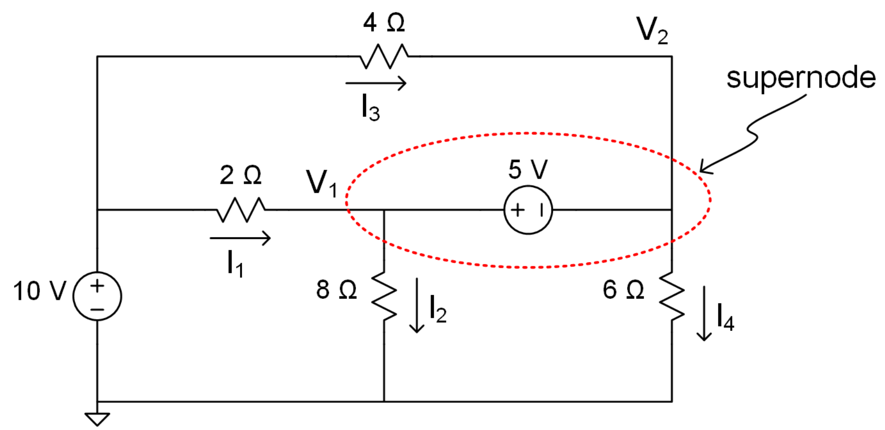
3) Generate a system of equations
   a) Apply KCL at each node and each supernode, using Ohm's Law to express branch currents in terms of node voltages
   b) Apply KVL to relate the voltages of the nodes that form the supernodes

4) Solve the resulting simultaneous system of equations using substitution, calculator, Cramer's Rule, etc.

5) Use Ohm's Law and node voltages to determine branch currents

Supernode – Example

- **Step 1**: Identify and label all nodes in the circuit
  - Any supernodes are identified and labeled in this step
- **Step 2**: Assign and label all branch currents
- **Step 3a**: Apply KCL at all nodes and all supernodes
  - Here we have only the one supernode:



$$I_1 - I_2 + I_3 - I_4 = 0$$

$$\frac{10\ V - V_1}{2\ \Omega} - \frac{V_1}{8\ \Omega} + \frac{10\ V - V_2}{4\ \Omega} - \frac{V_2}{6\ \Omega} = 0$$

$$V_1\left(\frac{1}{2\ \Omega} + \frac{1}{8\ \Omega}\right) + V_2\left(\frac{1}{4\ \Omega} + \frac{1}{6\ \Omega}\right) = 7.5\ A$$

Supernode – Example

- **Step 3b**: Apply KVL to relate the voltages of the nodes that form the supernode

$$V_1 - 5V - V2 = 0$$

$$V_1 - V_2 = 5\ V$$

- **Step 4**: Solve the resulting system of equations

$$V_1 \cdot 625\ mS + V_2 \cdot 416.7\ mS = 7.5\ A$$

$$V_1 - V_2 = 5\ V$$

  - Putting these into matrix form:

$$\begin{bmatrix} 625\ mS & 416.7\ mS \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 7.5\ A \\ 5\ V \end{bmatrix}$$

Supernode – Example

$$\begin{bmatrix} 625\ mS & 416.7\ mS \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 7.5\ A \\ 5\ V \end{bmatrix}$$

- Note that coefficient matrix on the left-hand side is no longer a conductance matrix
  - Second-row elements are dimensionless
  - Mix of KCL and KVL equations

- Solve using your method of choice
- Here, solved using MATLAB

$$V_1 = 9.2\ V$$

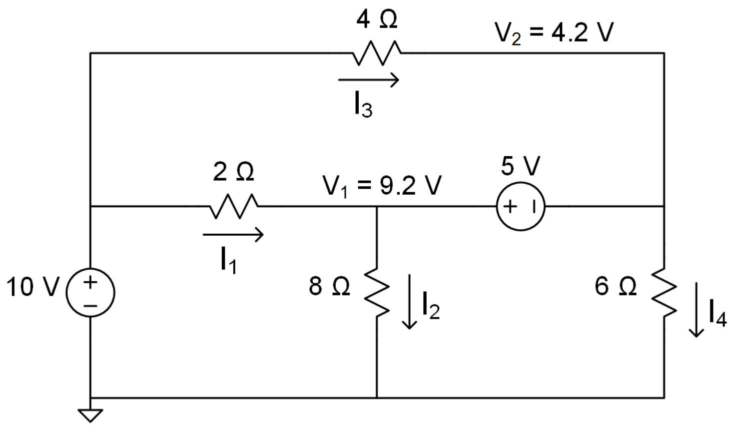
$$V_2 = 4.2\ V$$

Command Window

```
>> A = [0.625,0.41667;1,-1]

A =

    0.6250    0.4167
    1.0000   -1.0000

>> b = [7.5;5]

b =

    7.5000
    5.0000

>> V = A\b

V =

    9.2000
    4.2000
```

Supernode – Example

- **Step 5**: Use Ohm's law and branch currents to determine node voltages



$$I_1 = \frac{10\ V - 9.2\ V}{2\ \Omega} = \frac{0.8\ V}{2\ \Omega} = 0.4\ A$$

$$I_2 = \frac{9.2\ V}{8\ \Omega} = 1.15\ A$$

$$I_3 = \frac{10\ V - 4.2\ V}{4\ \Omega} = \frac{5.8\ V}{4\ \Omega} = 1.45\ A$$

$$I_4 = \frac{4.2\ V}{6\ \Omega} = 0.7\ A$$

$$I_1 = 0.4\ A$$

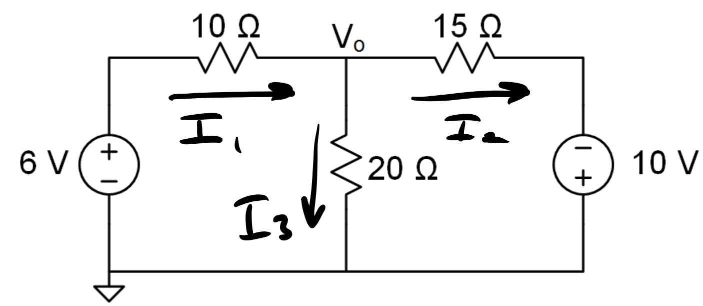
$$I_2 = 1.15\ A$$

$$I_3 = 1.45\ A$$

$$I_4 = 0.7\ A$$

Example Problems

Apply nodal analysis to determine V_o in the following circuit.



KCL at V_o:

$$I_1 - I_2 - I_3 = 0$$

$$\frac{6V - V_o}{10\Omega} - \frac{(V_o - (-10V))}{15\Omega} - \frac{V_o}{20\Omega} = 0$$

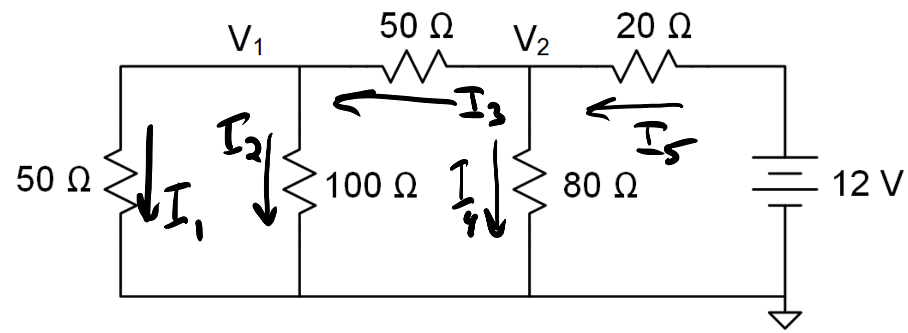
$$V_o\left(\frac{1}{10\Omega} + \frac{1}{15\Omega} + \frac{1}{20\Omega}\right) = \frac{6V}{10\Omega} - \frac{10V}{15\Omega}$$

$$V_o(216.7\,mS) = -66.7\,mA$$

$$V_o = \frac{-66.7\,mA}{216.7\,mS}$$

$$\underline{V_o = -307.7\,mV}$$

Apply nodal analysis to determine V_1 and V_2.



KCL at V_1:

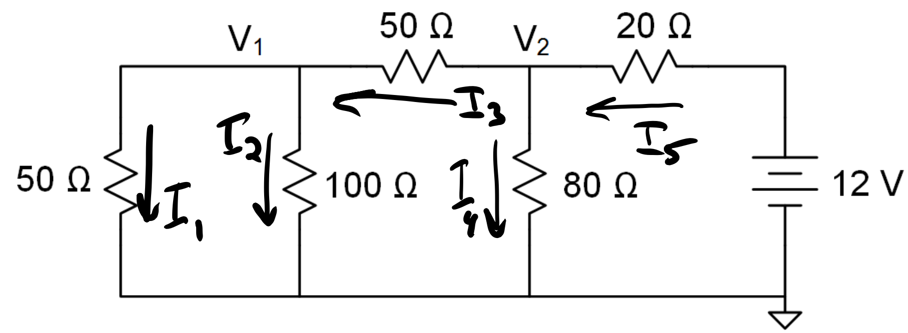
$$-I_1 - I_2 + I_3 = 0$$

$$-\frac{V_1}{50\Omega} - \frac{V_1}{100\Omega} + \frac{V_2 - V_1}{50\Omega} = 0$$

$$V_1\left(\frac{1}{50\Omega} + \frac{1}{100\Omega} + \frac{1}{50\Omega}\right) + V_2\left(-\frac{1}{50\Omega}\right) = 0$$

$$V_1(50mS) + V_2(-20mS) = 0 \qquad (1)$$

KCL at V_2:



$$-I_3 - I_4 + I_5 = 0$$

$$-\frac{(V_2 - V_1)}{50\Omega} - \frac{V_2}{80\Omega} + \frac{12V - V_2}{20\Omega} =$$

$$V_1\left(-\frac{1}{50\Omega}\right) + V_2\left(\frac{1}{50\Omega} + \frac{1}{80\Omega} + \frac{1}{20\Omega}\right) = \frac{12V}{20\Omega}$$

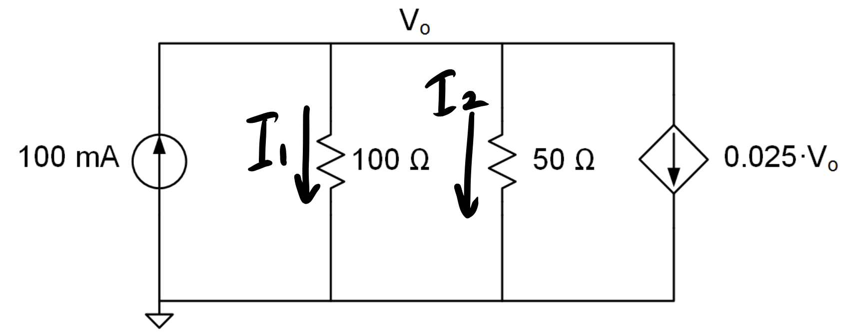
$$V_1(-20mS) + V_2(82.5mS) = 600mA \quad (2)$$

System of equations

$$\begin{bmatrix} 50mS & -20mS \\ -20mS & 82.5mS \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 600mA \end{bmatrix}$$

Solving → $V_1 = 3.22V$ $V_2 = 8.05V$

Apply nodal analysis to determine V_o.



KCL at V_o:

$$100mA - I_1 - I_2 - 0.25\,V_o = 0$$

$$100mA - \frac{V_o}{100\Omega} - \frac{V_o}{50\Omega} - 0.25\,V_o = 0$$

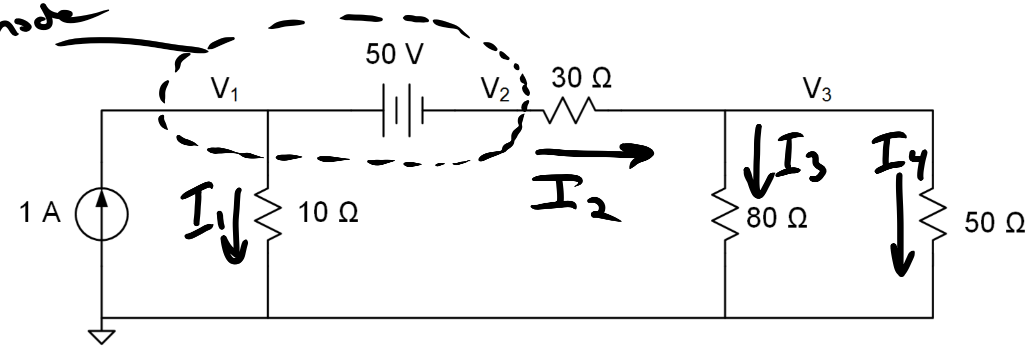
$$V_o\left(\frac{1}{100\Omega} + \frac{1}{50\Omega} + 0.025\,S\right) = 100mA$$

$$V_o\,(55mS) = 100mA$$

$$V_o = \frac{100mA}{55mS}$$

$$\underline{V_o = 1.82\,V}$$

Apply nodal analysis to determine V_1, V_2, and V_3.



KCL at the supernode:

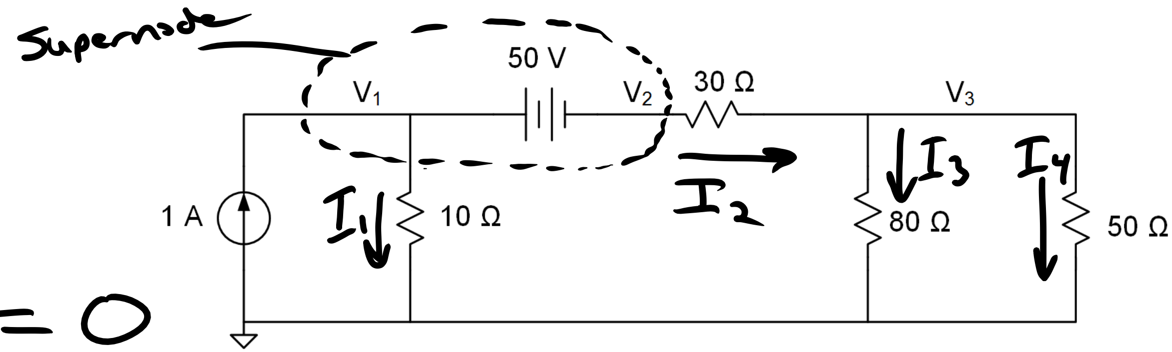
$$1A - I_1 - I_2 = 0$$

$$1A - \frac{V_1}{10\Omega} - \frac{(V_2 - V_3)}{30\Omega} = 0$$

$$V_1\left(\frac{1}{10\Omega}\right) + V_2\left(\frac{1}{30\Omega}\right) + V_3\left(-\frac{1}{30\Omega}\right) = 1A$$

$$V_1\,(100mS) + V_2\,(33.3mS) + V_3\,(-33.3mS) = 1\,A \qquad (1)$$

KCL at V_3



$$I_2 - I_3 - I_4 = 0$$

$$\frac{V_2 - V_3}{30\Omega} - \frac{V_3}{80\Omega} - \frac{V_3}{50\Omega} = 0$$

$$V_2\left(\frac{1}{30\Omega}\right) + V_3\left(-\frac{1}{30\Omega} - \frac{1}{80\Omega} - \frac{1}{50\Omega}\right) = 0$$

$$V_2(33.3\,mS) + V_3(-65.8\,mS) = 0 \qquad (2)$$

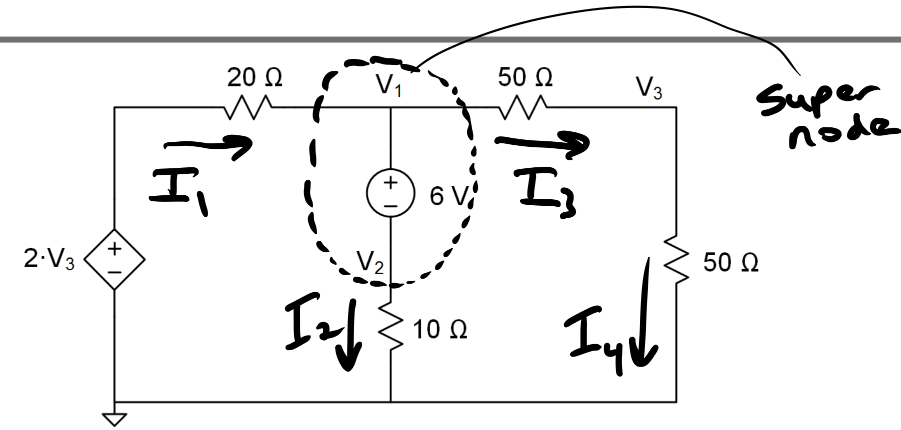
KVL at the supernode

$$V_1 - V_2 = 50\,V \qquad (3)$$

$$\begin{bmatrix} 100mS & 33.3mS & -33.3mS \\ 0 & 33.3mS & -65.8mS \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1A \\ 0 \\ 50V \end{bmatrix} \longrightarrow$$

$$\boxed{\begin{aligned} V_1 &= 15.65\,V \\ V_2 &= -34.35\,V \\ V_3 &= -17.39\,V \end{aligned}}$$

Apply nodal analysis to determine V_1, V_2, and V_3.



KCL at the supernode:

$$I_1 - I_2 - I_3 = 0$$

$$\frac{2V_3 - V_1}{20\Omega} - \frac{V_2}{10\Omega} - \frac{V_1 - V_3}{50\Omega} = 0$$

$$V_1\left(-\frac{1}{20\Omega} - \frac{1}{50\Omega}\right) + V_2\left(-\frac{1}{10\Omega}\right) + V_3\left(\frac{2}{20\Omega} + \frac{1}{50\Omega}\right) = 0$$

$$V_1(-70\,mS) + V_2(-100\,mS) + V_3(120\,mS) = 0 \qquad (1)$$

KCL at V_3 :

$$I_3 - I_4 = 0$$

$$\frac{V_1 - V_3}{50\Omega} - \frac{V_3}{50\Omega} = 0$$

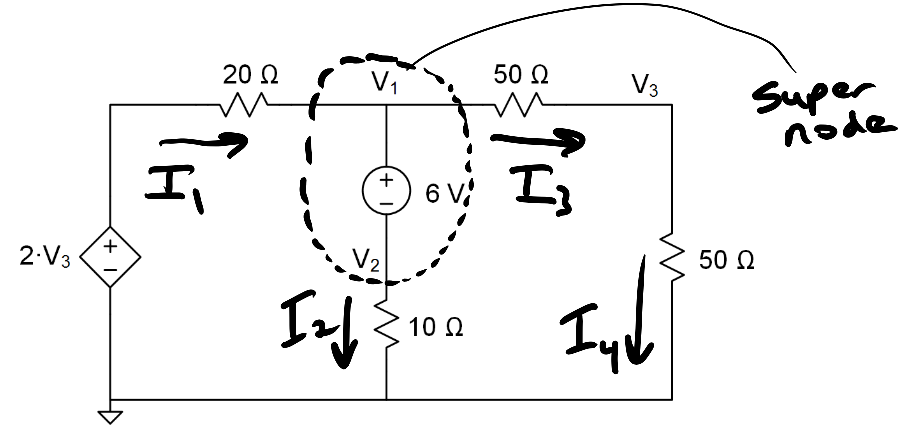
$$V_1(20mS) + V_3(-40mS) = 0 \qquad (2)$$

KVL at supernode

$$V_1 - V_2 = 6V \qquad (3)$$

$$\begin{bmatrix} -70mS & -100mS & 120mS \\ 20mS & 0 & -40mS \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6V \end{bmatrix}$$

$\longrightarrow$ $V_1 = 5.45V$, $V_2 = -0.55V$, $V_3 = 2.73V$



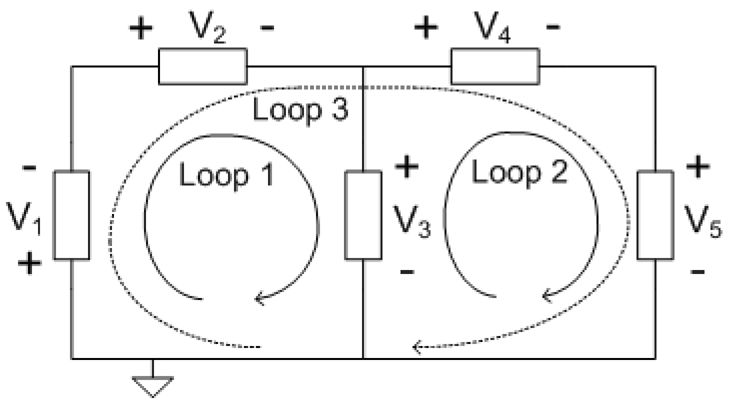
Mesh Analysis

Mesh Analysis

- ***Mesh analysis***
  - Systematic application of ***KVL***
  - Generate a system of equations
    - Mesh currents are the unknown variables
    - Number of equations equals number of unknown mesh currents
  - Solve equations to determine mesh currents
  - Determine branch currents as linear combinations of mesh currents
  - Apply Ohm's law to determine node voltages

Meshes

- What is a ***mesh***?
  - A mesh is a loop that does not contain any other loops

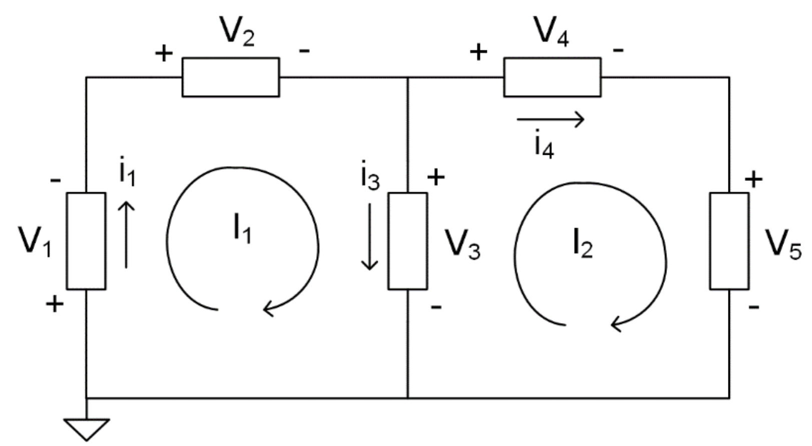


  - Loop 1 and Loop 2 are meshes, Loop 3 is not

Mesh Currents

- What is a ***mesh current***?
  - Fictitious circulating current in a mesh
  - Components of the branch currents
    - Branch currents are linear combinations of mesh currents, e.g.:

$$i_1 = I_1$$
$$i_3 = I_1 - I_2$$
$$i_4 = I_2$$



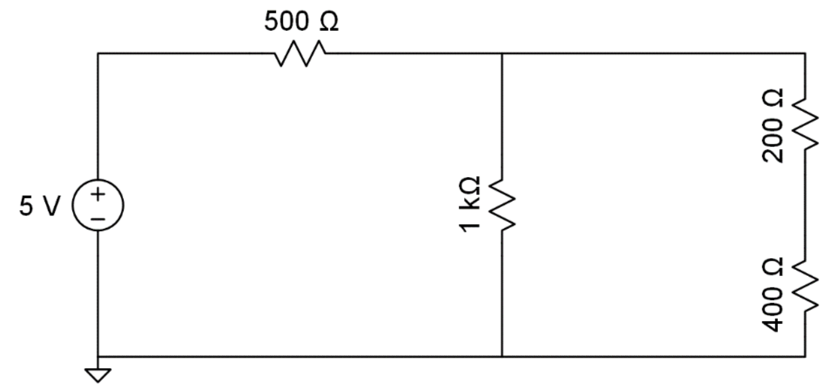
- ***Conventions***:
  - Denote mesh currents with uppercase I
  - Denote branch currents with lowercase i
  - Mesh current direction is clockwise

Mesh Analysis – step-by-step procedure

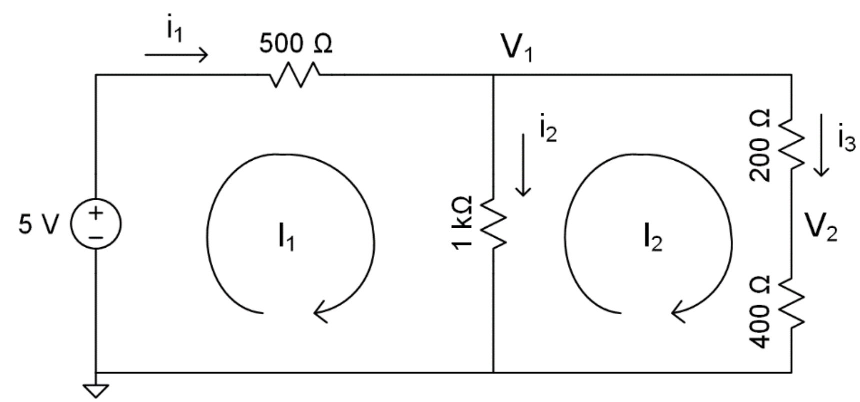
1) Identify and label all:
   - Mesh currents, I_n
   - Branch currents, i_n
   - Unknown node voltages
2) Apply KVL around each mesh
   - Follow CW direction of the mesh current
   - Use Ohm’s law to express voltage drops in terms of mesh currents
3) Solve the resulting simultaneous system of equations using Gaussian elimination, calculator, MATLAB, etc.
4) Determine branch currents from the mesh currents
5) Use Ohm’s Law and branch currents to determine node voltages

Mesh Analysis – Example

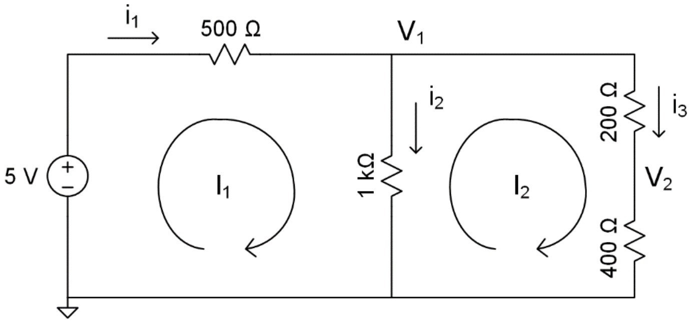
- Use ***mesh analysis*** to determine all
  - Node voltages
  - Branch currents



- **Step 1**: Identify and label all mesh currents, branch currents, and unknown node voltages
  - Two unknown mesh currents
  - Three distinct branch currents
  - Two unknown node voltages



Mesh Analysis – Example



- **Step 2**: Apply KVL around each mesh
  - KVL around mesh 1:

$$5\,V - I_1 \cdot 500\,\Omega - I_1 \cdot 1\,k\Omega + I_2 \cdot 1\,k\Omega = 0$$

  - Note that there are two components to the voltage across the $1\,k\Omega$ resistor
    - A drop due to I_1
    - A rise due to I_2
  - KVL around mesh 2:

$$-I_2 \cdot 1\,k\Omega + I_1 \cdot 1\,k\Omega - I_2 \cdot 200\,\Omega - I_2 \cdot 400\,\Omega = 0$$

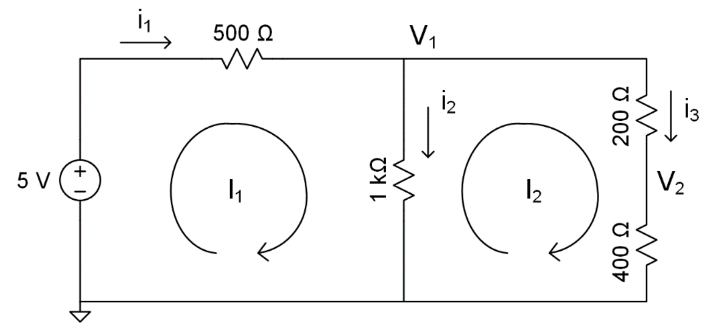
  - Again, note the two voltage components across the $1\,k\Omega$ resistor

Mesh Analysis – Example

- **Step 3**: Solve the resulting system of mesh equations
  - Cleaning up the two equations:

$$I_1 \cdot 1.5\ k\Omega - I_2 \cdot 1\ k\Omega = 5\ V$$

$$-I_1 \cdot 1\ k\Omega + I_2 \cdot 1.6\ k\Omega = 0$$



- Organizing the system of two equations into matrix form:

$$\begin{bmatrix} 1.5\ k\Omega & -1\ k\Omega \\ -1\ k\Omega & 1.6\ k\Omega \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 5\ V \\ 0 \end{bmatrix}$$

$$\mathbf{R\,I} = \mathbf{V}$$

  - Solving in MATLAB yields:

$$I_1 = 5.71\ mA$$

$$I_2 = 3.57\ mA$$

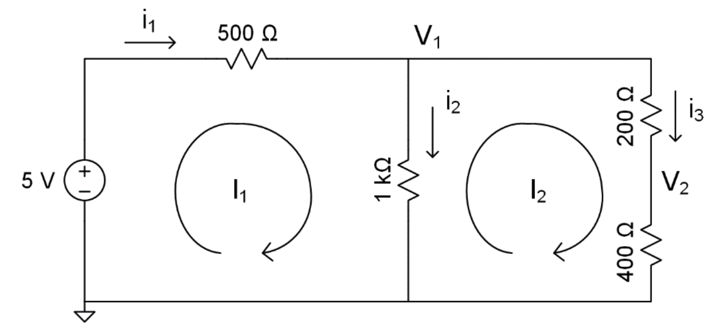
  - ***Mesh currents***, not branch currents

Command Window

```
>> R = [1.5e3,-1e3;-1e3,1.6e3]

R =

     1.5000e+003    -1.0000e+003
    -1.0000e+003     1.6000e+003

>> v = [5;0]

v =

     5.0000e+000
     0.0000e+000

>> I = R\v

I =

     5.7143e-003
     3.5714e-003
```

Mesh Analysis – Example

- **<u>Step 4</u>**: Determine branch currents from mesh currents
  - Branch current, i_1, is the same as mesh current, I_1



$$i_1 = I_1 = 5.71\,mA$$

  - Branch current, i_2, is a combination of the two opposing mesh currents
    - In the same direction as I_1
    - In the opposite direction of I_2

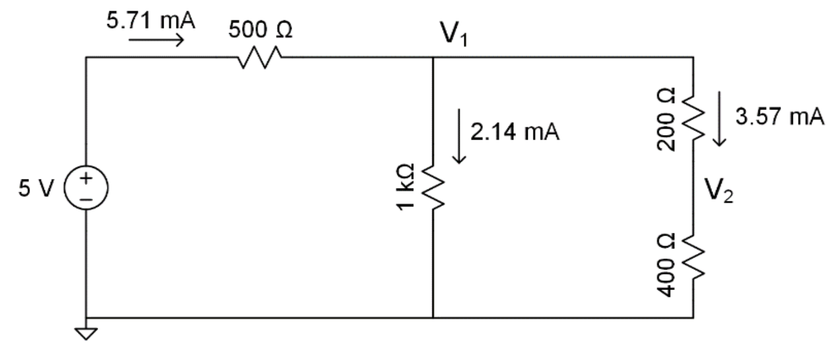
$$i_2 = I_1 - I_2 = 5.71\,mA - 3.57\,mA = 2.14\,mA$$

  - Branch current, i_3, is the same as mesh current, I_2

$$i_3 = I_2 = 3.57\,mA$$

Mesh Analysis – Example

$$i_1 = 5.71\ mA$$
$$i_2 = 2.14\ mA$$
$$i_3 = 3.57\ mA$$



- **Step 5**: Use Ohm's law and branch currents to determine node voltages

$$V_1 = 2.14\ mA \cdot 1\ k\Omega = 2.14\ V$$

$$V_2 = 3.57\ mA \cdot 400\ \Omega = 1.43\ V$$

$$V_1 = 2.14\ V$$
$$V_2 = 1.43\ V$$

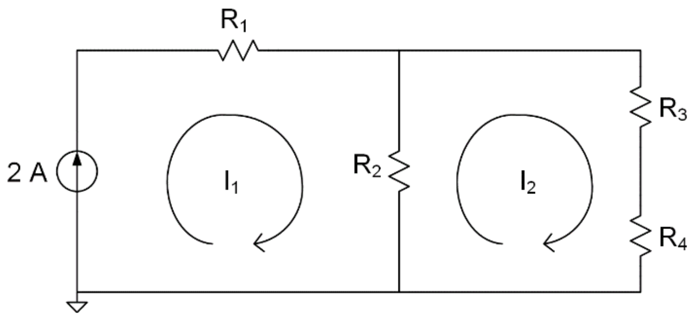
- Note that these results agree with those obtained through nodal analysis

Supermeshes

Mesh Analysis – Current Sources

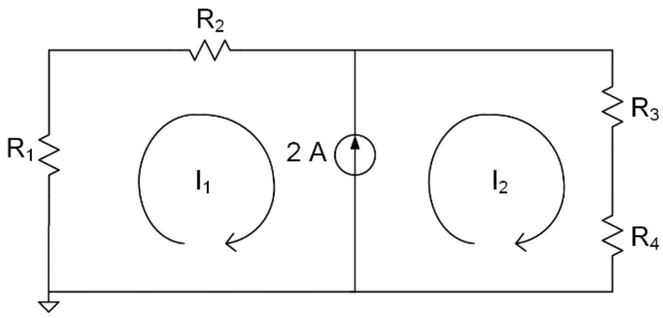
- Sometimes, we may want to perform ***mesh analysis*** on a circuit containing ***current sources***
- Two possible scenarios:

- Current source is part of only one mesh



- Here, $I_1 = 2\ A$
- Only one unknown mesh current, I_2
  - Only one mesh equation
- Mesh analysis proceeds as usual

- Current source is part of two meshes



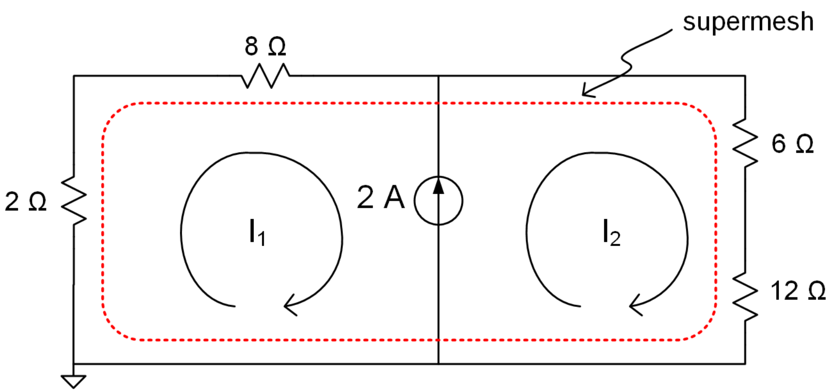
- Can't apply KVL around either mesh
  - Don't know the voltage drop across the current source

Supermesh

- Current source shared by two meshes poses a problem
  - Need to apply KVL around each mesh, but don’t know the voltage across the current source

- Solution:
  - Form a ***supermesh*** around the periphery of the two meshes that share the current source
  - Apply KVL around the supermesh
    - One equation for the two unknown mesh currents
  - Apply KCL to a node on the branch common to the two meshes in the supermesh
    - This provides the second required equation for the two unknown mesh currents

Supermesh – Example



- Meshes 1 and 2 are combined to form a ***supermesh***
- Circuit has two unknown mesh currents, I_1 and I_2
  - System of two equations is required
- KVL will be applied around the supermesh
  - Only one equation will result
- Additional required equation obtained by applying KCL to a node on the branch common to both meshes
- If multiple supermeshes intersect, they should be joined into a single supermesh

Mesh Analysis with Supermeshes – Step-by-Step

1) Identify and label all:
   - Mesh currents, I_n
   - Branch currents, i_n
   - Unknown node voltages

2) Generate a system of equations
   a) Apply KVL around each supermesh and each mesh that is not part of a supermesh
   b) Apply KCL at a node on each branch common to two meshes in each supermesh

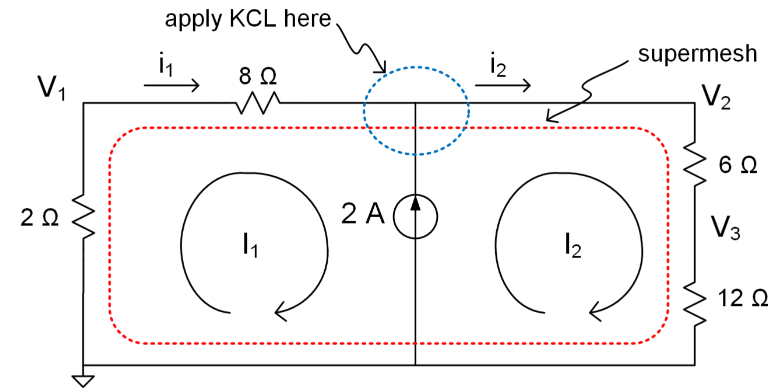
3) Solve the resulting simultaneous system of equations using Gaussian elimination, calculator, MATLAB, etc.

4) Determine branch currents from the mesh currents

5) Use Ohm's Law and branch currents to determine node voltages

Supermesh – Example

- **Step 1**: Identify and label all mesh currents, branch currents, and unknown node voltages
  - Any supermeshes are identified and labeled in this step



- **Step 2a**: Apply KVL around each mesh and each supermesh
  - Only one supermesh, and no other meshes

$$-I_1 \cdot 2\ \Omega - I_1 \cdot 8\ \Omega - I_2 \cdot 6\Omega - I_2 \cdot 12\ \Omega = 0$$

- **Step 2b**: Apply KCL at a node on the branch common to the two meshes in the supermesh

$$I_1 - I_2 + 2A = 0$$

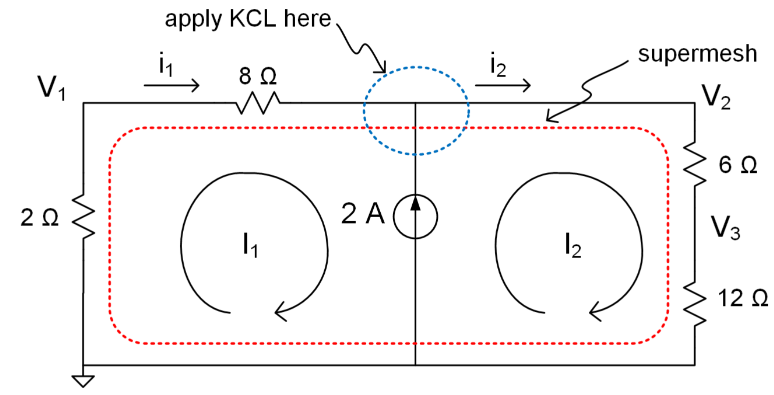
- These are the two equations needed to solve for the two unknown mesh currents, I_1 and I_2

Supermesh – Example

- **<u>Step 3</u>**: Solve the resulting system of equations
  - Rearranging the equations:

$$I_1 \cdot 10\ \Omega + I_2 \cdot 18\ \Omega = 0$$

$$-I_1 + I_2 = 2\ A$$



- In matrix form, the system of equations is

$$\begin{bmatrix} 10\ \Omega & 18\ \Omega \\ -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\ A \end{bmatrix}$$

  - Note that, similar to the supernode analysis, the two equations now have different units

- Solving in MATLAB yields

$$I_1 = -1.29\ A$$

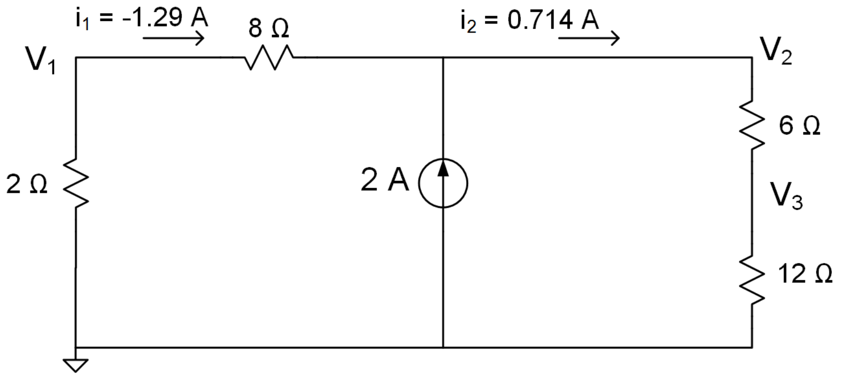
$$I_2 = 0.714\ A$$

Supermesh – Example

- **Step 4**: Determine branch currents from the mesh currents
  - Very simple in this example

$$i_1 = I_1 = -1.29\ A$$

$$i_2 = I_2 = 0.714\ A$$



- **Step 5**: Use Ohm's law and branch currents to determine node voltages

$$V_1 = -i_1 \cdot 2\ \Omega = 1.29\ A \cdot 2\ \Omega = 2.57\ V$$

$$V_2 = i_2 \cdot 18\ \Omega = 0.714\ A \cdot 18\ \Omega = 12.86\ V$$

$$V_3 = i_2 \cdot 12\ \Omega = 0.714\ \mathrm{A} \cdot 12\ \Omega = 8.57\ \mathrm{V}$$

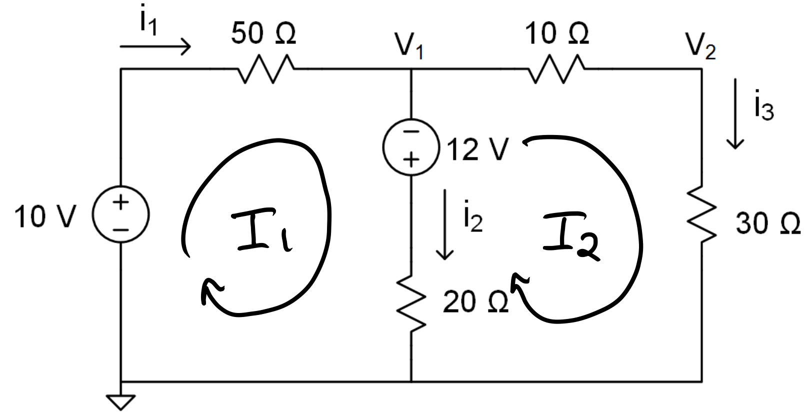
- Results of the mesh analysis:

$$i_1 = -1.29\ A$$
$$i_2 = 0.714\ A$$

$$V_1 = 2.57\ V$$
$$V_2 = 12.86\ V$$
$$V_3 = 8.57\ V$$

Example Problems

Apply mesh analysis to determine V_1, V_2, i_1, i_2, and i_3 in the following circuit.



KVL around Mesh 1:

$$10V - I_1 \cdot 50\Omega + 12V - I_1 \cdot 20\Omega + I_2 \cdot 20\Omega = 0$$

$$I_1(50\Omega + 20\Omega) + I_2(-20\Omega) = 22V$$

$$I_1(70\Omega) + I_2(-20\Omega) = 22V \qquad (1)$$

KVL around mesh 2:

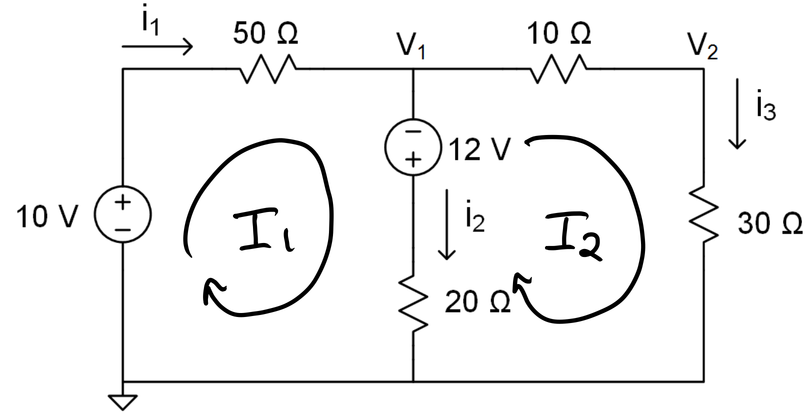
$$-I_2 \cdot 20\Omega + I_1 \cdot 20\Omega - 12V - I_2 \cdot 10\Omega - I_2 \cdot 30\Omega = 0$$

$$I_1(20\Omega) + I_2(-20\Omega - 10\Omega - 30\Omega) = 12V$$

$$I_1(20\Omega) + I_2(-60\Omega) = 12V \qquad (2)$$

In matrix form

$$\begin{bmatrix} 70\Omega & -20\Omega \\ 20\Omega & -60\Omega \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 22V \\ 12V \end{bmatrix}$$



Solving yields

$I_1 = 284.2 mA$ and $I_2 = -105.3\ mA$

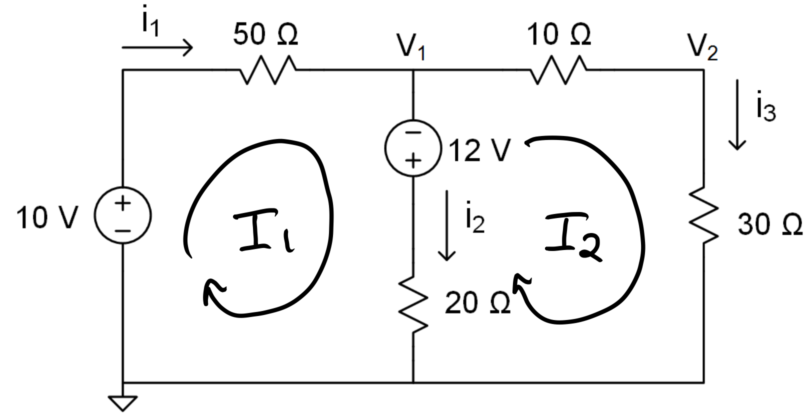
Branch currents are

$i_1 = I_1 = 284.2\ mA$

$i_2 = I_1 - I_2 = 284.2 mA + 105.3 mA = 389.5 mA$

$i_3 = I_2 = -105.3 mA$

Node voltages:



$V_1 = 10V - i_1 \cdot 50\Omega$

$V_1 = 10V - 284.2mA \cdot 50\Omega$

$V_1 = -4.21\ V$

$V_2 = i_3 \cdot 30\Omega = -105.3\ mA \cdot 30\Omega$

$V_2 = -3.16V$

$V_1 = -4.21V$

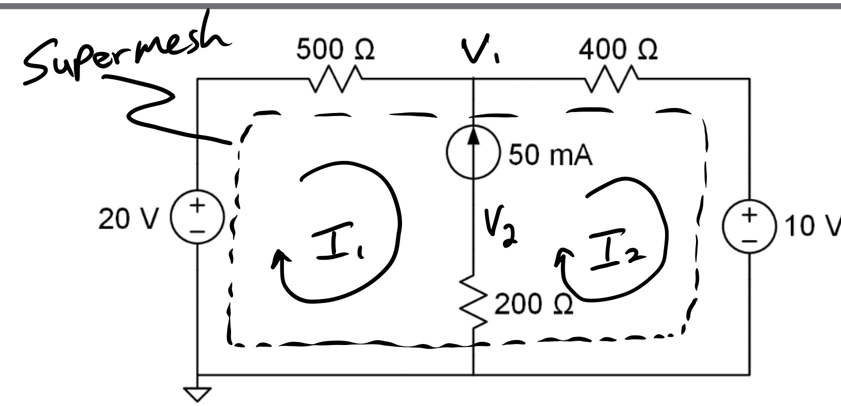
$V_2 = -3.16\ V$

$i_1 = 284.2\ mA$

$i_2 = 389.5\ mA$

$i_3 = -105.3\ mA$

Apply mesh analysis to determine the power supplied/absorbed by each of the sources in the following circuit.



KVL around super mesh:

$$20V - I_1 \cdot 500\Omega - I_2 \cdot 400\Omega - 10V = 0$$

$$I_1 \cdot 500\Omega + I_2 \cdot 400\Omega = 10V \qquad (1)$$

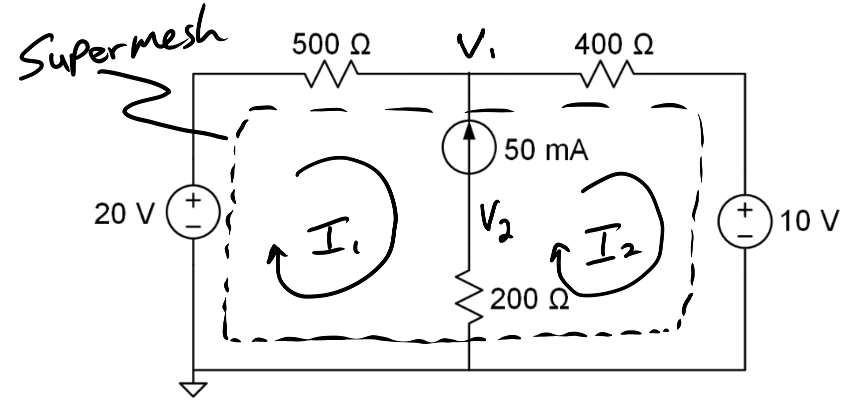
KCL at V_1:

$$I_1 + 50mA - I_2 = 0$$

$$I_1 - I_2 = -50mA \qquad (2)$$

$$\begin{bmatrix} 500\Omega & 400\Omega \\ 1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10V \\ -50mA \end{bmatrix} \longrightarrow \begin{matrix} I_1 = -11.1 mA \\ I_2 = 38.9 mA \end{matrix}$$

Node voltages



$$V_1 = 20\,V - I_1 \cdot 500\,\Omega$$

$$V_1 = 20\,V + 11.1\,mA \cdot 500\,\Omega$$

$$V_1 = 25.56\,V$$

$$V_2 = -50\,mA \cdot 200\,\Omega = -10\,V$$

Power of 20V source

$$P_{20V} = 20V(-I_1) = 20\,V(11.1\,mA) = \underline{222\,mW}$$ (absorbed)

Power of 10V source

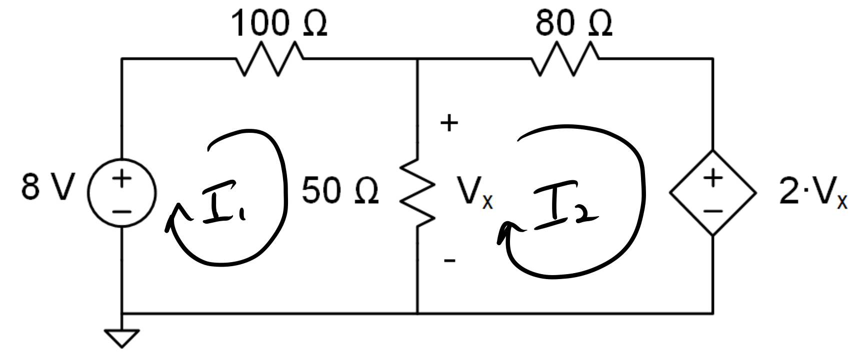
$$P_{10V} = 10V(I_2) = 10\,V(38.9\,mA) = \underline{389\,mW}$$ (absorbed)

Power of 50mA source

$$P_{50mA} = (V_2 - V_1) \cdot 50\,mA = (-10\,V - 25.56\,V) \cdot 50\,mA$$

$$P_{50mA} = -35.56\,V \cdot 50\,mA = \underline{-1.78\,W}$$ (supplied)

Apply mesh analysis to determine V_x.



KVL around mesh 1

$$8V - I_1 \cdot 100\Omega - I_1 \cdot 50\Omega + I_2 \cdot 50\Omega = 0$$

$$I_1(100\Omega + 50\Omega) + I_2(-50\Omega) = 8V$$

$$I_1(150\Omega) + I_2(-50\Omega) = 8V \qquad (1)$$

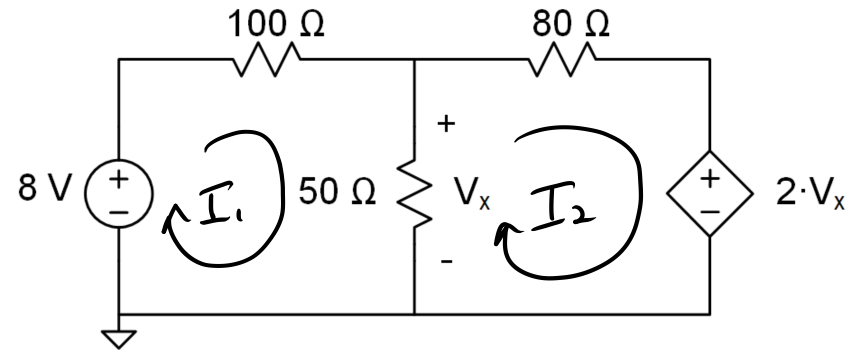
KVL around mesh 2

$$I_1\, 50\Omega - I_2\, 50\Omega - I_2 \cdot 80\Omega - 2V_x = 0$$

$$V_x = I_1 \cdot 50\Omega - I_2 \cdot 50\Omega$$

$$I_1\, 50\Omega - I_2\, 50\Omega - I_2 \cdot 80\Omega - 2 \cdot I_1\, 50\Omega + 2 I_2 \cdot 50\Omega = 0$$

$$I_1(-50\Omega) + I_2(-30\Omega) = 0 \quad (2)$$



In matrix form:

$$\begin{bmatrix} 150\Omega & -50\Omega \\ -50\Omega & -30\Omega \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 8V \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} I_1 = 34.3\,mA \\ I_2 = -57.1\,mA \end{matrix}$$

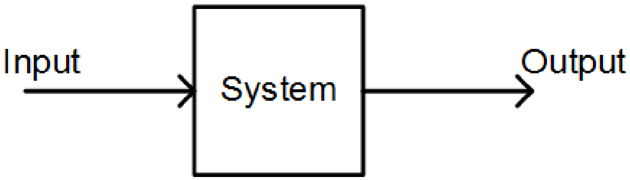
$$V_x = (I_1 - I_2)\cdot 50\Omega = (34.3mA + 57.1mA)\,50\Omega$$

$$\underline{V_x = 4.57V}$$

Linearity & Superposition

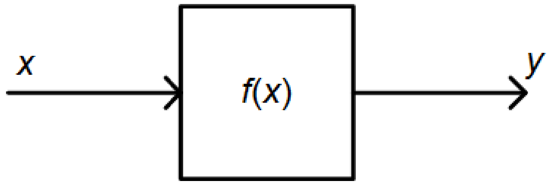
Systems

- ***System***
  - Some entity – component, group of components – with inputs and outputs
    - Electrical component
    - Electrical circuit
    - Motor, engine, robot, aircraft, etc. …



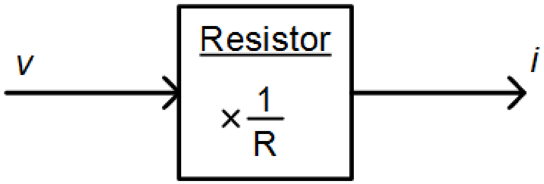
- Can think of the system as a ***mathematical function*** that operates on the input to provide the output

$$y = f(x)$$



- A resistor is a system with voltage as the input and current as the output (or vice versa)

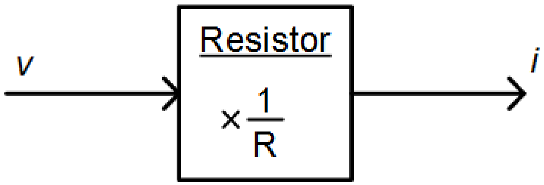
$$i = \frac{1}{R}v$$



Linear Systems

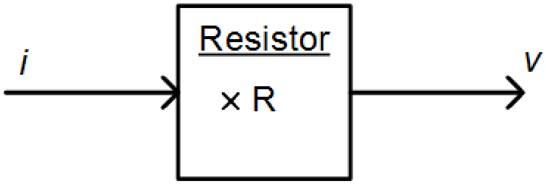
- ***Linear system***
  - A system whose ***constitutive relationship is linear***
    - Function relating input to output is an ***equation for a line***
- An ideal resistor is an example of a linear system
  - Voltage in, current out:

$$i = \frac{1}{R} \cdot v$$



    - A line with slope $1/R$
  - Current in, voltage out:

$$v = R \cdot i$$

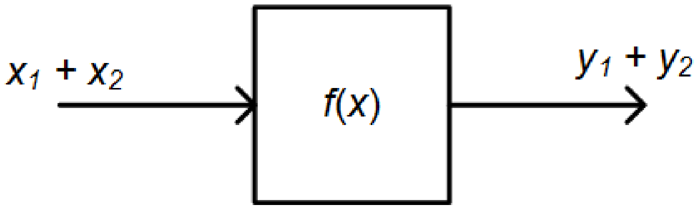


    - A line with slope R

Superposition

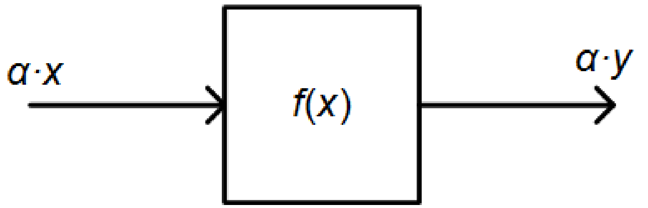
- Linear systems obey the principle of ***superposition***
- Two components to the superposition principle:
  - ***Additivity***

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$



  - ***Homogeneity***

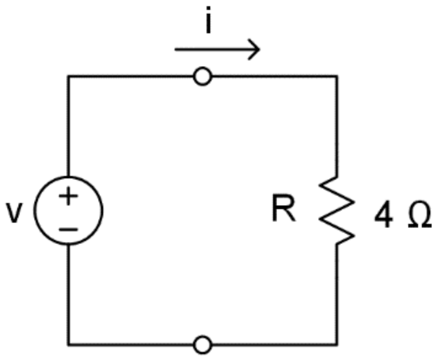
$$f(\alpha \cdot x) = \alpha \cdot f(x)$$



Superposition

- Consider an 4 Ω resistor

$$i = \frac{v}{R}$$



- $v_1 = 2\ V$

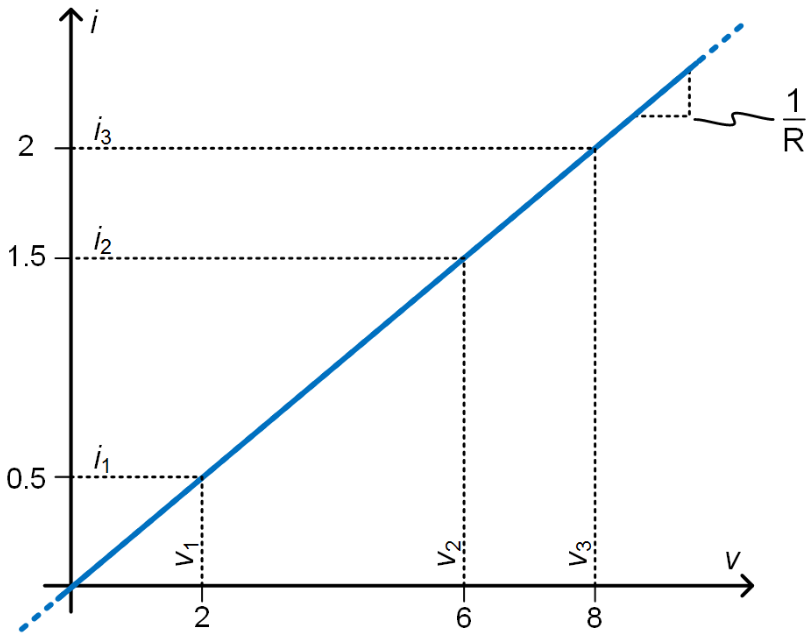
$$i_1 = \frac{2\ V}{4\ \Omega} = 0.5\ A$$

- $v_2 = 6\ V$

$$i_2 = \frac{3 \cdot 2\ V}{4\ \Omega} = 3 \cdot 0.5\ A = 1.5\ A$$

- $v_3 = 8\ V$

$$i_3 = \frac{2\ V + 6\ V}{4\ \Omega} = 0.5\ A + 1.5\ A = 2\ A$$

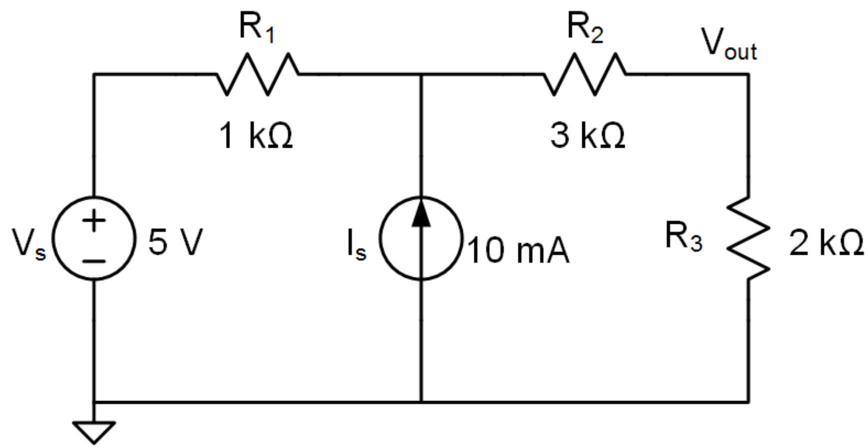


Superposition – Electrical Circuits

- Superposition applied to electrical circuits
  - Tool for analyzing networks with ***multiple sources***
- For example:
  - Output, V_{out}, is some linear combination of the inputs:

$$V_{out} = a_1 V_s + a_2 I_s$$

  - a_1 and a_2 are constants
    - If we know them, we know V_{out}



- To determine a_1
  - Set $I_s = 0$
  - Analyze the circuit to determine V_{out}
- To determine a_2
  - Set $V_s = 0$
  - Analyze the circuit to determine V_{out}

Superposition

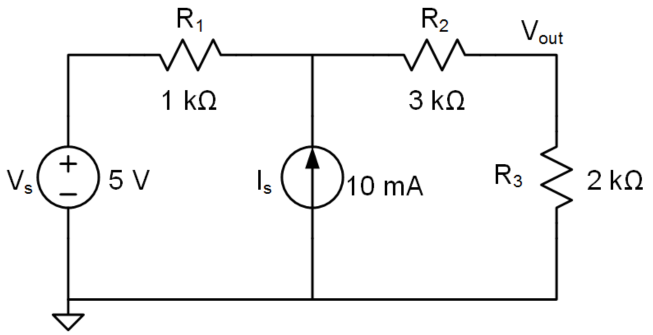
The output of a multiple-input system is the sum of the outputs due to each independent source acting individually

- Circuit analysis using superposition:
  - Set all *independent* sources to zero, except for one
  - Determine the output component due to that source
  - Repeat for all independent sources
  - Sum all output components to find the total output

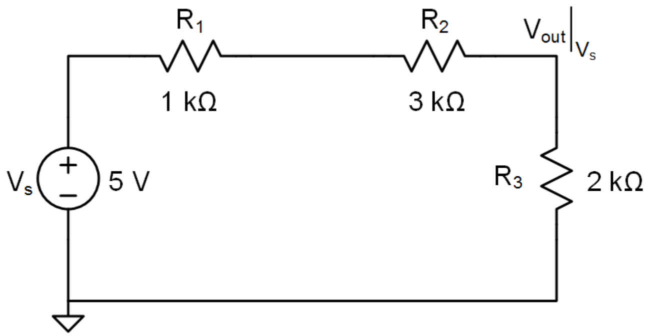
- Setting sources to zero:
  - Voltage sources become short circuits ($v = 0$)
  - Current sources become open circuits ($i = 0$)

Superposition - Example

- Apply superposition to determine the output voltage, V_{out}



- First, set the current source to zero
  - Replace it with an open circuit
  - Analyze the circuit to determine the output components due to the voltage source acting alone:



$$V_{out}\Big|_{V_s}$$

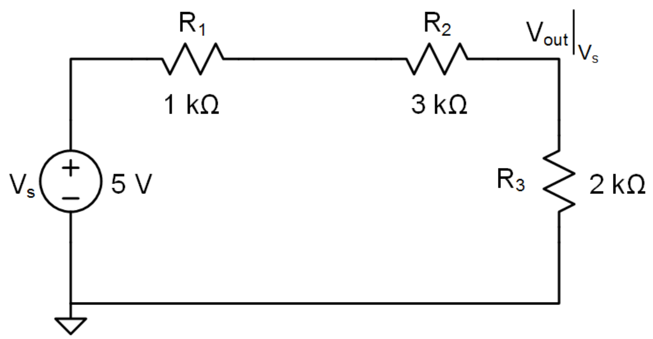
Superposition - Example

- A simple ***voltage-divider*** circuit

$$V_{out}\Big|_{V_s} = V_s \frac{R_3}{R_1 + R_2 + R_3}$$

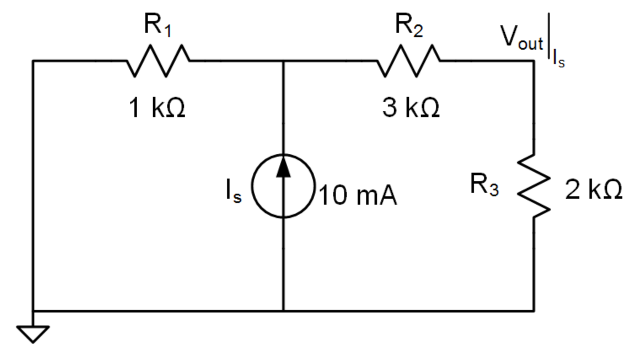
$$V_{out}\Big|_{V_s} = 5\ V \frac{2\ k\Omega}{6\ k\Omega}$$

$$V_{out}\Big|_{V_s} = 1.67\ V$$



- Next, set the voltage source to zero
  - Replace it with a short circuit
  - Analyze the circuit to determine the output components due to the current source acting alone:

$$V_{out}\Big|_{I_s}$$



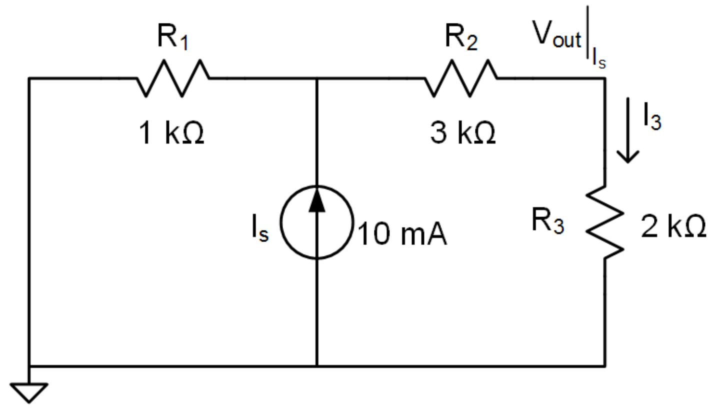
Superposition - Example

- In this case, we have a ***current-divider*** circuit
  - First, determine the current, I_3, flowing through R_3

$$I_3 = I_s \frac{R_1}{R_1 + R_2 + R_3}$$

$$I_3 = 10\ mA \frac{1\ k\Omega}{6\ k\Omega}$$

$$I_3 = 1.67\ mA$$



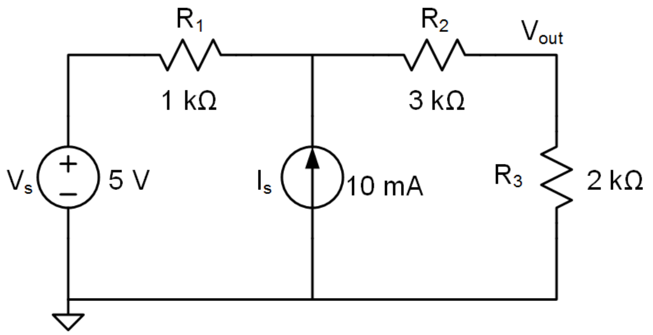
- Applying Ohm's law to R_3 gives the output voltage due to the current source

$$V_{out}\Big|_{I_s} = I_3 R_3 = 1.67\ mA \cdot 2\ k\Omega$$

$$V_{out}\Big|_{I_s} = 3.33\ V$$

Superposition - Example

- The total output due to both sources is the sum of the individual output components



$$V_{out} = V_{out}\Big|_{V_s} + V_{out}\Big|_{I_s}$$

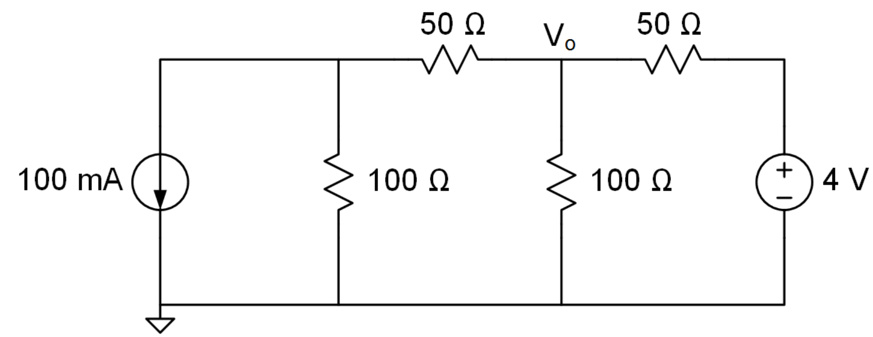
$$V_{out} = 1.67\ V + 3.33\ V$$

$$\boxed{V_{out} = 5\ V}$$

- ***Comments***:
  - Superposition applies to circuits with any number of sources and any mix of voltage and/or current sources
  - Becomes a more useful tool as circuits get more complex
  - Applies to ***all types of linear systems*** – not just electrical

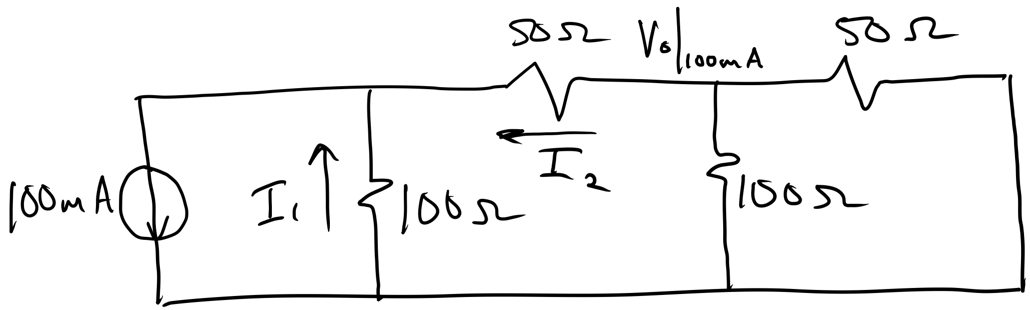
Example Problems

Apply superposition to determine V_o in the following circuit.



$$V_o = V_o\big|_{100mA} + V_o\big|_{4V}$$

First, set the 4V source to zero



Output voltage is

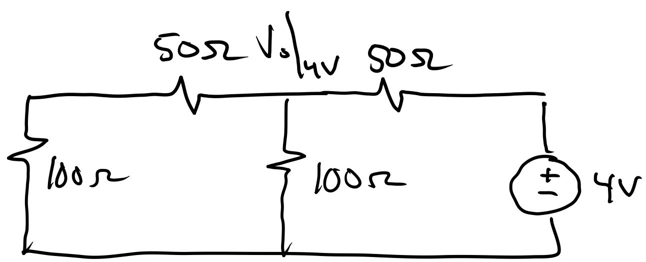
$$V_o\big|_{100mA} = -I_2\,(50\Omega \| 100\Omega) = -I_2 \cdot 33.3\Omega$$

Apply Current division to get I_2

$$I_2 = 100mA \cdot \frac{100\Omega}{100\Omega + 50\Omega + 33.3\Omega} = 54.5mA$$

$$V_o\big|_{100mA} = -54.5mA \cdot 33.3\Omega = \underline{-1.82\,V}$$

Next, set the 100mA source to zero



Applying voltage division

$$V_o\big|_{4V} = 4V \frac{(50\Omega + 100\Omega) \| 100\Omega}{50\Omega + (50\Omega + 100\Omega) \| 100\Omega} = 4V \frac{60\Omega}{50\Omega + 60\Omega}$$

$$V_o\big|_{4V} = 2.18\,V$$

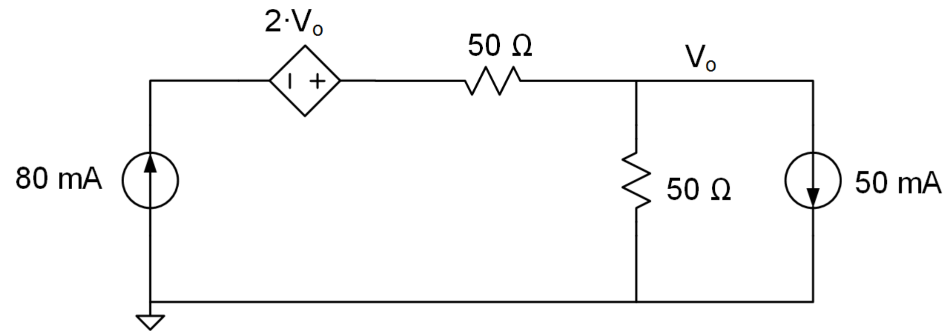
Finally, applying superposition

$$V_o = V_o\big|_{100mA} + V_o\big|_{4V}$$

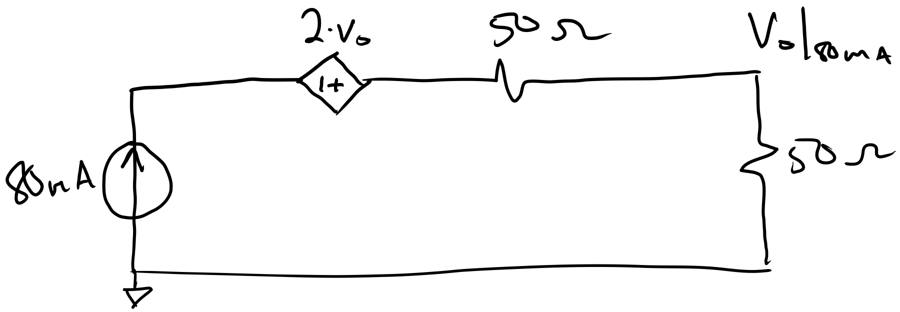
$$V_o = -1.82\ V + 2.18\ V$$

$$\boxed{V_o = 364\ mV}$$

Apply superposition to determine V_o in the following circuit.



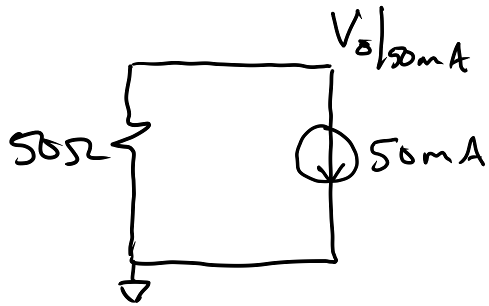
First, set the 50mA source to zero



$$V_o\Big|_{80mA} = 80mA \cdot 50\,\Omega$$

$$V_o\Big|_{80mA} = 4V$$

Next, set the 80mA source to zero



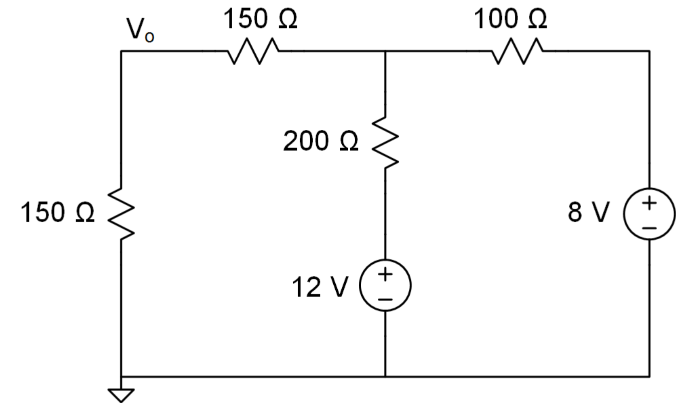
$$V_o|_{50mA} = -50mA \cdot 50\Omega = -2.5V$$

Apply superposition

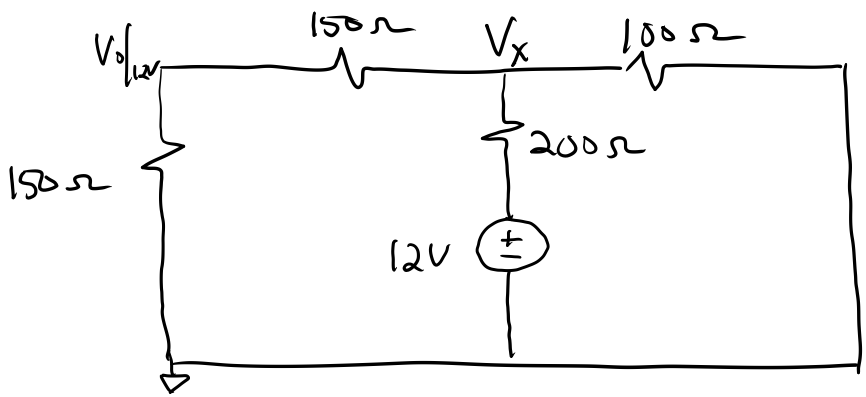
$$V_o = V_o|_{80mA} + V_o|_{50mA} = 4V - 2.5V$$

$$\boxed{V_o = 1.5\ V}$$

Apply superposition to determine V_o in the following circuit.



First, set the 8V source to zero



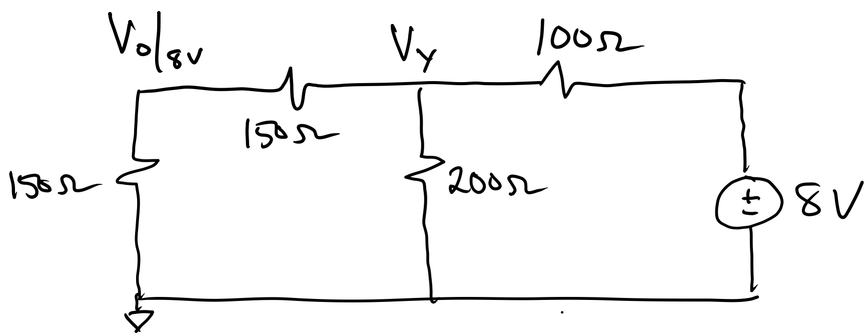
Apply voltage division twice. First, find V_x. Then

$$V_o\big|_{12V} = V_x \frac{150\,\Omega}{150\,\Omega + 150\,\Omega} = \frac{V_x}{2}$$

$$V_x = 12V \frac{100\Omega \| (150\Omega + 150\Omega)}{200\Omega + 100\Omega \| (150\Omega + 150\Omega)} = 12V \frac{75\Omega}{275\Omega} = 3.27V$$

$$V_o\big|_{12V} = \frac{V_x}{2} = \frac{3.27V}{2} = \underline{1.64V}$$

Next, set the 12V source to zero



Again, apply voltage division twice. First, find V_y. The

$$V_o\big|_{8V} = V_y \frac{150\Omega}{150\Omega + 150\Omega} = \frac{V_y}{2}$$

$$V_y = 8V \frac{200\Omega \| 300\Omega}{100\Omega + 200\Omega \| 300\Omega} = 8V \frac{120\Omega}{220\Omega} = 4.36V$$

$$V_o\big|_{8V} = \frac{V_y}{2} = \frac{4.36V}{2} = 2.18V$$

Applying superposition

$$V_o = V_o\big|_{12V} + V_o\big|_{8V} = 1.64V + 2.18V$$

$$\boxed{V_o = 3.82V}$$

Thévenin & Norton Equivalents

Thévenin Equivalent Circuits

- Thévenin's theorem:

Any two-terminal linear network of resistors and sources can be represented as single resistor in series with a single independent voltage source

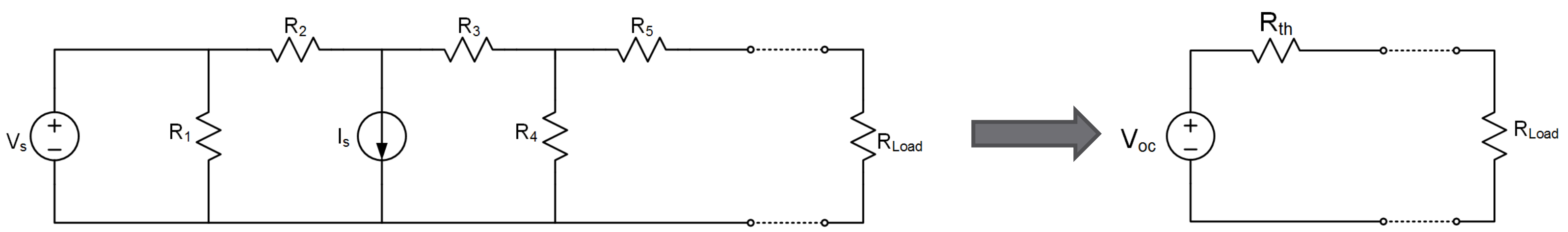
- The resistor is the ***Thévenin equivalent resistance***, R_{th}
- The voltage source is the ***open-circuit voltage***, V_{oc}

Léon Charles Thévenin, 1857 – 1926

Thévenin Equivalent Circuits

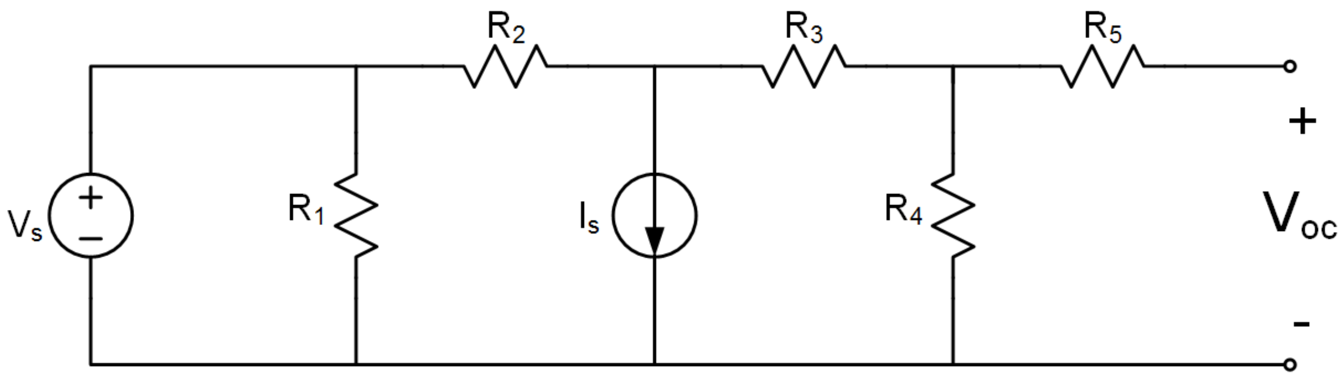
- Simplifies the analysis of complex networks
  - Quickly determine current, voltage, or power to any load connected to the network terminals

Complex network → Thévenin equivalent network



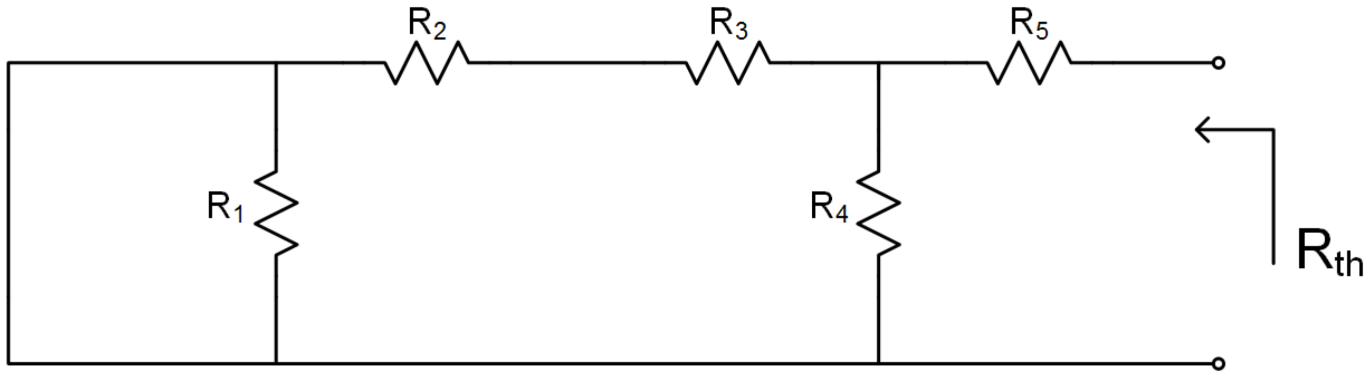
Open-Circuit Voltage - V_{oc}

- ***Open-circuit voltage***, V_{oc}
  - The terminal voltage with no load attached
- Determine V_{oc} by using most convenient method
  - Ohm's Law
  - Kirchhoff's Laws
  - Voltage or current divider
  - Nodal or mesh analysis



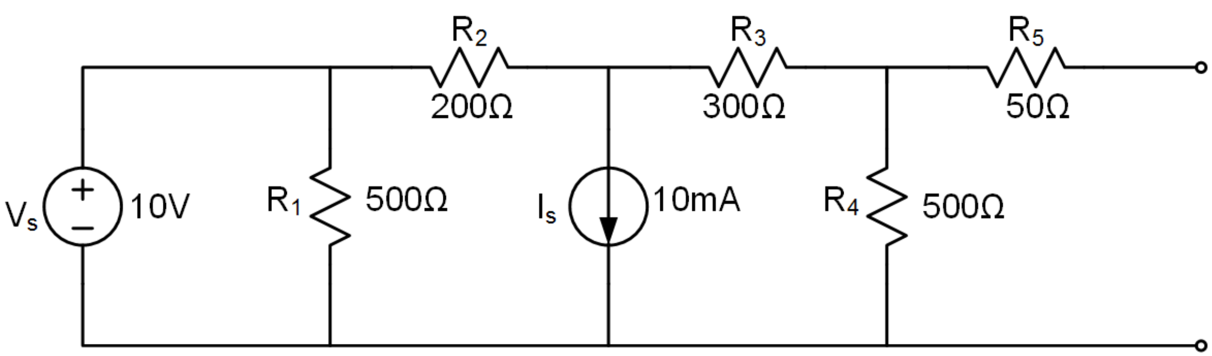
Thévenin Resistance - R_{th}

- ***Thévenin equivalent resistance***, R_{th}
  - Resistance seen between the two terminals with all independent sources set to zero
    - Voltage sources → short circuits
    - Current sources → open circuits



Thévenin Equivalent – Example

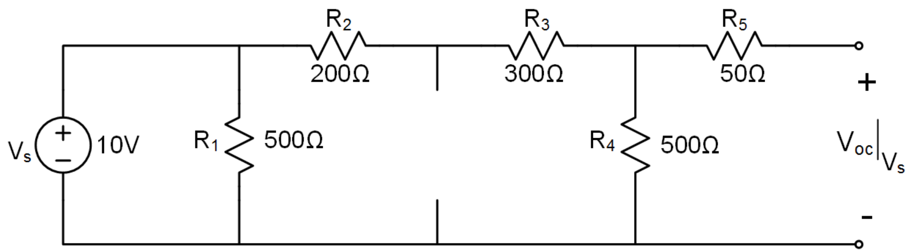
- For a 100 Ω load connected to the following network, determine:
  - Load current, I_L
  - Load voltage, V_L

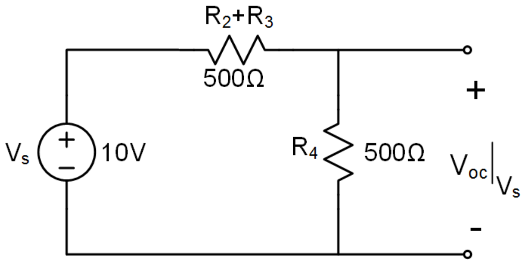


- Transform to a Thévenin equivalent circuit, then connect a 100 Ω load
  - I_L and V_L are then easily determined using Ohm's Law

Thévenin Equivalent – Example

- Analyze the circuit using any convenient technique
  - Nodal analysis would be a reasonable choice
  - Two independent sources – we'll use superposition



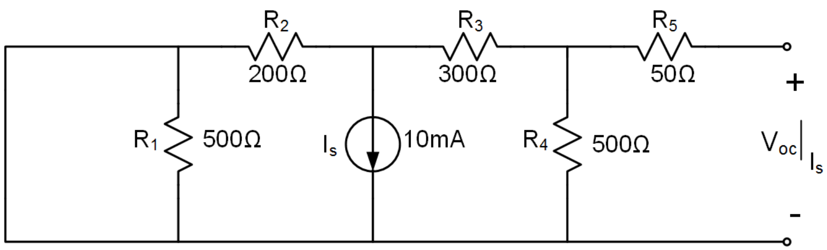


- First, find V_{oc} due to V_s
  - R_1 is in parallel with a voltage source, so it can be neglected
  - No current flows through R_5 so it can be neglected
  - Circuit reduces to a simple voltage divider

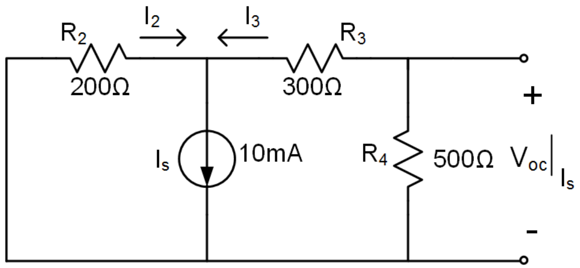
$$V_{oc}\Big|_{V_s} = 10\ V \cdot \frac{500\ \Omega}{1000\ \Omega} = 5\ V$$

Thévenin Equivalent – Example

- Next, find V_{oc} due to I_s
  - R_1 gets shorted, so it can be neglected
  - No current flows through R_5 so it can be neglected



- Circuit reduces to a simple current divider
  - Find I_3 to determine the terminal voltage



$$I_3 = 10\,mA\,\frac{200\,\Omega}{1000\,\Omega} = 2\,mA$$

  - Terminal voltage is negative due to current direction

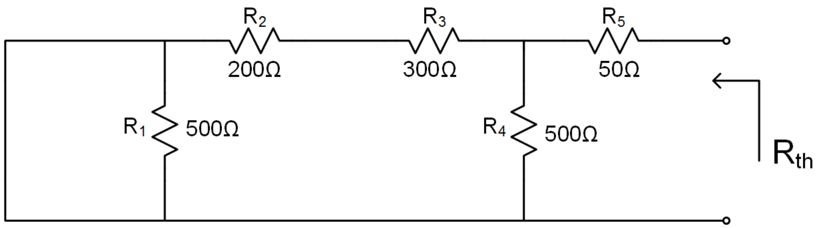
$$V_{oc}\Big|_{I_s} = -I_3 R_4 = -2\,mA \cdot 500\,\Omega = -1\,V$$

  - Open-circuit voltage is the sum of the individual components

$$\boxed{V_{oc} = V_{oc}\Big|_{V_s} + V_{oc}\Big|_{I_s} = 5\,V - 1\,V = 4\,V}$$

Thévenin Equivalent – Example

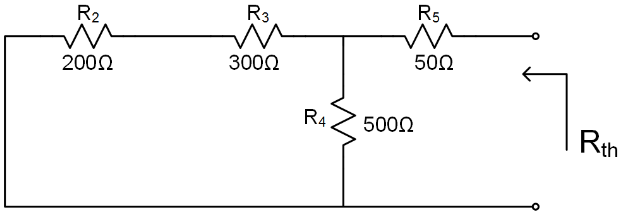
- Next, determine the Thévenin equivalent resistance, R_{th}
  - Set independent sources to zero
    - $V_s \rightarrow$ short circuit ($V = 0$)
    - $I_s \rightarrow$ open circuit ($I = 0$)
  - Determine equivalent resistance between the terminals
- R_1 is shorted
  - In parallel with a short circuit
- Combine other series and parallel resistors



$$R_{th} = R_5 + R_4||(R_2 + R_3)$$

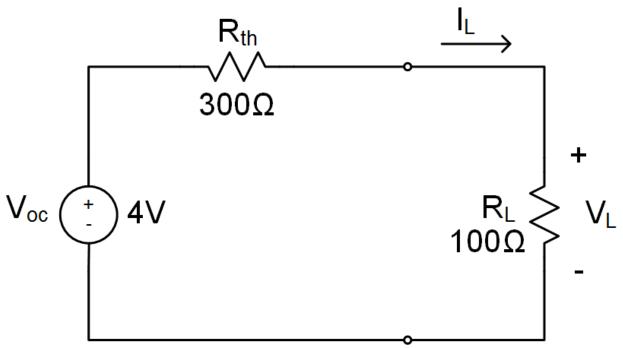
$$R_{th} = 50\ \Omega + 500\ \Omega||(200\ \Omega + 300\ \Omega)$$

$$\boxed{R_{th} = 300\ \Omega}$$



Thévenin Equivalent – Example

- The Thévenin equivalent circuit with a 100 Ω load connected:



- Voltage division gives the load voltage

$$V_L = V_{oc}\frac{R_L}{R_{th}+R_L} = 4\ V\,\frac{100\ \Omega}{400\ \Omega}$$

$$\boxed{V_L = 1\ V}$$

- Ohm's law gives the load current

$$I_L = \frac{V_L}{R_L} = \frac{1\ V}{100\ \Omega}$$

$$\boxed{I_L = 10\ mA}$$

Norton Equivalent Circuits

- Norton's theorem:

Any two-terminal linear network of resistors and sources can be represented as single resistor in parallel with a single independent current source

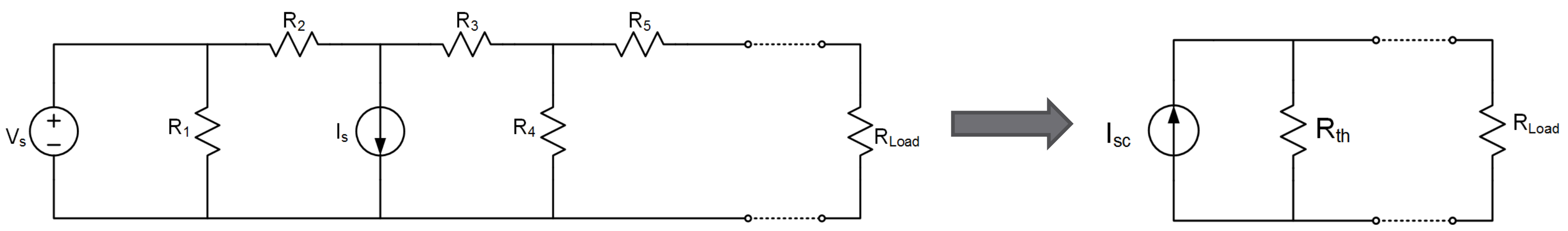
- The resistor is the ***Thévenin equivalent resistance***, R_{th}
- The current source is the ***short-circuit current***, I_{sc}

Edward Lawry Norton, 1898 – 1983

Norton Equivalent Circuits

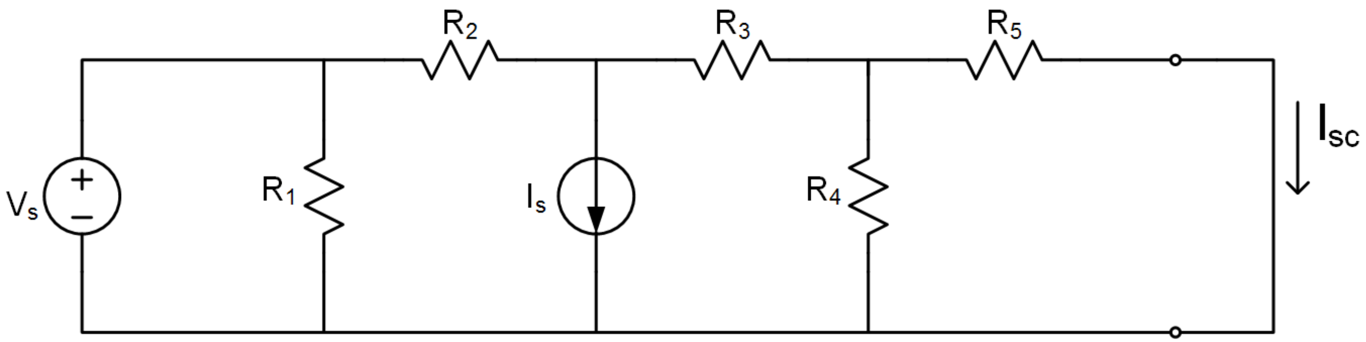
- An extension of Thévenin's Theorem
- Motivated by the development of vacuum tubes
  - More appropriately modeled with current sources
  - Same is true of the successors to tubes: transistors

Complex network → Norton equivalent network



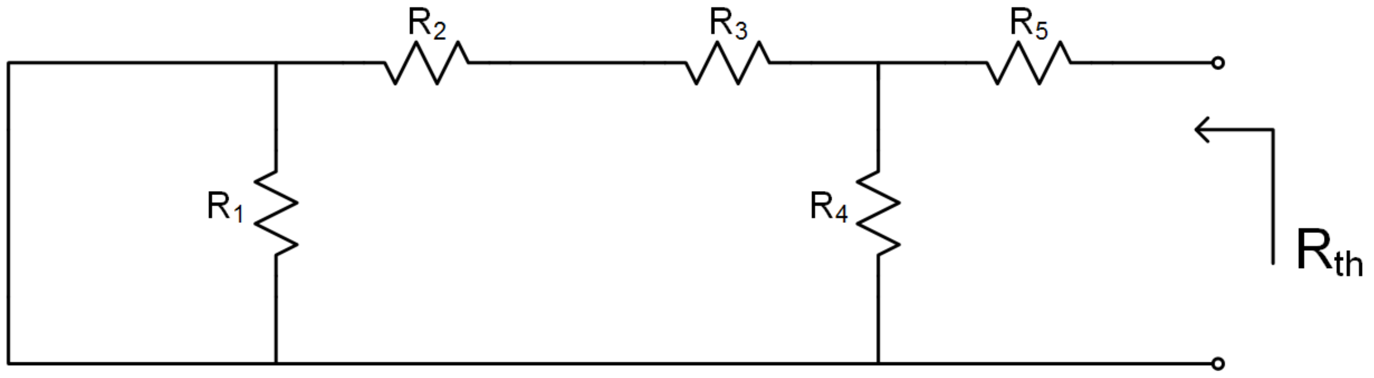
Short-Circuit Current- I_{sc}

- ***Short-circuit current***, I_{sc}
  - The current that flows between the ***short-circuited*** terminals
- Determine I_{sc} by using most convenient method
  - Ohm's Law
  - Kirchhoff's Laws
  - Voltage or current divider
  - Nodal or mesh analysis



Thévenin Resistance - R_{th}

- ***Thévenin equivalent resistance***, R_{th},
  - The same for a Norton equivalent circuit as for a Thévenin equivalent circuit
  - The resistance seen between the two terminals with all independent sources set to zero

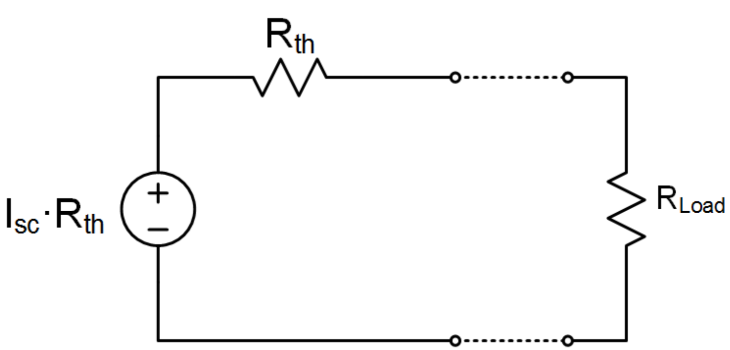


Thévenin and Norton Equivalents

- Easily convert between Thévenin and Norton equivalent circuits

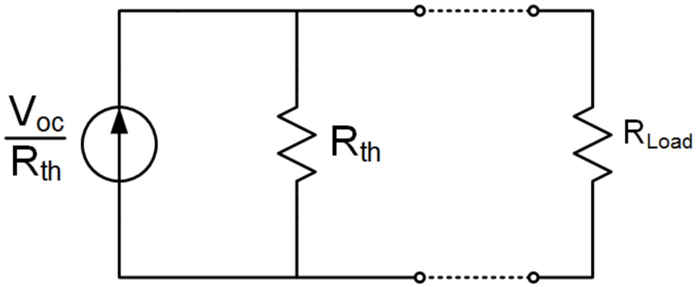
Thévenin

$$V_{oc} = I_{sc} \cdot R_{th}$$



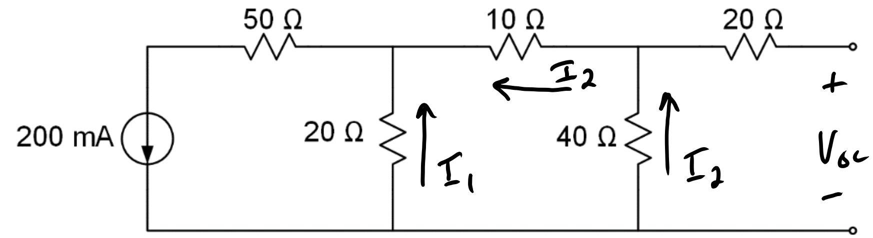
Norton

$$I_{sc} = \frac{V_{oc}}{R_{th}}$$



Example Problems

Determine both the Thévenin and Norton equivalents for the following circuit.



First, find V_{oc}

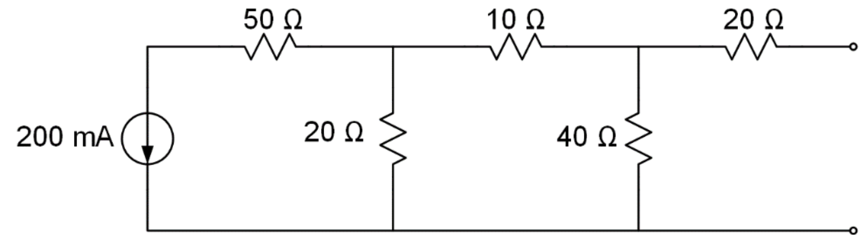
$$V_{oc} = -I_2 \cdot 40\Omega$$

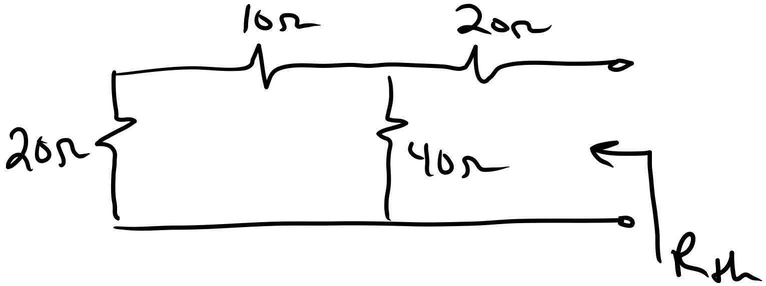
- No current through 20Ω resistor at terminals, so it has no impact on V_{oc}.
- Apply current division

$$I_2 = 200\,mA \frac{20\Omega}{20\Omega + 10\Omega + 40\Omega} = 57.14\,mA$$

$$V_{oc} = -57.14\,mA \cdot 40\Omega = -2.29\,V$$

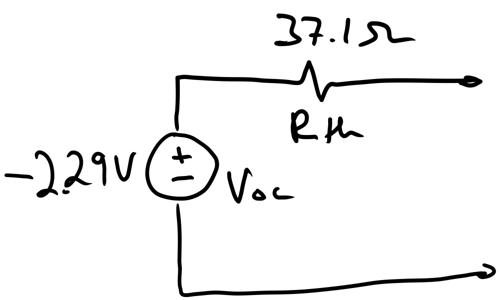
Next, find R_{th}



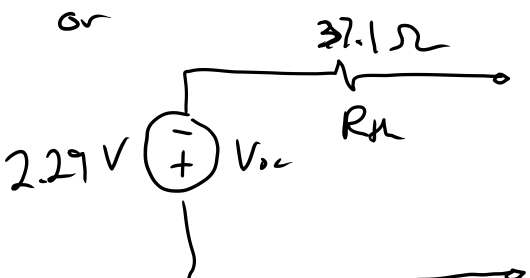


$$R_{th} = 20\Omega + 40\Omega \,//\, (10\Omega + 20\Omega) = \underline{37.1\Omega}$$

Thevenin equivalent:



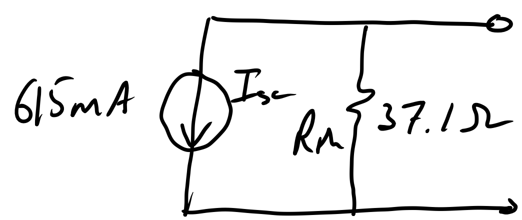
or



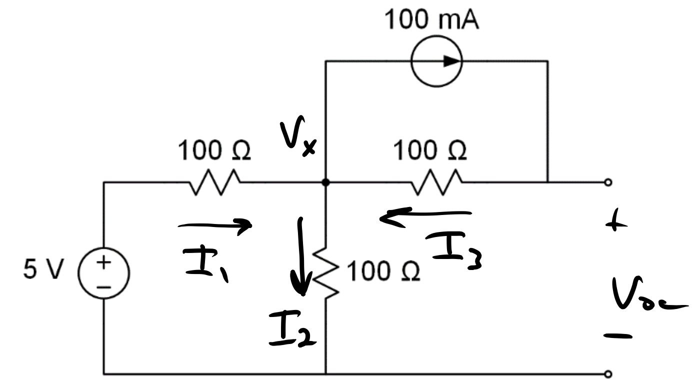
Norton equivalent

$$I_{sc} = \frac{V_{oc}}{R_{th}} = \frac{-2.29\,V}{37.1\,\Omega}$$

$$I_{sc} = -61.5\,mA$$



Determine the Thévenin equivalent for the following circuit.



First, find V_{oc}. Apply KCL at V_x:

$$I_1 - I_2 + I_3 = 0$$

$$\frac{5V - V_x}{100\Omega} - \frac{V_x}{100\Omega} + \frac{V_{oc} - V_x}{100\Omega} - 100mA = 0$$

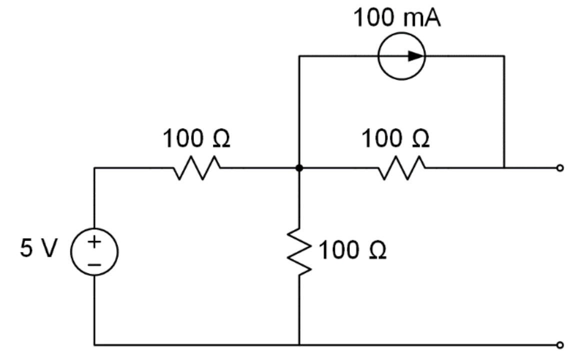
$$V_x\left(-\frac{1}{100\Omega} - \frac{1}{100\Omega} - \frac{1}{100\Omega}\right) + V_{oc}\left(\frac{1}{100\Omega}\right) = 100mA - \frac{5V}{100\Omega}$$

$$V_x(-30mS) + V_{oc}(10mS) = 50mA \qquad (1)$$

KCL at V_{oc}

$$100mA - I_3 = 0$$

$$100mA - \frac{V_{oc} - V_x}{100\Omega} = 0$$



$$V_{oc} - V_x = 100\,mA \cdot 100\,\Omega$$

$$V_{oc} = V_x + 10\,V \qquad (2)$$

Substitute (2) → (1)

$$V_x(-30\,mS) + V_x(10\,mS) + 10\,V(10\,mS) = 50\,mA$$

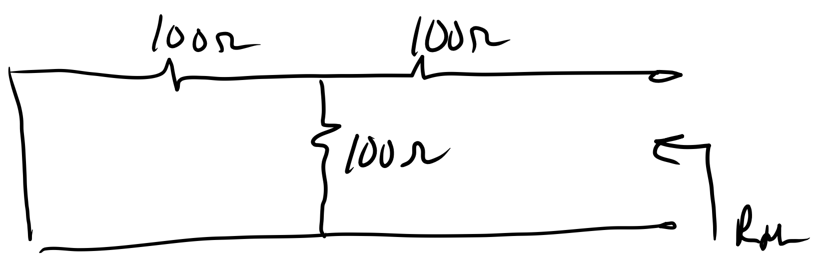
$$V_x(-20\,mS) = 50\,mA - 100\,mA = -50\,mA$$

$$V_x = \frac{-50\,mA}{-20\,mS} = 2.5\,V$$

Substitute V_x into (2)

$$V_{oc} = 2.5\,V + 10\,V = \underline{12.5\,V}$$

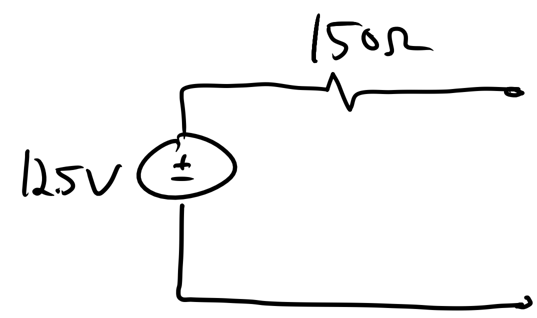
Finally, find R_{th}:



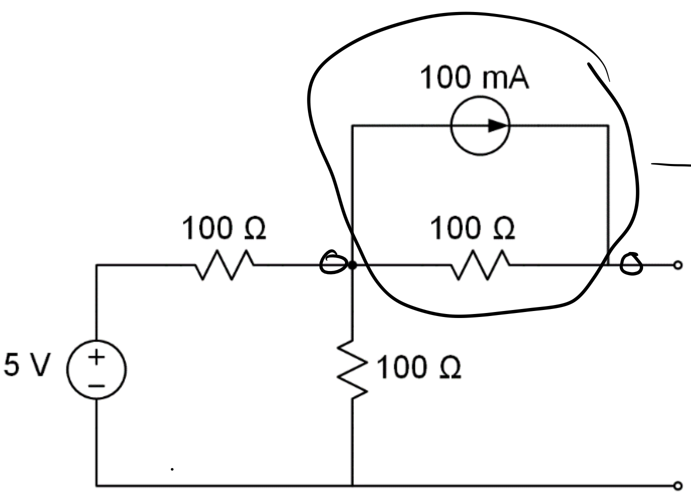
$$R_{th} = 100\,\Omega + (100\,\Omega \,||\, 100\,\Omega)$$

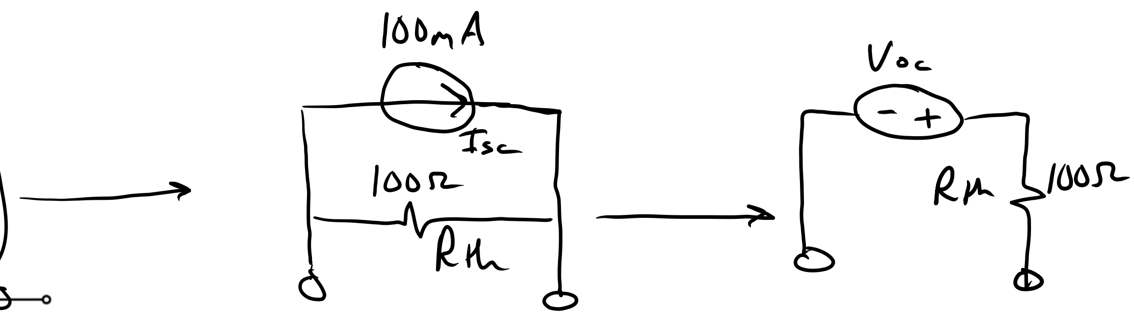
$$\underline{R_{th} = 50\,\Omega}$$

Thevenin equivalent:



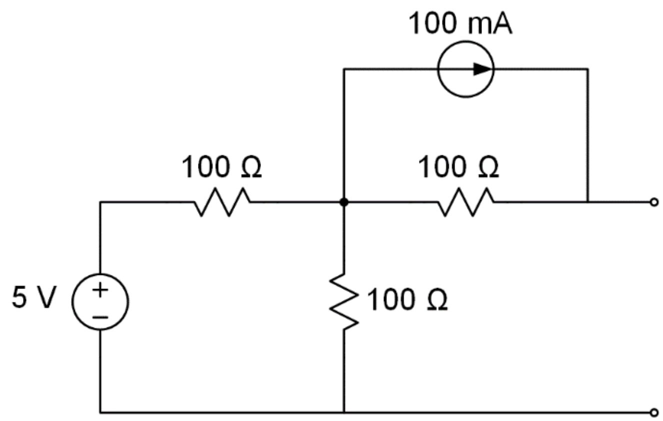
Alternatively, could do a source transformation on the current source

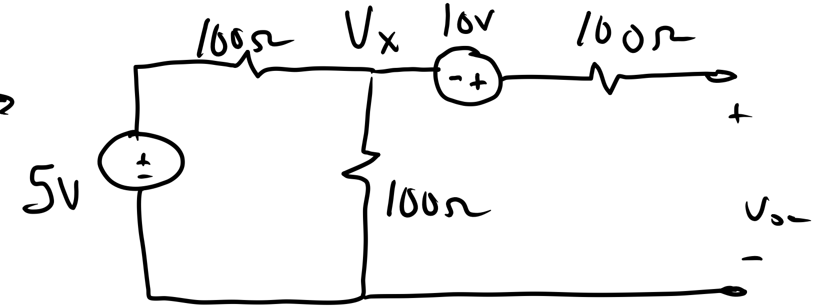




$$V_{oc} = I_{sc} \cdot R_{th} = 100mA \cdot 100\Omega$$

$$V_{oc} = 10\ V$$





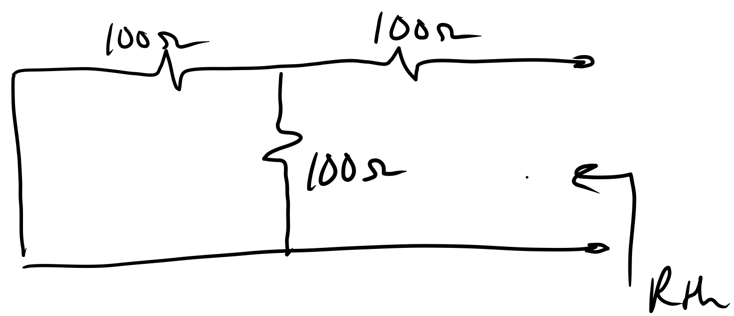
Now, apply voltage division to get V_x

$$V_x = 5V \frac{100\Omega}{100\Omega + 100\Omega} = 2.5V$$

And, clearly now, we have

$$V_{oc} = V_x + 10V = 12.5V \rightarrow \underline{V_{oc} = 12.5V}$$

R_{th} is the same as before



$$R_{th} = 100\Omega + 100\Omega \,//\, 100\Omega$$

$$\underline{R_{th} = 150\Omega}$$