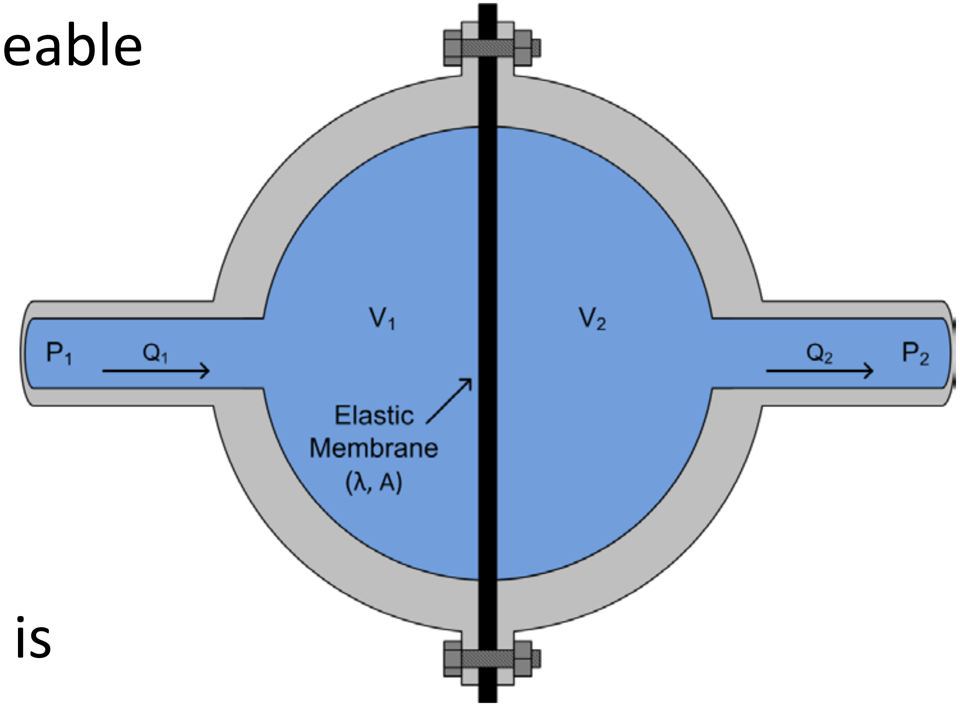
SECTION 5:
CAPACITANCE & INDUCTANCE

ENGR 201 – Electrical Fundamentals I

Fluid Capacitor

Fluid Capacitor

- Consider the following device:
  - Two rigid hemispherical shells
  - Separated by an impermeable elastic membrane
    - Modulus of elasticity, λ
    - Area, A
  - Incompressible fluid
  - External pumps set pressure or flow rate at each port
  - Total volume inside shell is constant
  - Volume on either side of the membrane may vary



Fluid Capacitor – Equilibrium

- Equal pressures

$$\Delta P = P_1 - P_2 = 0$$

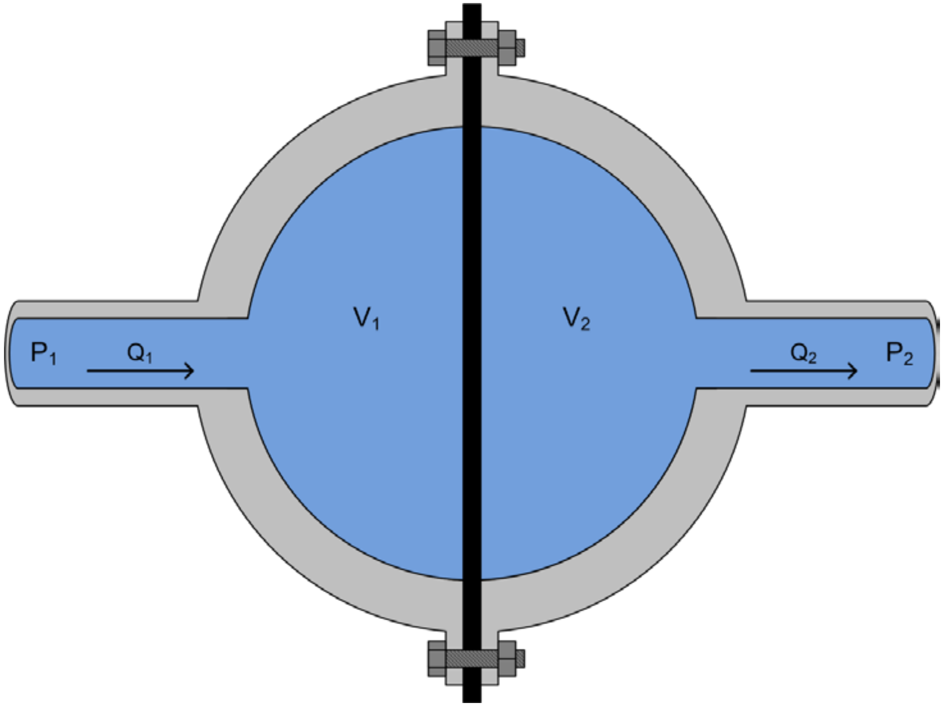
- No fluid flow

$$Q_1 = Q_2 = 0$$

- Membrane does not deform

- Equal volume on each side

$$V_1 = V_2 = \frac{V}{2}$$



Fluid Capacitor – $P_1 > P_2$

- Pressure differential

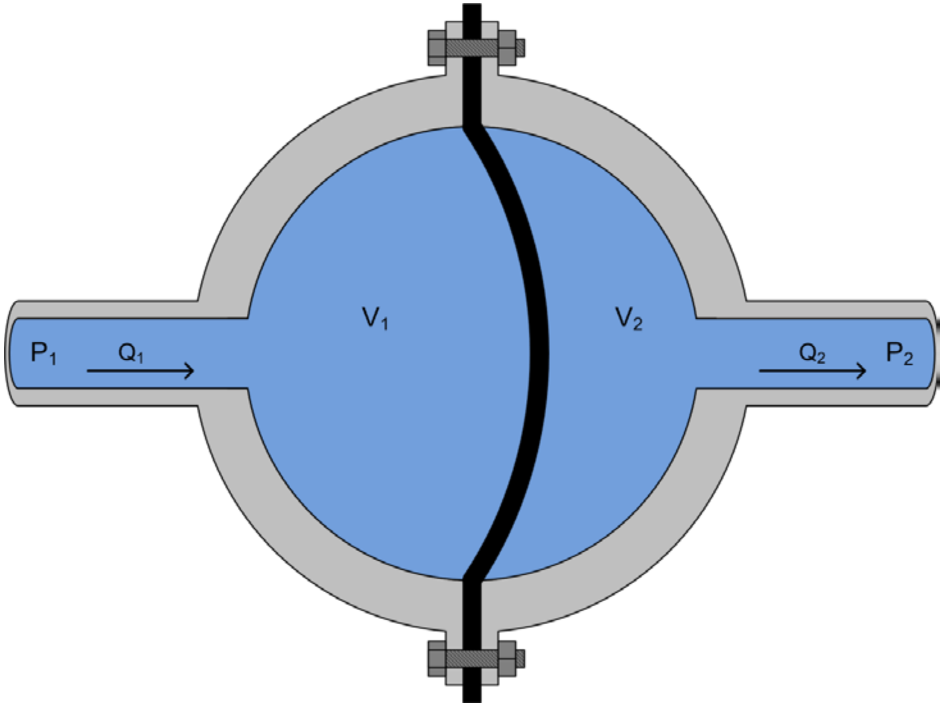
$$\Delta P = P_1 - P_2 > 0$$

- Membrane deforms
- Volume differential

$$\Delta V = V_1 - V_2 > 0$$

- Transient flow as membrane stretches, but...
- No steady-state flow
  - As $t \to \infty$

$$Q_1 = Q_2 = 0$$



Fluid Capacitor – $P_1 < P_2$

- Pressure differential

$$\Delta P = P_1 - P_2 < 0$$

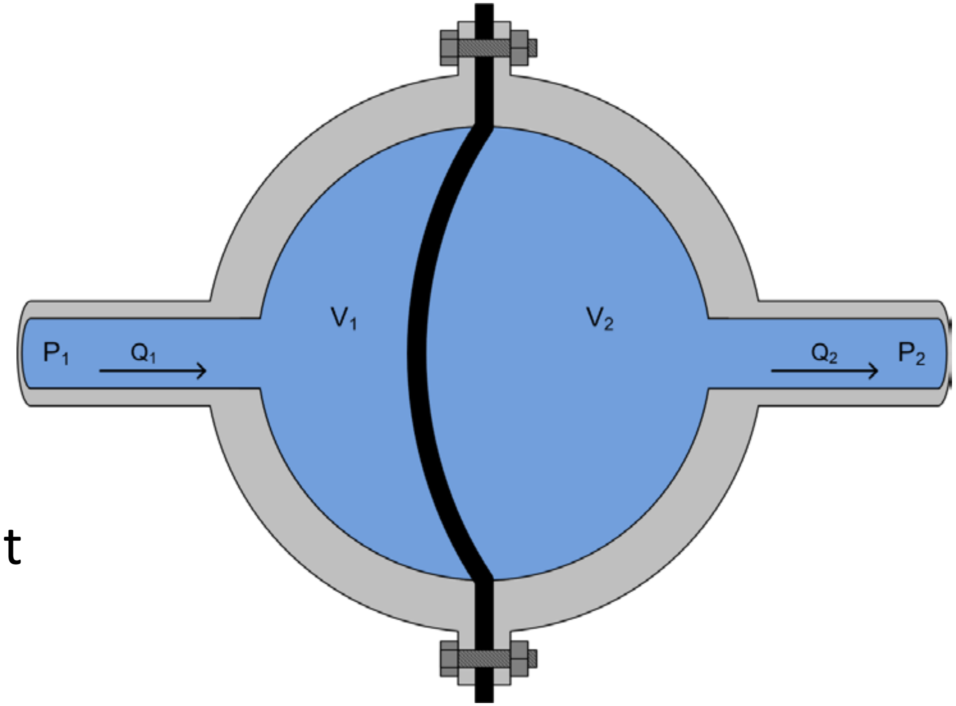
- Volume differential

$$\Delta V = V_1 - V_2 < 0$$

- ΔV proportional to:
  - Pressure differential
  - Physical properties, λ, A
- Total volume remains constant

$$V_1 + V_2 = V$$

- Again, no steady-state flow



Fluid Capacitor – Constant Flow Rate

- Constant flow rate forced into port 1

$$Q_1 \neq 0$$

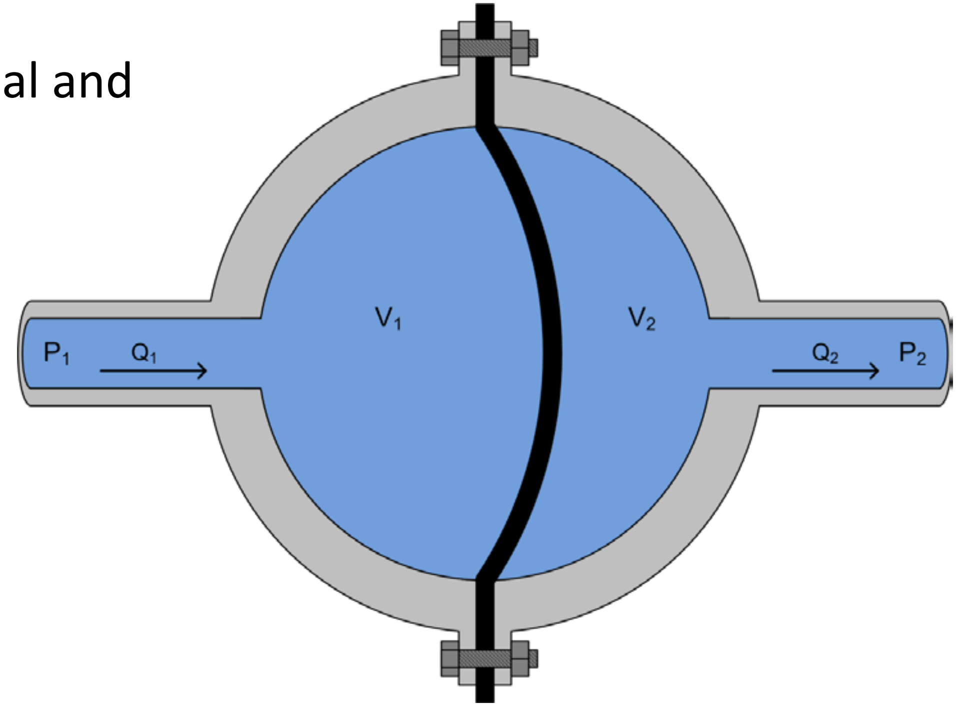
- Incompressible, so flows are equal and opposite

$$Q_1 = Q_2$$

- Volume on each side proportional to time

$$V_1 = \frac{V}{2} + Q_1 \cdot t$$

$$V_2 = \frac{V}{2} - Q_2 \cdot t = \frac{V}{2} - Q_1 \cdot t$$



- Volume differential proportional to time

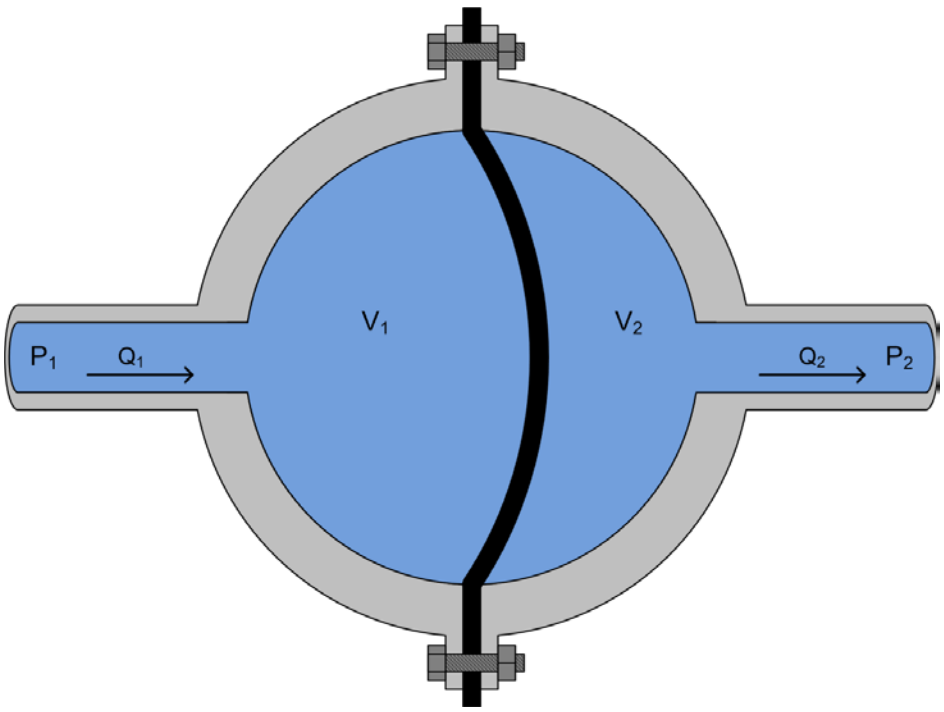
$$\Delta V = V_1 - V_2 = 2Q_1 \cdot t$$

Fluid Capacitor – Capacitance

- Define a relationship between differential volume and pressure
- ***Capacitance***

$$C = \frac{\Delta V}{\Delta P}$$

- Intrinsic device property
- Determined by physical parameters:
  - Membrane area, A
  - Modulus of elasticity, λ



Fluid Capacitor – DC vs. AC

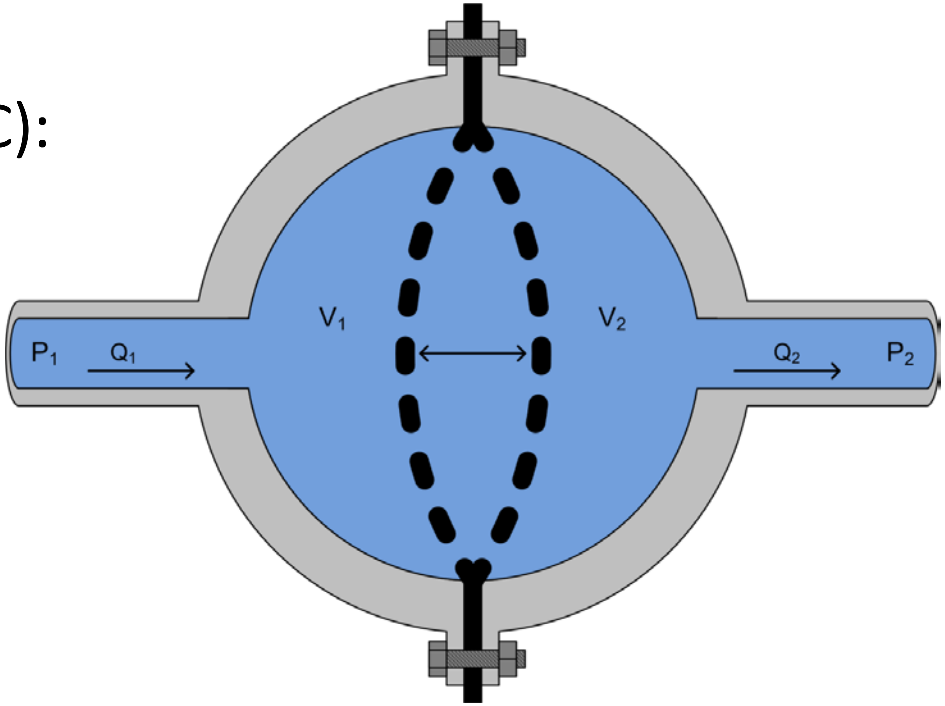
- In steady-state (DC), no fluid flows

$$Q_1 = Q_2 = 0$$

- Consider sinusoidal ΔP (AC):

$$\Delta P = P\sin(\omega t)$$

- Resulting flow rate is proportional to:
  - Rate of change of differential pressure
  - Capacitance



$$Q_1 = Q_2 = C\frac{dP}{dt} = \omega CP\cos(\omega t)$$

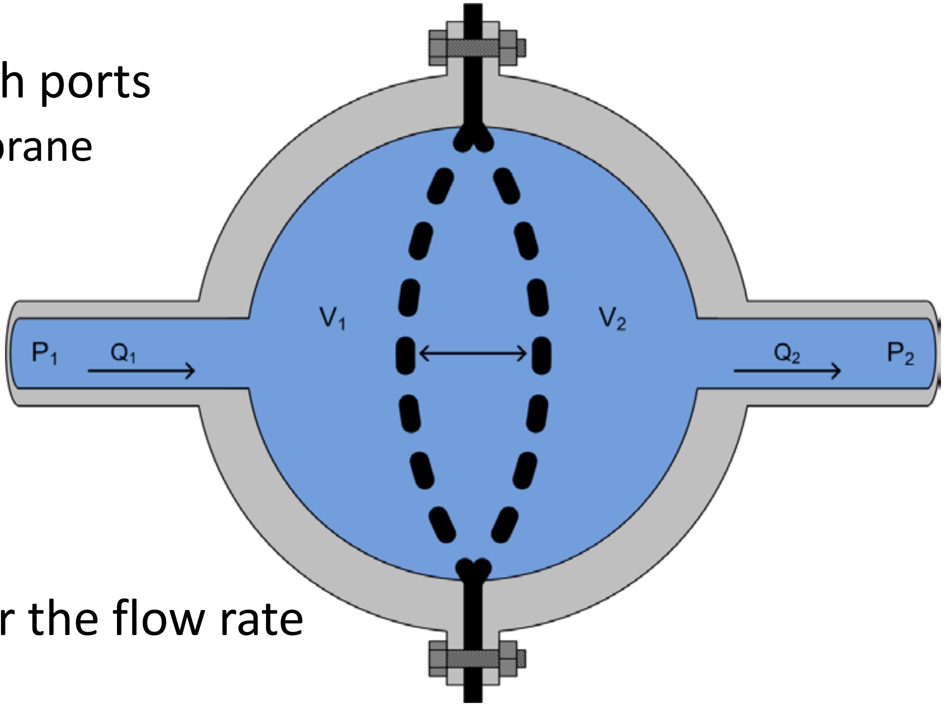
Fluid Capacitor – Time-Varying ΔP

- Equal and opposite flow at both ports

$$Q_1 = Q_2$$

- Not the same fluid flowing at both ports
  - Fluid cannot permeate the membrane

- ***Fluid appears to flow through the device***
  - Due to the displacement of the membrane
  - A ***displacement flow***



- The faster ΔP changes, the higher the flow rate

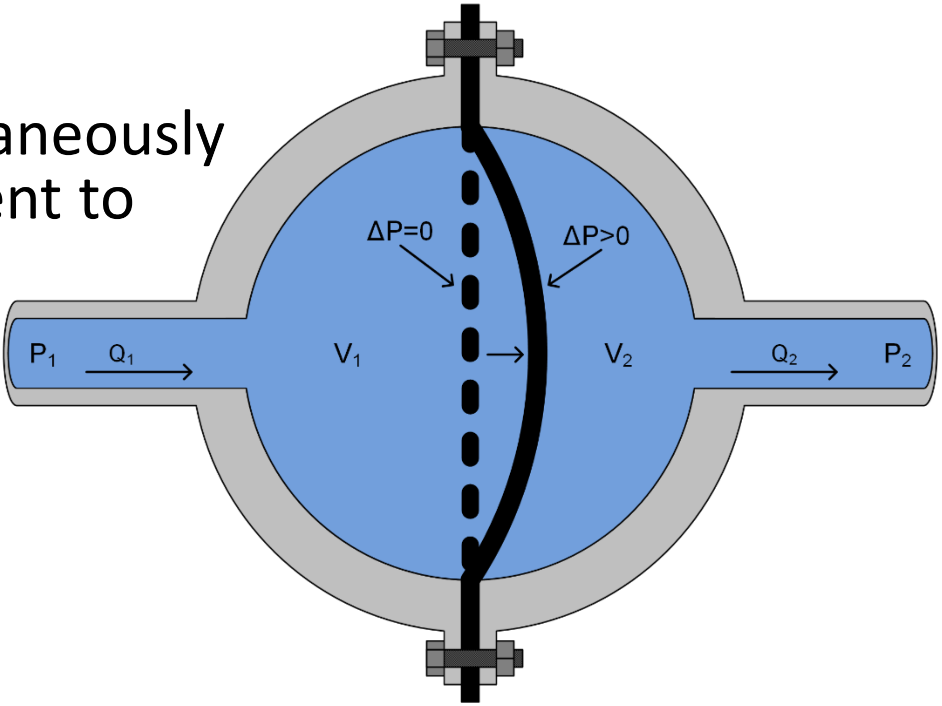
$$Q \propto \omega$$

- The larger the capacitance, the higher the flow rate

$$Q \propto C$$

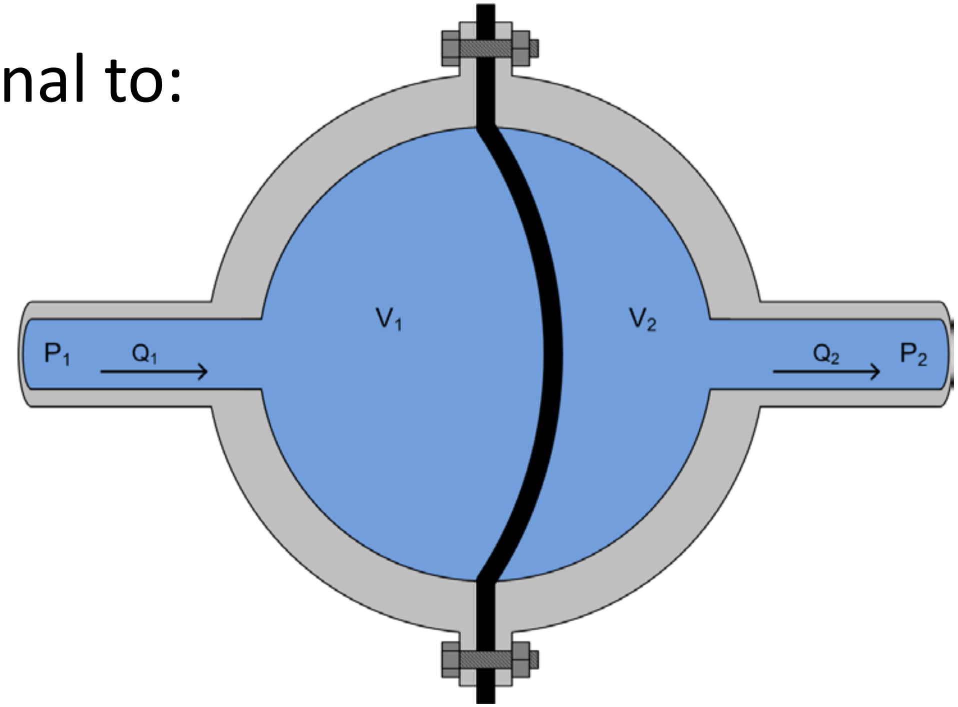
Fluid Capacitor – Changing ΔP

- A given ΔP corresponds to a particular membrane displacement
  - Forces must balance
- Membrane cannot instantaneously jump from one displacement to another
- Step change in displacement/pressure is impossible
  - Would require an infinite flow rate
- ***Pressure across a fluid capacitor cannot change instantaneously***



Fluid Capacitor – Energy Storage

- Stretched membrane ***stores energy***
  - ***Potential energy***
- Stored energy proportional to:
  - ΔP
  - ΔV
- Energy released as membrane returns
  - P and Q are supplied



Electrical Capacitors

Electrical Capacitor

- In the electrical domain, our “working fluid” is ***positive electrical charge***
- In either domain, we have a ***potential-driven flow***

Fluid Domain	Electrical Domain
Pressure – P	Voltage – V
Volumetric flow rate – Q	Current – I
Volume – V	Charge – Q

Electrical Capacitor

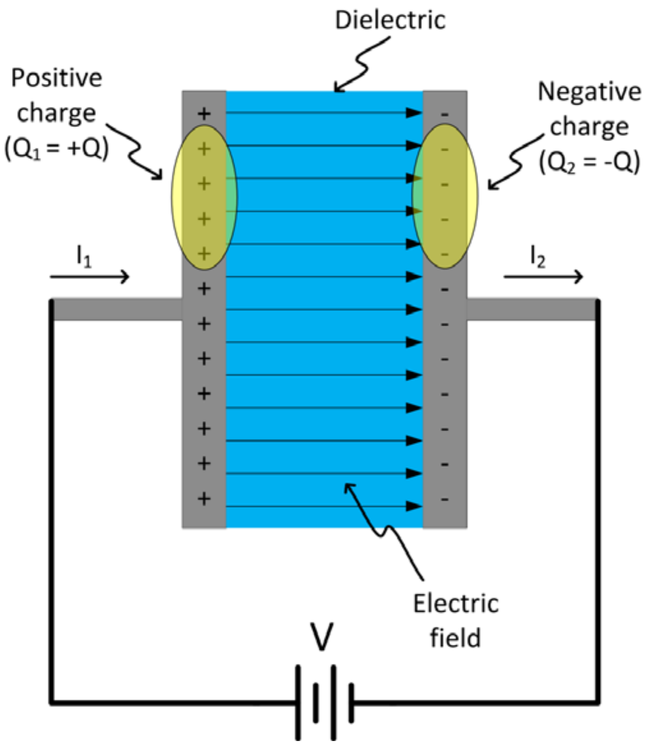
- Parallel-plate capacitor
  - Parallel metal plates
  - Separated by an insulator
- ***Applied voltage creates charge differential***
  - Equal and opposite charge

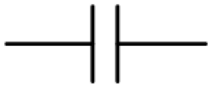
$$Q_1 = -Q_2$$

  - Zero net charge
- Equal current

$$I_1 = I_2$$

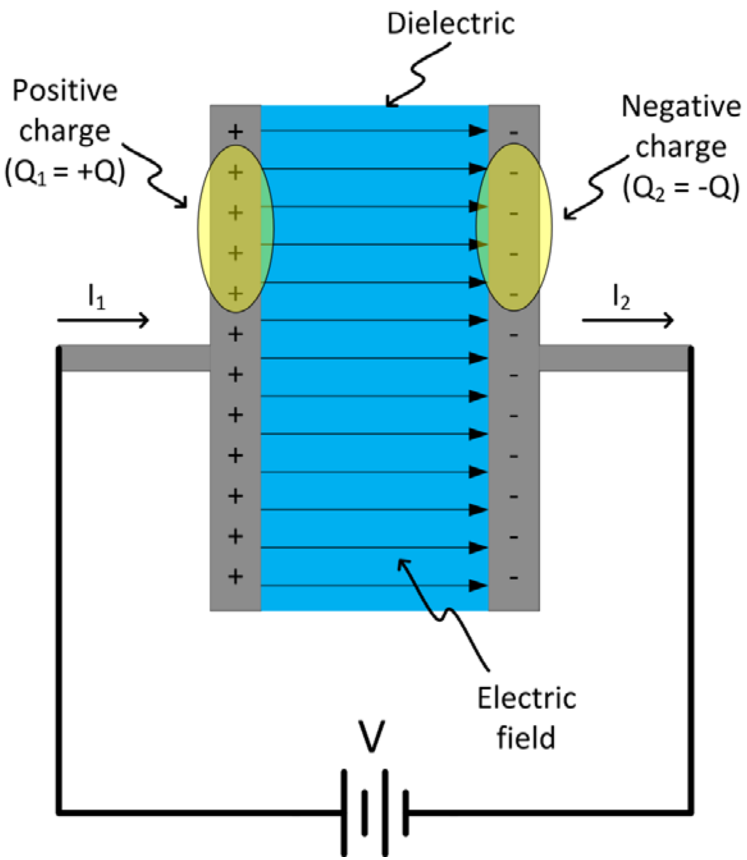
  - What flows in one side flows out the other



- Schematic symbol: 
- Units: Farads (F)

Electrical Capacitor – Electric Field

- Charge differential results in an ***electric field***, $\boldsymbol{E}$, in the dielectric
  - Units: V/m
- $|\boldsymbol{E}|$ is inversely proportional to dielectric thickness, d
- Above some maximum electric field strength, dielectric will ***break down***
  - Conducts electrical current
  - Maximum capacitor voltage rating

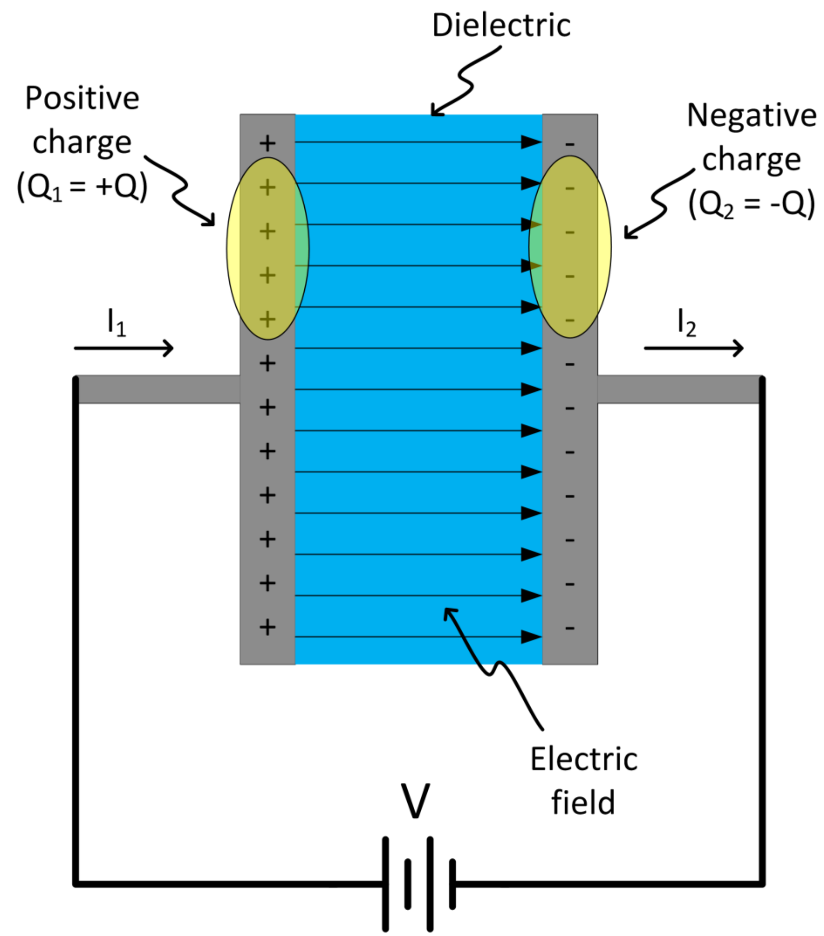


Electrical Capacitor - Capacitance

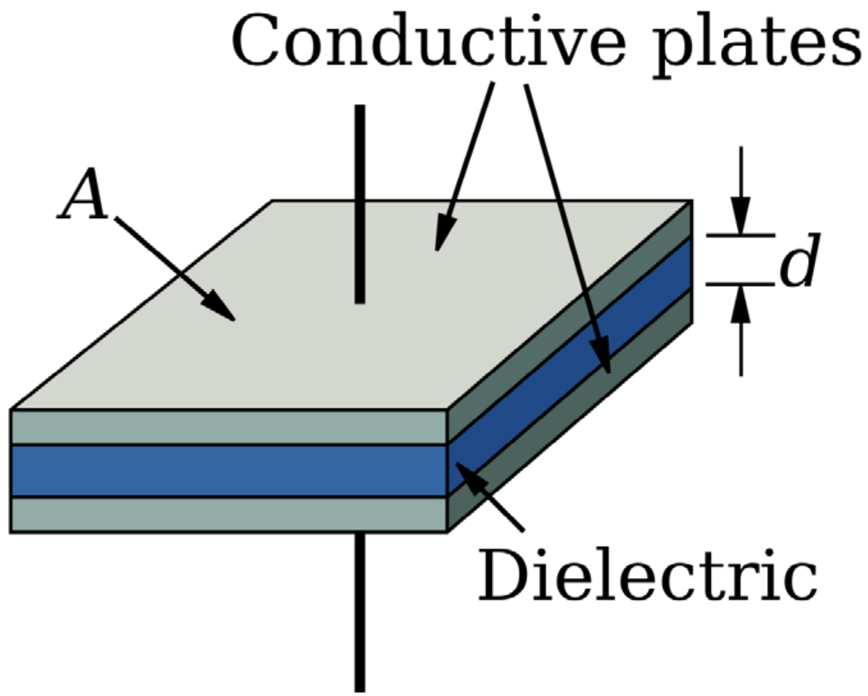
- ***Capacitance***
  - Ratio of charge to voltage

$$C = \frac{Q}{V}$$

- Intrinsic device property
- Proportional to physical parameters:
  - Dielectric thickness, d
  - Dielectric constant, ε
  - Area of electrodes, A



Parallel-Plate Capacitor



- Capacitance

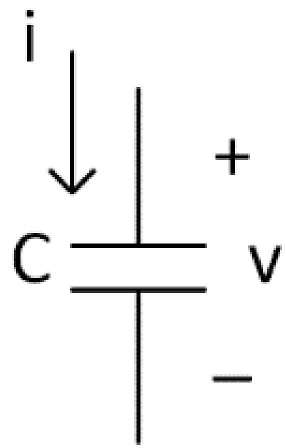
$$C = \frac{\varepsilon A}{d}$$

  - ε: dielectric permittivity
  - A: area of the plates
  - d: dielectric thickness

- Capacitance is maximized by using:
  - High-dielectric-constant materials
  - Thin dielectric
  - Large-surface-area plates

Capacitors – Voltage and Current

- Current through a capacitor is proportional to
  - Capacitance
  - Rate of change of the voltage



$$i(t) = C\frac{dv}{dt}$$

- Voltage across capacitor results from an accumulation of charge differential
  - Capacitor integrates current

$$v(t) = \frac{1}{C}\int i(t)dt$$

Voltage Change Across a Capacitor

- For a step change in voltage,

$$\frac{dv}{dt} = \infty$$

  - The corresponding current would be ***infinite***

- ***Voltage across a capacitor cannot change instantaneously***

- Current can change instantaneously, but voltage is the integral of current

$$\lim_{\Delta t \to 0} \Delta V = \lim_{\Delta t \to 0} \frac{1}{C} \int_{t_0}^{t_0 + \Delta t} i(t)dt = 0$$

Capacitors – Open Circuits at DC

- Current through a capacitor is proportional to the time rate of change of the voltage across the capacitor

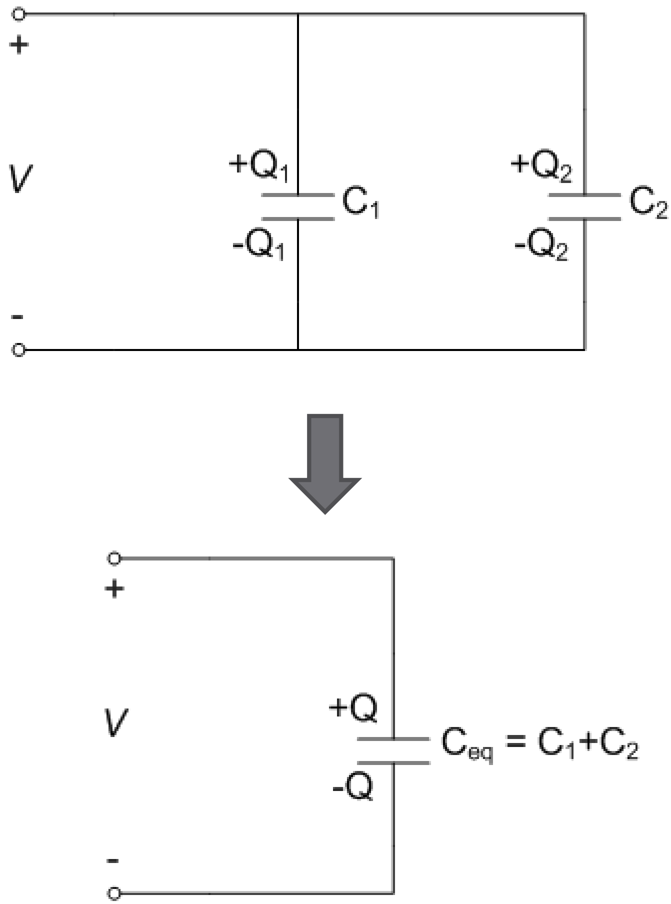
$$i(t) = C\frac{dv}{dt}$$

- A DC voltage does not change with time, so

$$\frac{dv}{dt} = 0 \quad \text{and} \quad i(t) = 0$$

- ***A capacitor is an open circuit at DC***

Capacitors in Parallel



- Total charge on two parallel capacitors is

$$Q = Q_1 + Q_2$$

$$Q = C_1 V + C_2 V$$

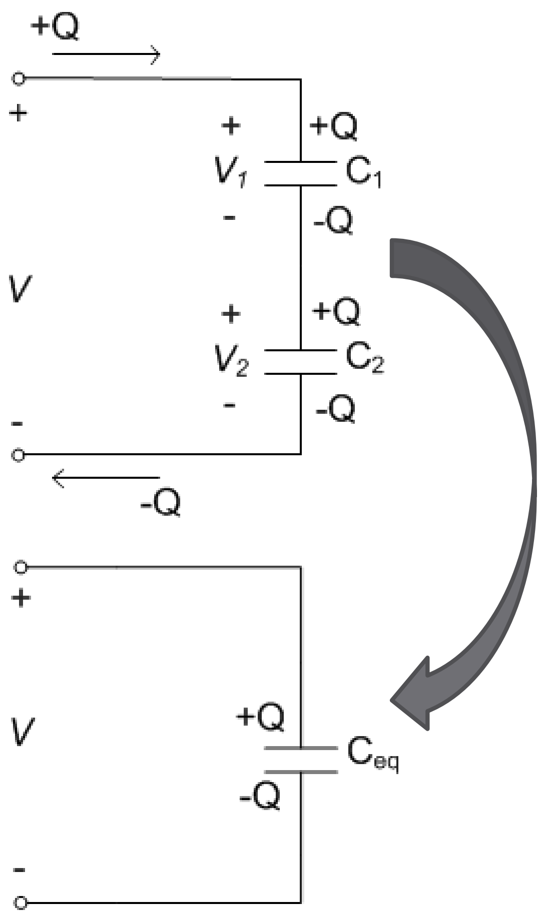
$$Q = (C_1 + C_2)V$$

$$Q = C_{eq} V$$

- ***Capacitances in parallel add***

$$C_{eq} = C_1 + C_2$$

Capacitors in Series



- Total voltage across the series combination is

$$V = V_1 + V_2$$

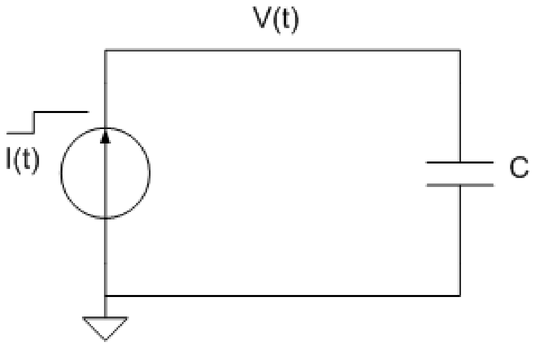
$$V = \frac{Q}{C_1} + \frac{Q}{C_2}$$

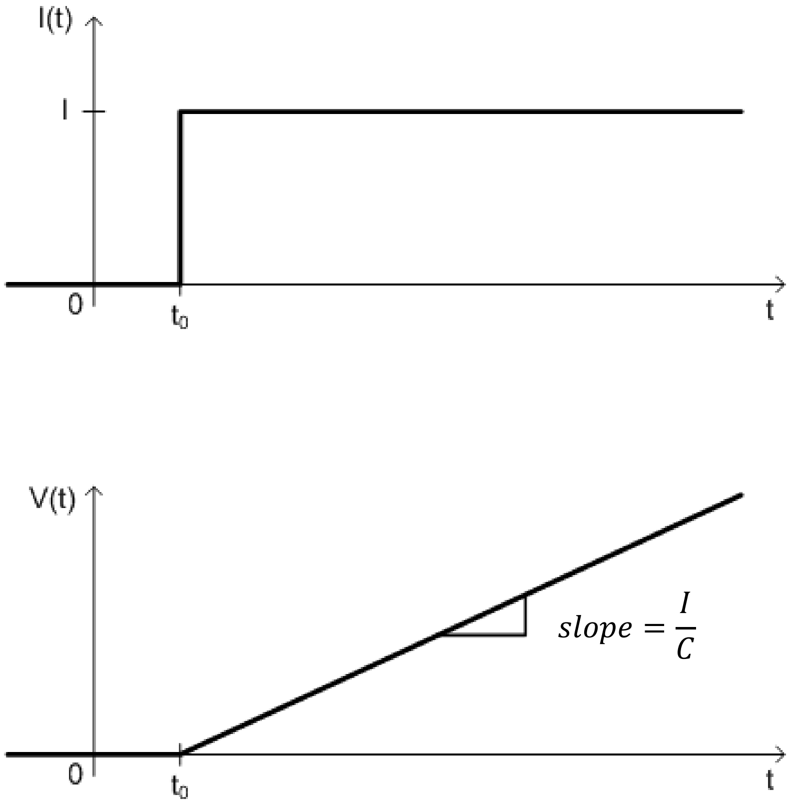
$$V = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_{eq}}$$

- ***The inverses of capacitors in series add***

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \frac{C_1 C_2}{C_1 + C_2}$$

Constant Current Onto a Capacitor





- Capacitor voltage increases linearly for constant current

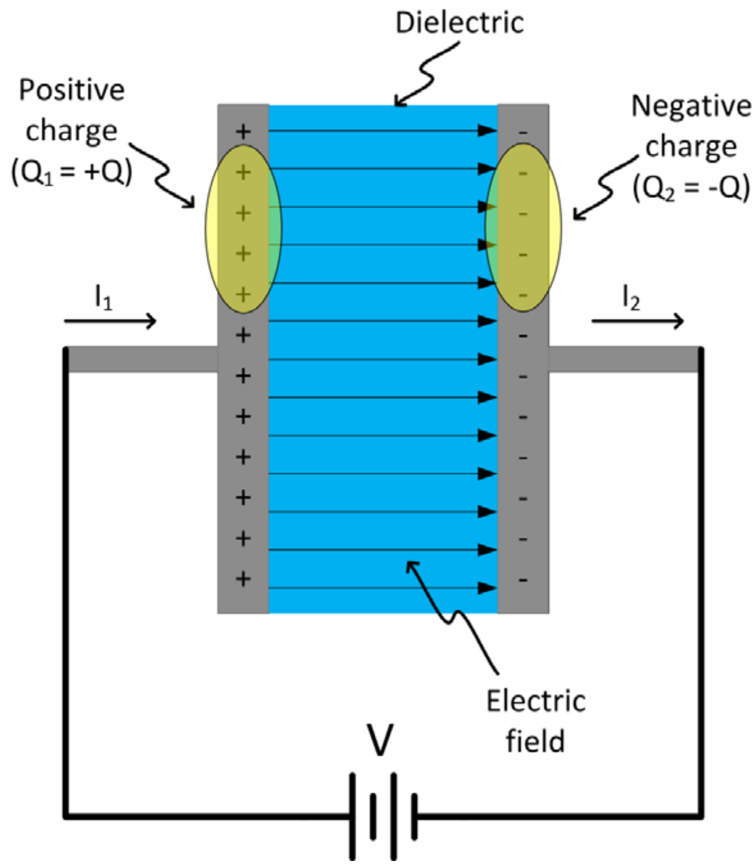
$$v(t) = \frac{I(t-t_0)}{C}, \quad t \geq t_0$$

Capacitor – Energy Storage

- Capacitors store ***electrical energy***
  - Energy stored in the ***electric field***
- Stored energy is proportional to:
  - Voltage
  - Charge differential

$$E = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

- Energy released as E-field collapses
  - V and I supplied



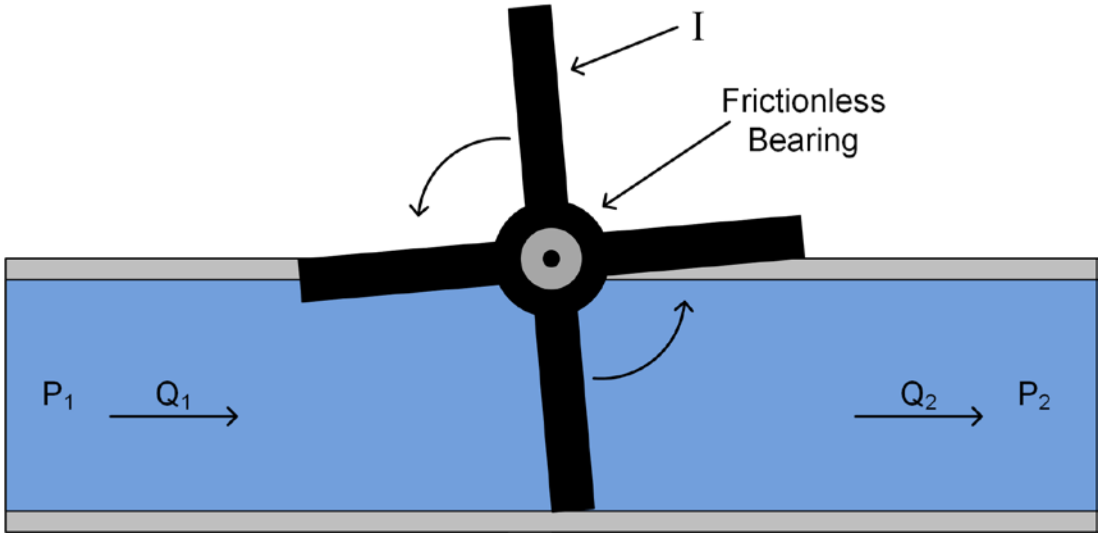
Fluid Inductor

Fluid Inductor

- Consider the following device:
  - Lossless pipe
  - Heavy paddle-wheel/turbine
    - Frictionless bearing
    - Moment of inertia, I
  - Incompressible fluid

    $$Q_1 = Q_2$$

  - Paddle wheel rotates at same rate as the flow



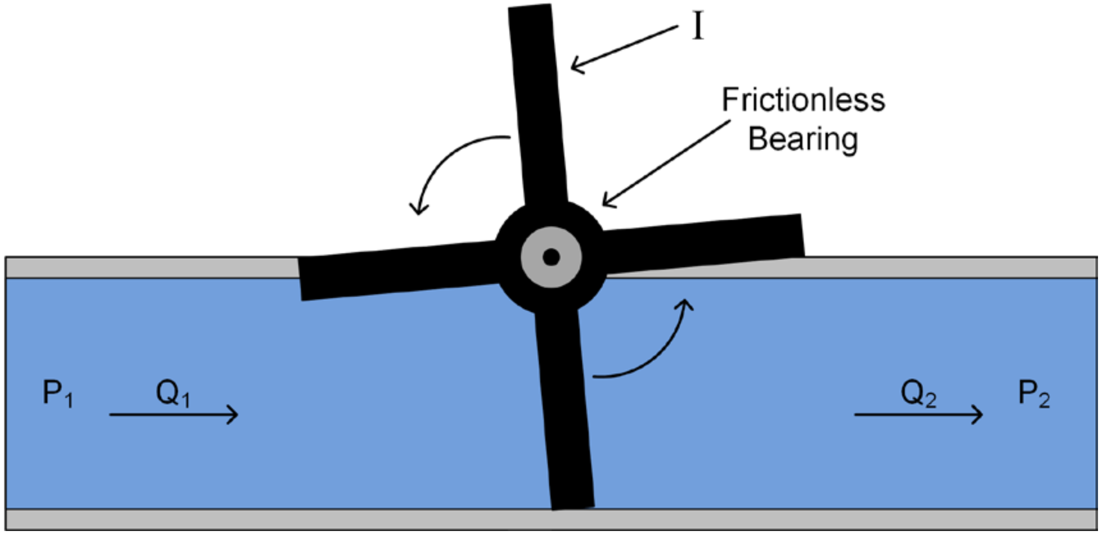
Fluid Inductor – Constant Flow Rate

- Constant flow rate

$$Q_1 = Q_2 = Q$$

- Constant paddle-wheel angular velocity
  - Zero acceleration
  - Zero net applied force
- Frictionless bearing
  - No force required to maintain rotation
- Zero pressure differential

$$\Delta P = P_1 - P_2 = 0$$



Fluid Inductor – Constant ΔP

- Constant pressure differential

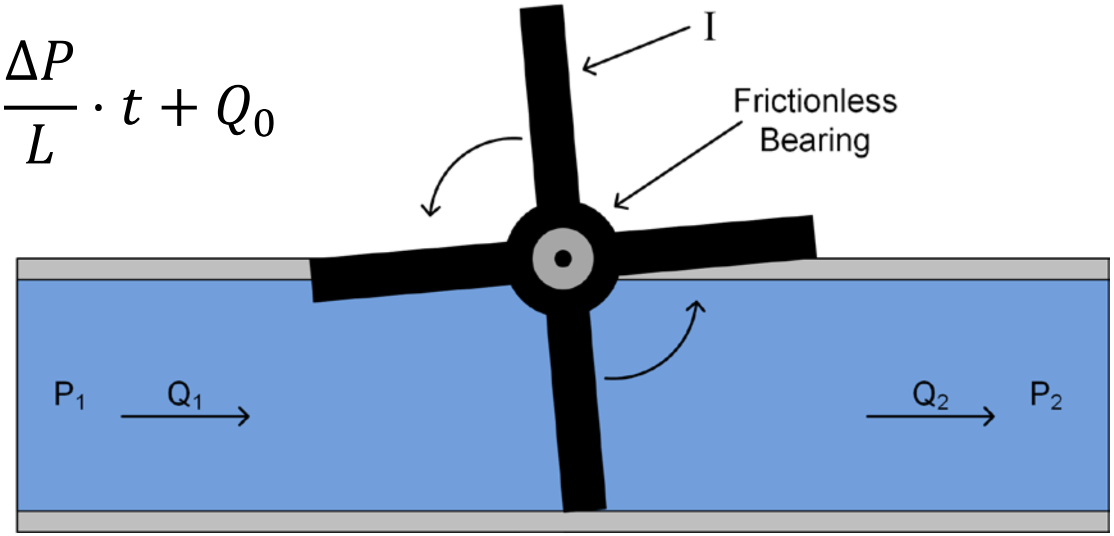
$$\Delta P = P_1 - P_2 \neq 0$$

- Constant applied torque
  - Constant angular acceleration
- Flow rate increases linearly with time

$$Q_1(t) = Q_2(t) = \frac{\Delta P}{L} \cdot t + Q_0$$

- L is ***inductance***
  - Intrinsic device property

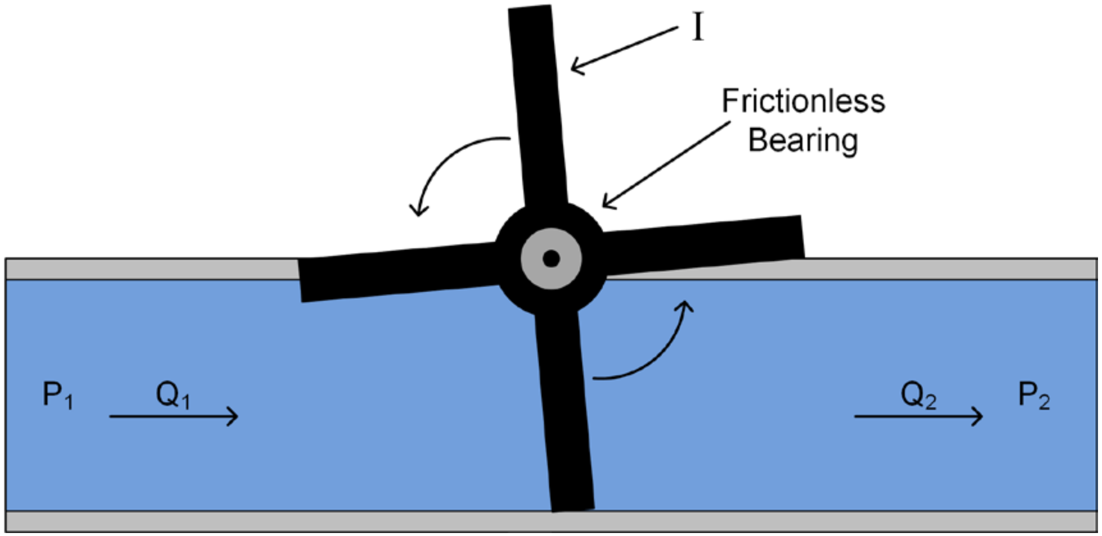
$$L \propto I$$



Fluid Inductor – Changing Flow Rate

- Paddle wheel has inertia (inductance)
  - Does not want to change angular velocity
- Changes in flow rate require:
  - Paddle-wheel acceleration
  - Torque
  - Pressure differential
- Pressure differential associated with changing flow rate:

$$\Delta P = L\frac{dQ}{dt}$$



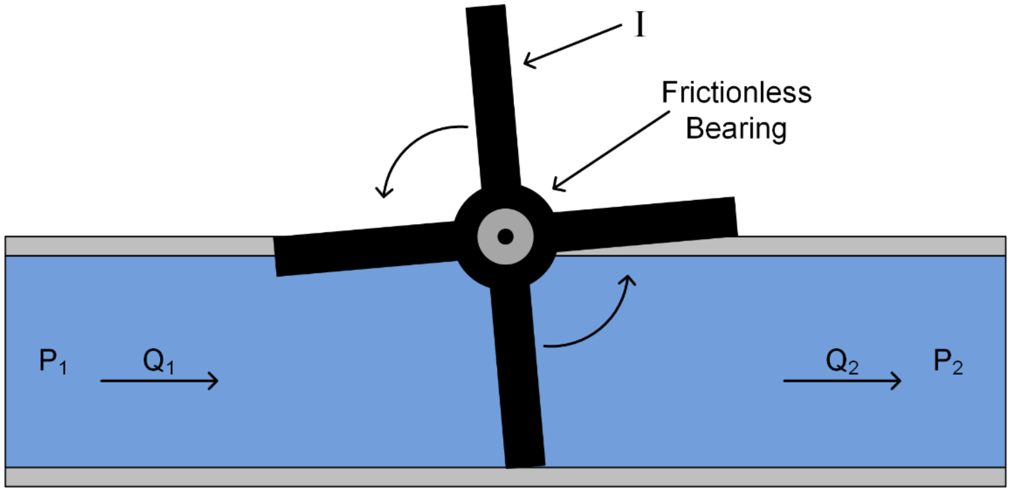
Fluid Inductor – AC vs. DC

- ***Steady state*** (***DC***) flow
  - No pressure differential

$$\Delta P = 0$$

- Consider a ***sinusoidal*** (***AC***) flow rate:

$$Q_1 = Q_2 = Q\sin(\omega t)$$



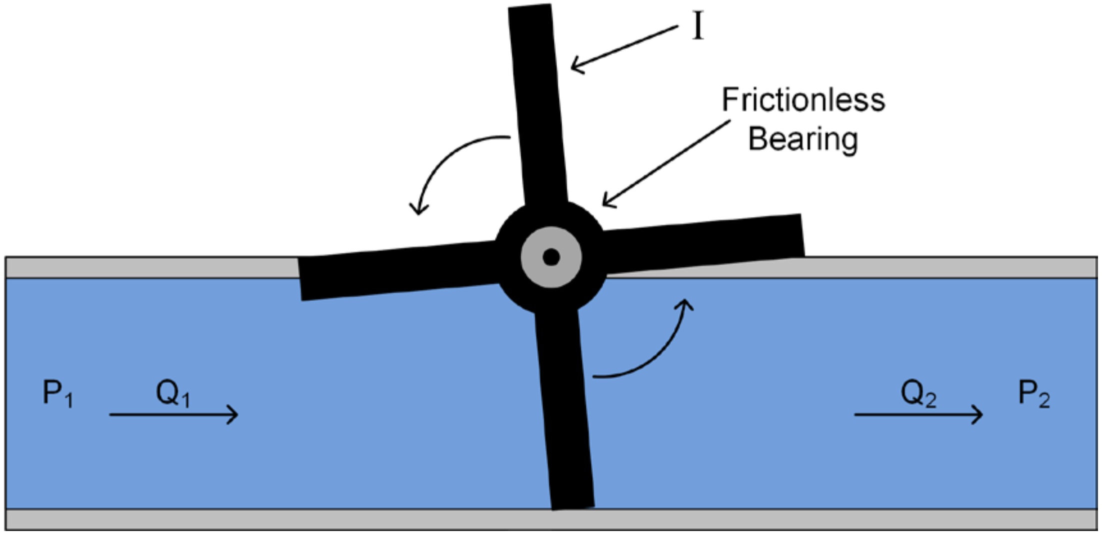
  - Sinusoidal acceleration
  - Sinusoidal torque
  - Sinusoidal pressure differential:

$$\Delta P = L\frac{dQ}{dt} = \omega LQ\cos(\omega t)$$

  - ΔP proportional to:
    - Inductance
    - Frequency

Fluid Inductor – Changing Flow Rate

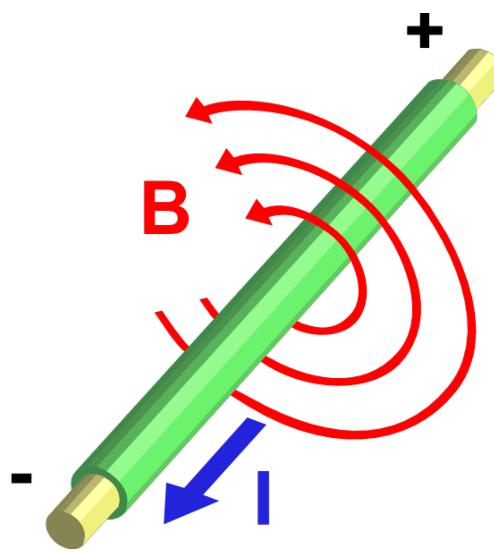
- Each flow rate has a corresponding angular velocity
  - Changes in flow rate require changes in angular velocity
- Angular velocity cannot change instantaneously from one value to another:
  - Must accelerate continuously through all intermediate values
- ***Flow rate through a fluid inductor cannot change instantaneously***
  - Would require:
    - $dQ/dt = \infty$
    - $\Delta P = \infty$

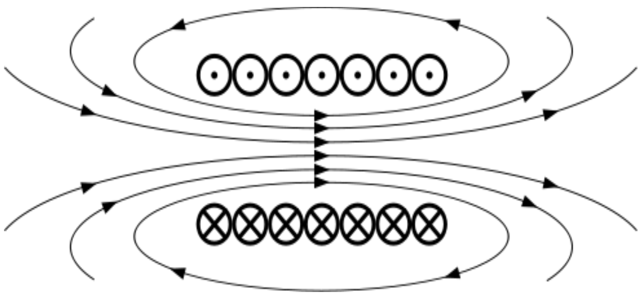


Electrical Inductors

Electrical Inductors

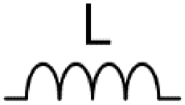
- ***Inductance***
  - Electrical property impeding changes in electrical current
- ***Ampere's law***
  - Electrical current induces a magnetic field surrounding the conductor in which it flows
- ***Electromagnetic induction***
  - As current changes, the changing magnetic field induces a voltage that ***opposes the changing current***



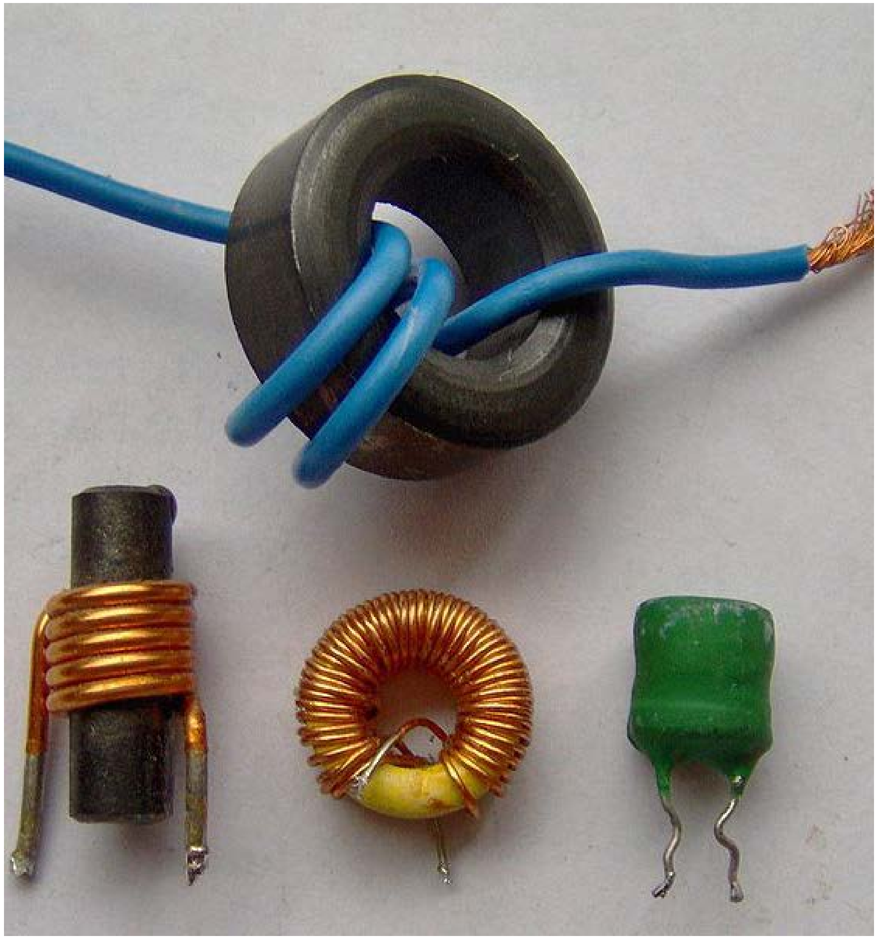


Inductors

- ***Inductors***
  - Electrical components that store energy in a magnetic field
  - Coils of wire
  - Often wrapped around a magnetic core
  - Magnetic fields from current in adjacent turns sum
  - Inductance is proportional to the number of turns
- Schematic symbol:

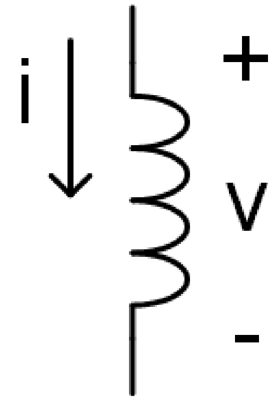


- Units: ***henries*** (H)



Inductors – Voltage and Current

- Voltage across an inductor is proportional to:
  - Inductance
  - Rate of change of the current



$$v(t) = L\frac{di}{dt}$$

- Current through inductor builds gradually with applied voltage
  - Inductor integrates voltage

$$i(t) = \frac{1}{L}\int v(t)dt$$

Current Change Through an Inductor

- For a step change in current,

$$\frac{di}{dt} = \infty$$

  - The corresponding voltage would be ***infinite***

- ***Current through an inductor cannot change instantaneously***

- Voltage can change instantaneously, but current is the integral of voltage

$$\lim_{\Delta t \to 0} \Delta i = \lim_{\Delta t \to 0} \frac{1}{L} \int_{t_0}^{t_0 + \Delta t} v(t)dt = 0$$

Inductors – Short Circuits at DC

- Voltage across an inductor is proportional to the time rate of change of the current through the inductor

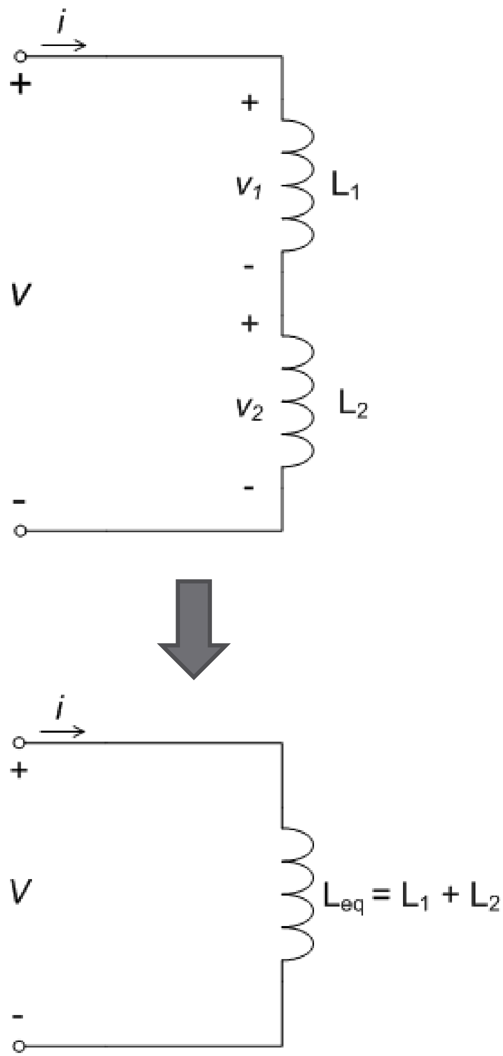
$$v(t) = L\frac{di}{dt}$$

- A DC current does not change with time, so

$$\frac{di}{dt} = 0 \quad \text{and} \quad v(t) = 0$$

- ***An inductor is a short circuit at DC***

Inductors in Series



- Total voltage across the series combination is

$$V = V_1 + V_2$$

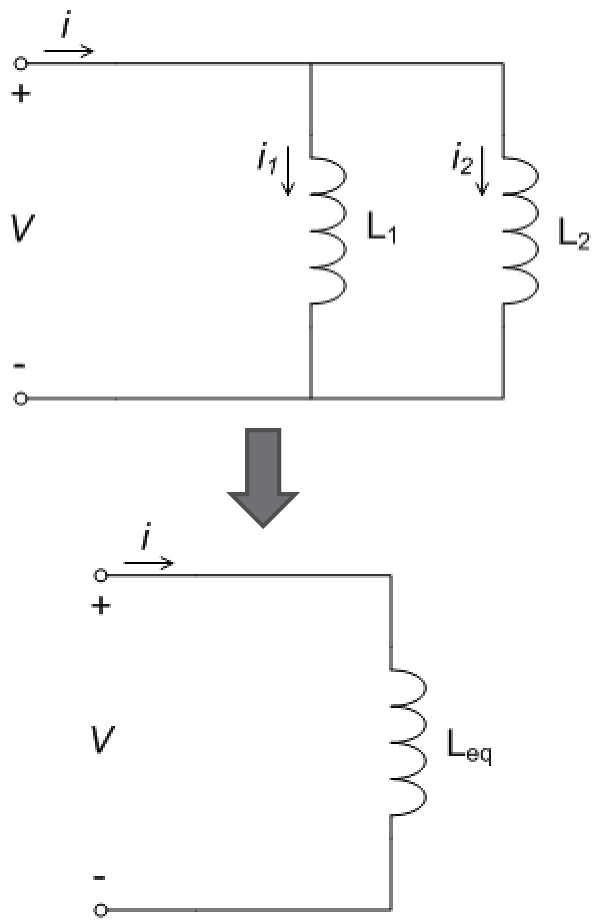
$$V = L_1 \frac{di}{dt} + L_2 \frac{di}{dt}$$

$$V = \frac{di}{dt}(L_1 + L_2) = L_{eq}\frac{di}{dt}$$

- ***Inductances in series add***

$$\boxed{L_{eq} = L_1 + L_2}$$

Inductors in Parallel



- Voltage across the two parallel inductors:

$$v = L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt}$$

so

$$\frac{di_1}{dt} = \frac{v}{L_1}, \qquad \frac{di_2}{dt} = \frac{v}{L_2}$$

- Voltage across the equivalent inductor:

$$v = L_{eq} \frac{di}{dt} = L_{eq} \left(\frac{di_1}{dt} + \frac{di_2}{dt} \right)$$

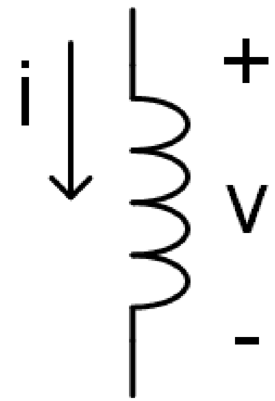
$$v = L_{eq} \left(\frac{v}{L_1} + \frac{v}{L_2} \right)$$

- ***Inverses of inductors in parallel add***

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} \right)^{-1} = \frac{L_1 L_2}{L_1 + L_2}$$

Inductor – Energy Storage

- Inductors store ***magnetic energy***
  - Energy stored in the ***magnetic field***
- Stored energy is proportional to:
  - Current
  - Inductance
  - Magnetic flux: $\lambda = LI$



$$E = \frac{1}{2}LI^2$$

- Energy released as magnetic field collapses
  - V and I supplied

Example Problems

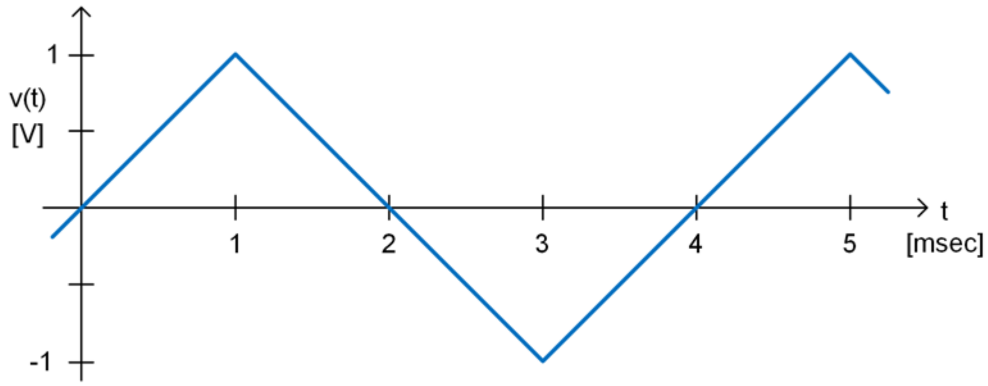
A typical US home consumes about 1000 kWh/month.

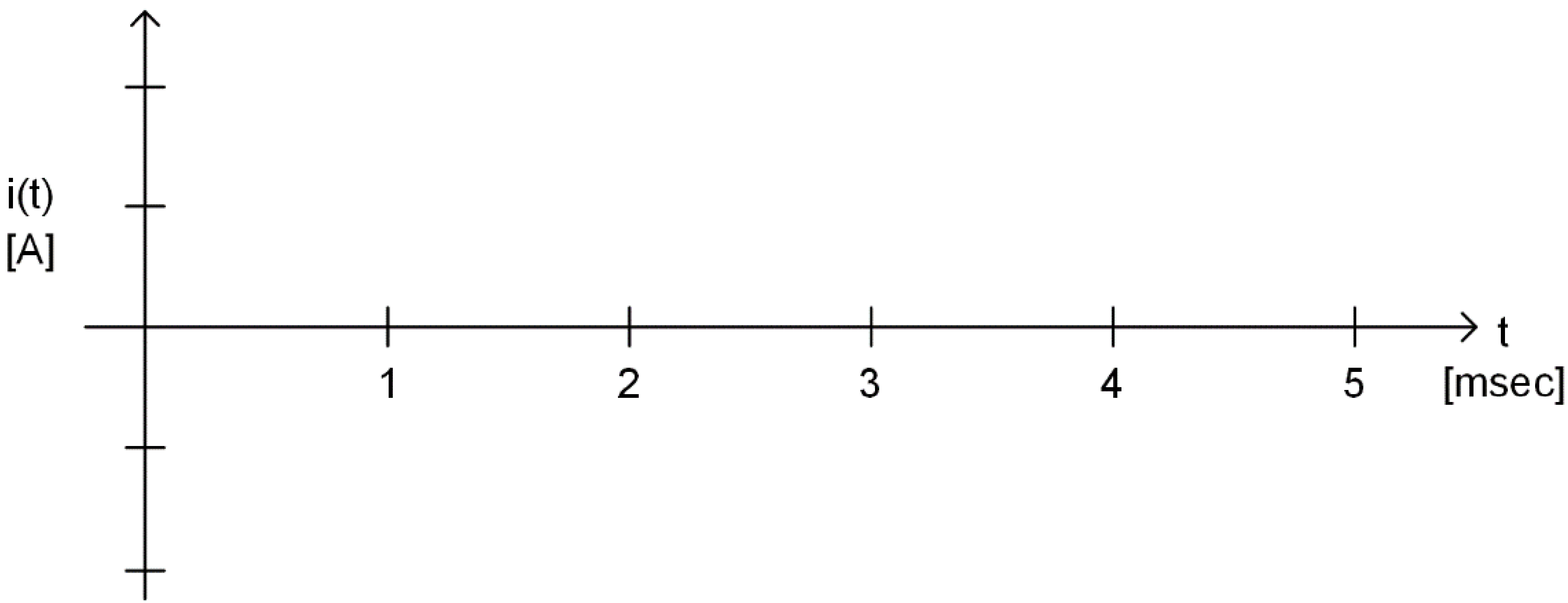
a) To what voltage would a 1 F capacitor need to be charged in order to store this amount of energy?

b) If the fully-charged voltage is limited to 200 V, how much capacitance would be required to store this amount of energy?

A 10 V, 1 kHz, sinusoidal voltage is applied across a 1 μF capacitor. How much current flows through the capacitor?

The following voltage is applied across a 1μF capacitor. Sketch the current through the capacitor.




i(t)
[A]
t
1
2
3
4
5
[msec]

A typical US home consumes about 1000 kWh/month.

a) How much current would be required in order to store this amount of energy in a 1 H inductor ?

b) If the maximum current is limited to 200 A, how much inductance would be required to store this amount of energy?

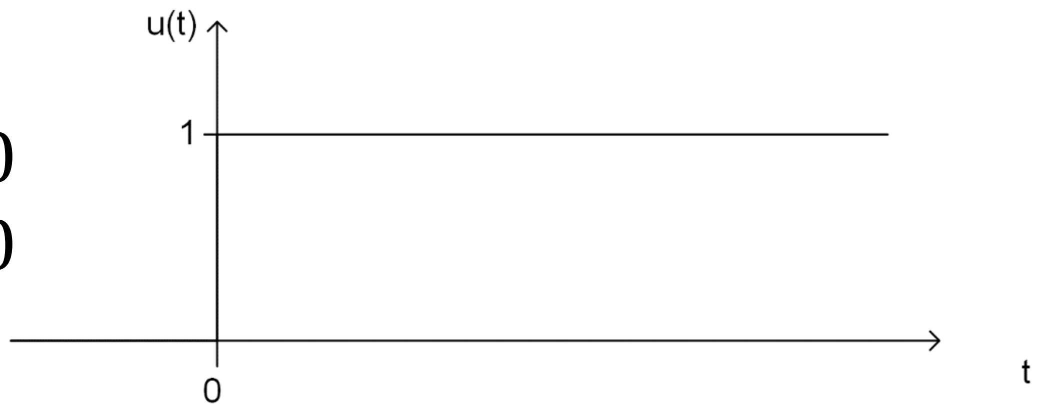
If 10 VDC is applied across a 500 mH inductor, how long will it take the current to reach 10 A?

RC Circuits

Step Response

- ***Step response***
  - Response of a dynamic system (not necessarily electrical) to a step function input
- ***Unit step function*** or ***Heaviside step function***:

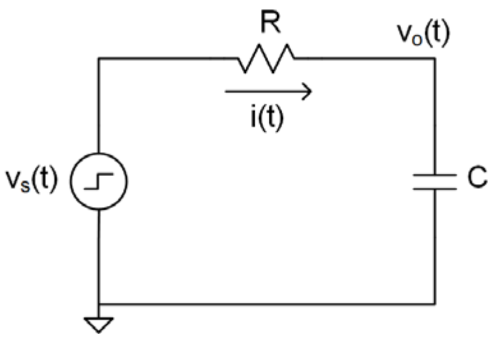
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

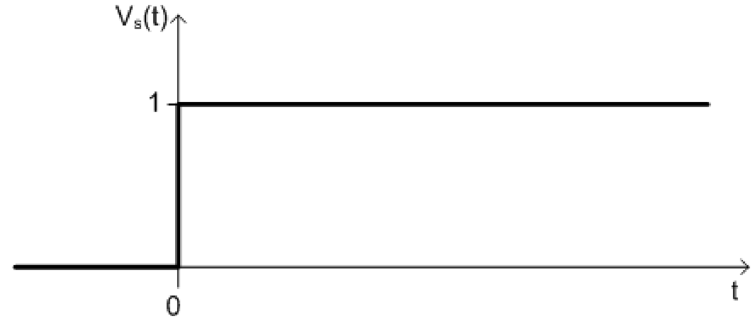


- To characterize an electrical network, a ***voltage step*** can be applied as an input

RC Circuit – Step Response

- Step response of this RC circuit is the output voltage in response to a step input: $v_s(t) = 1V \cdot u(t)$

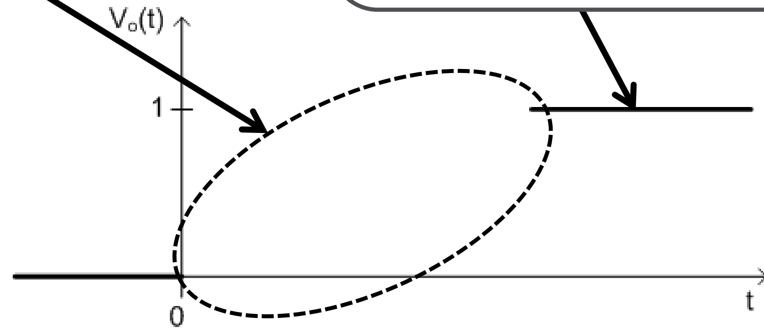




- For $t \gg 0$,
  - $v_s(t) \rightarrow$ DC
  - $C \rightarrow$ open circuit
  - $v_o(t) \rightarrow 1\ V$

How does $v_o(t)$ get from $v_o(0) = 0V$ to $v_o(\infty) = 1\ V$?

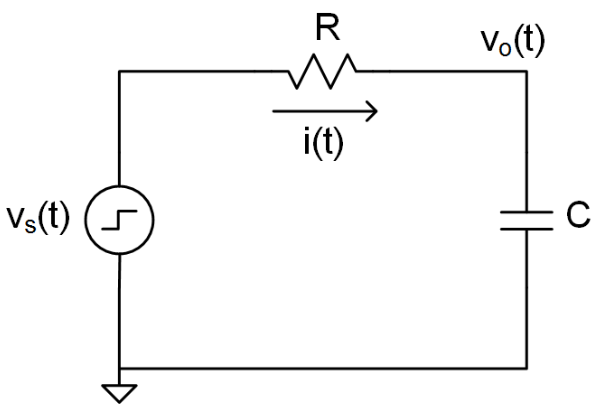
- For $t < 0$,
  - $v_s(t) = 0\ V$
  - $v_o(t) = 0\ V$



RC Circuit – Step Response

- To determine the step response, apply KVL around the circuit

$$v_s(t) - i(t)R - v_{o(t)} = 0$$



- Current, $i(t)$, is the capacitor current

$$i(t) = C\frac{dv_o}{dt}$$

- Substituting in for $i(t)$ and $v_s(t)$

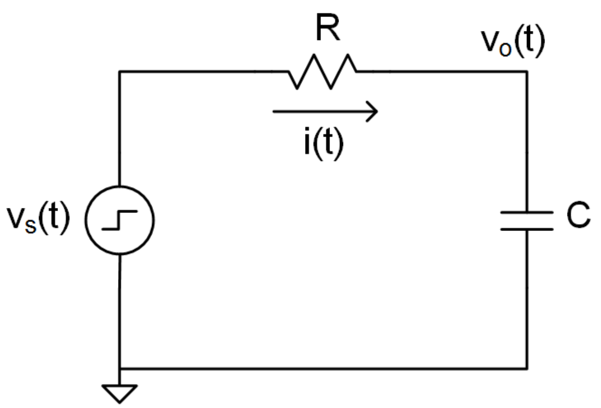
$$1\,V \cdot u(t) - RC\frac{dv_o}{dt} - v_{o(t)} = 0$$

- Rearranging

$$\frac{dv_o}{dt} + \frac{1}{RC}v_{o(t)} = \frac{1\,V}{RC} \cdot u(t)$$

RC Circuit – Step Response

$$\frac{dv_o}{dt} + \frac{1}{RC} v_{o(t)} = \frac{1\,V}{RC} \cdot u(t)$$



- A ***first-order linear, ordinary, non-homogeneous differential equation***
- Solution for $v_o(t)$ is the sum of two solutions:
  - ***Complementary*** solution
  - ***Particular*** solution
- The ***complementary solution*** is the solution to the homogeneous equation
  - Set the input (***forcing function***) to zero
  - The circuit's ***natural response***

$$\frac{dv_o}{dt} + \frac{1}{RC} v_{o(t)} = 0$$

RC Step Response – Homogeneous Solution

$$\frac{dv_o}{dt} + \frac{1}{RC} v_{o(t)} = 0$$

- For a first-order ODE of this form, we assume a solution of the form

$$v_{o_c}(t) = K_0 e^{\lambda t}$$

then

$$\frac{dv_o}{dt} = \lambda K_0 e^{\lambda t}$$

and

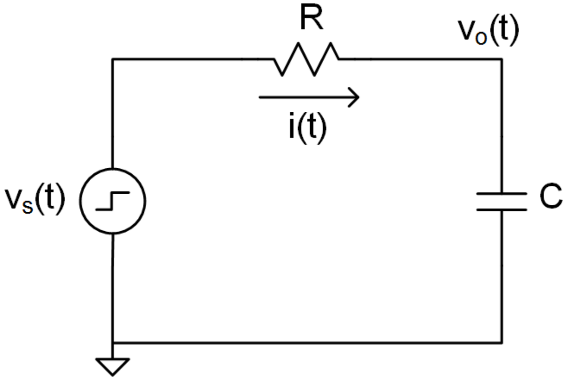
$$\lambda K_0 e^{\lambda t} + \frac{1}{RC} K_0 e^{\lambda t} = 0$$

so

$$\lambda + \frac{1}{\mathrm{RC}} = 0 \;\;\rightarrow\;\; \lambda = -\frac{1}{RC} = -\frac{1}{\tau}$$

and

$$v_{o_c}(t) = K_0 e^{-\frac{t}{RC}} = K_0 e^{-\frac{t}{\tau}}$$



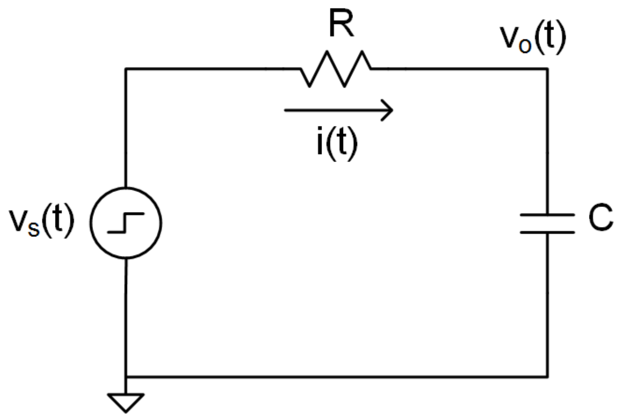
RC Step Response – Homogeneous Solution

$$v_{o_c}(t) = K_0 e^{-\frac{t}{\tau}}$$

- The ***complementary solution***
  - τ is the circuit ***time constant***

    $$\boxed{\tau = RC}$$

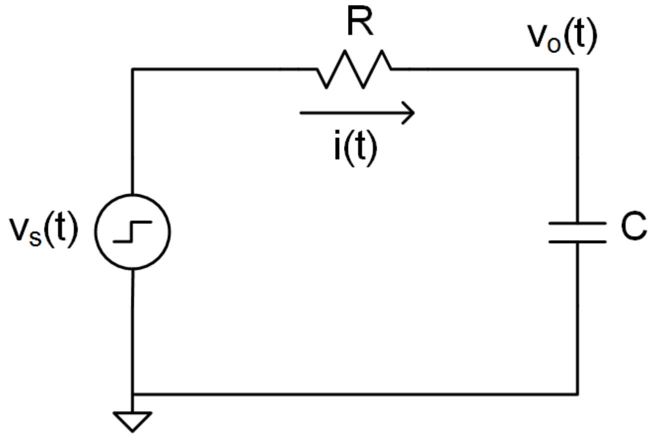
  - K_0 is an unknown constant
    - To be determined through application of ***initial conditions***
- Next, find the ***particular solution***, $v_{o_p}(t)$



RC Step Response – Particular Solution

- For a step input, the ***particular solution*** is the circuit's ***steady-state response***
  - As $t \rightarrow \infty$
  - Long after the input step
- In steady state:
  - $v_s(t) = 1\,V$ (DC)
  - Capacitor → open circuit
  - $i(t) = 0$
  - $v_0(t) = v_s(t) = 1\,V$
- The ***particular solution***:

$$v_{o_p}(t) = 1\,V$$



RC Step Response

- ***Step response***
  - Solution to the non-homogeneous equation
  - Sum of the complementary and particular solutions

$$v_o(t) = v_{o_c}(t) + v_{o_p}(t)$$

$$v_o(t) = K_0 e^{-\frac{t}{\tau}} + 1\ V$$

- Next, determine K_0 by applying an ***initial condition***
- For $t < 0$
  - $v_s(t < 0) = 0\ V$
  - $v_o(t < 0) = 0\ V$
- At $t = 0$
  - $v_s(0) = 1\ V$
  - Capacitor voltage cannot change instantaneously
  - $v_o(0) = 0\ V$ – this is the ***initial condition***

RC Step Response

- Apply the initial condition

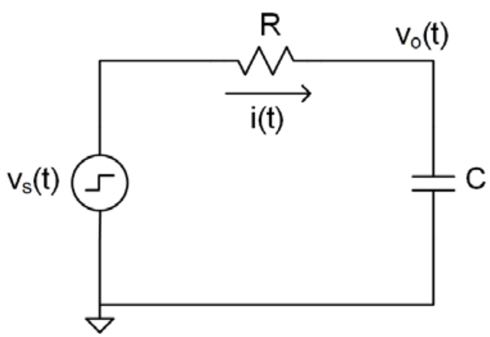
$$v_o(0) = K_0 e^{-\frac{0}{\tau}} + 1\,V = 0\,V$$

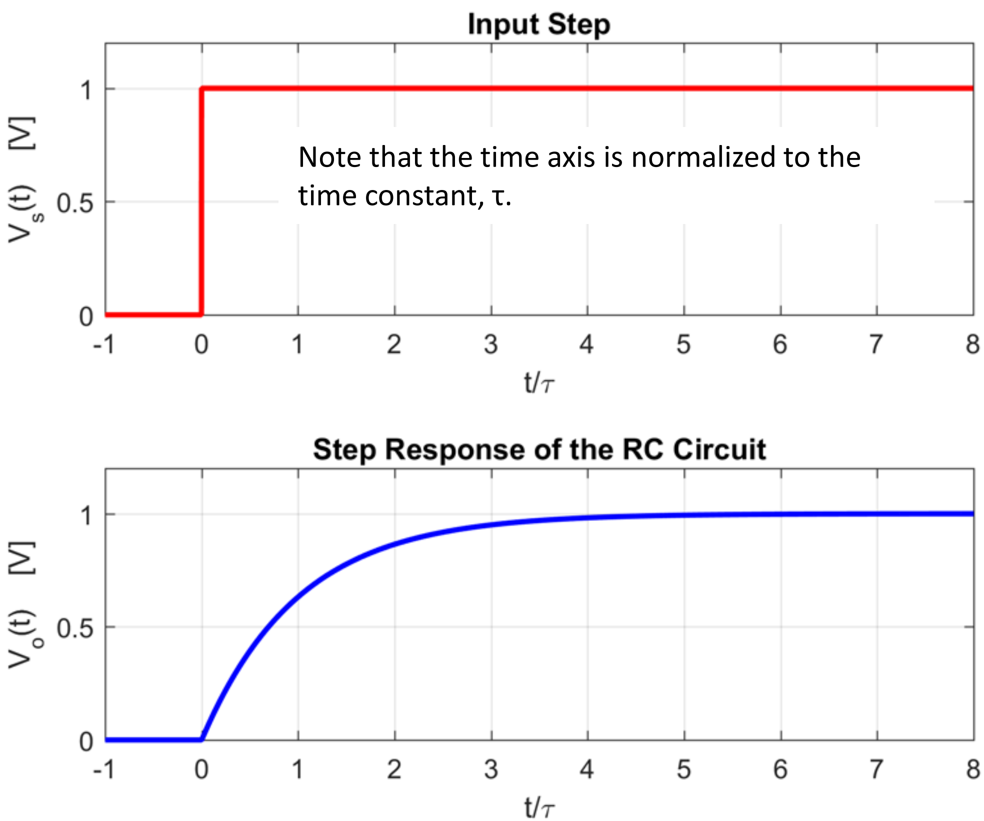
$$v_o(0) = K_0 + 1\,V = 0\,V$$

$$K_0 = -1\,V$$

- The step response is

$$v_o(t) = -1\,V e^{-\frac{t}{\tau}} + 1\,V$$





Step Response – General Solution

$$v_o(t) = -1\,V e^{-\frac{t}{\tau}} + 1\,V$$

- This solution assumes an input that steps from $0\,V$ to $1\,V$ at $t = 0$
  - These are also the initial and final values of v_o
- Suppose the input steps between two arbitrary voltage levels:

$$v_i(t) = \begin{cases} V_i & t < 0 \\ V_f & t \geq 0 \end{cases}$$

  - Now, the initial condition is

$$v_o(0) = V_i$$

  - The particular solution is the steady-state value, which is now

$$v_{o_p}(t) = v_o(t \rightarrow \infty) = V_f$$

  - Solution to the non-homogeneous equation is

$$v_o(t) = K_o e^{-\frac{t}{\tau}} + V_f$$

Step Response – General Solution

- Apply the initial condition to determine K_0

$$v_o(0) = K_o e^{-\frac{0}{\tau}} + V_f = V_i$$

$$K_0 = V_i - V_f$$

- Substituting in for K_0 gives the general voltage step response:

$$\boxed{v_o(t) = V_f + \left(V_i - V_f\right)e^{-\frac{t}{\tau}}}$$

Step Response – General Solution

- General RC circuit step response:

$$v_o(t) = V_f + \left(V_i - V_f\right)e^{-\frac{t}{\tau}}$$

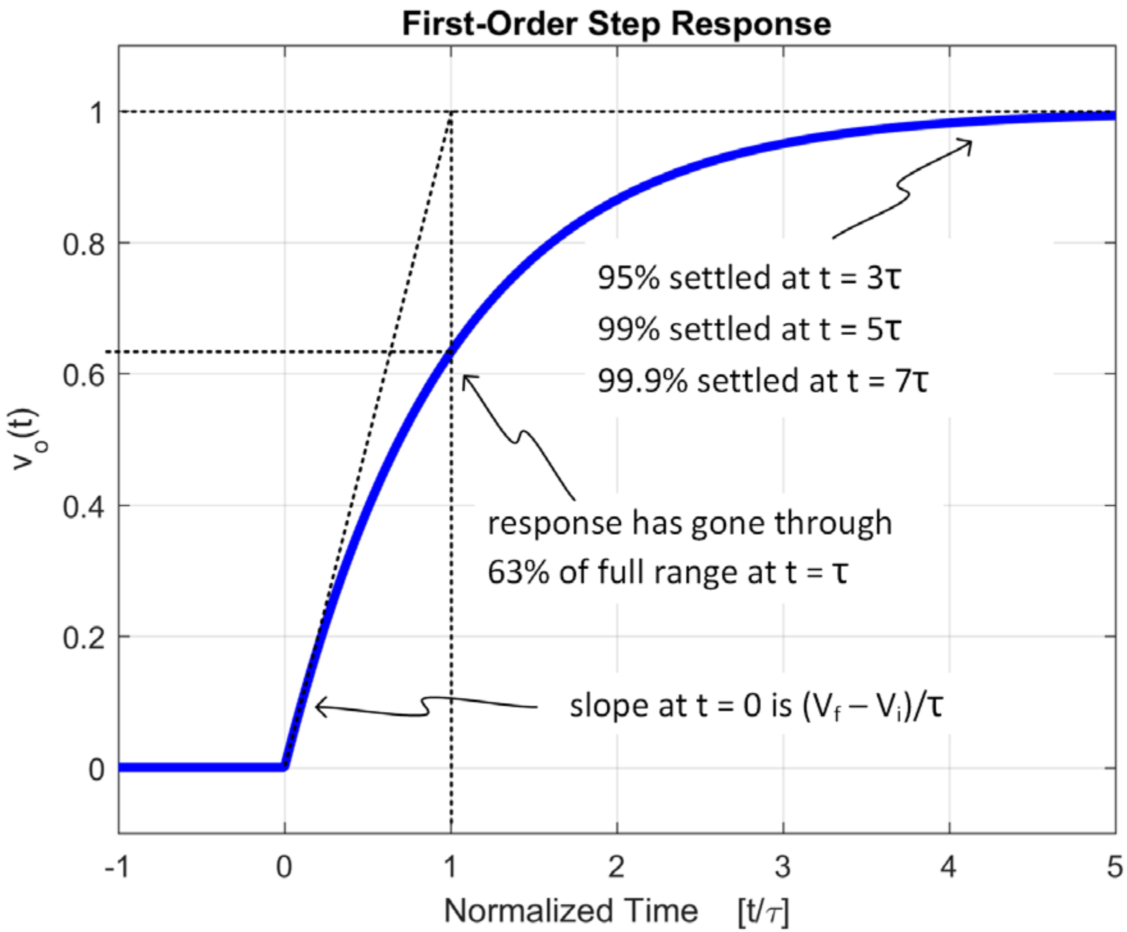
- General step response for any ***first-order linear system*** with a finite steady-state value:

$$y(t) = Y_f + \left(Y_i - Y_f\right)e^{-\frac{t}{\tau}}$$

  - Not necessarily an electrical system
  - $y(t)$ is any quantity of interest (voltage, current, temperature, pressure, displacement, etc.)
  - $Y_i = y(0)$ is the ***initial condition***
  - $Y_f = y(t \rightarrow \infty)$ is the ***steady-state value***

First-Order Step Response

- Initial slope is inversely proportional to time constant
- Response completes 63% of transition after one time constant
- Almost completely settled after 7τ

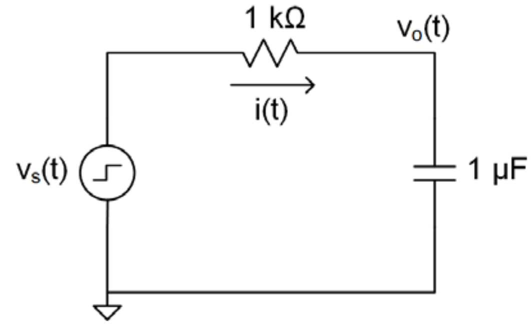


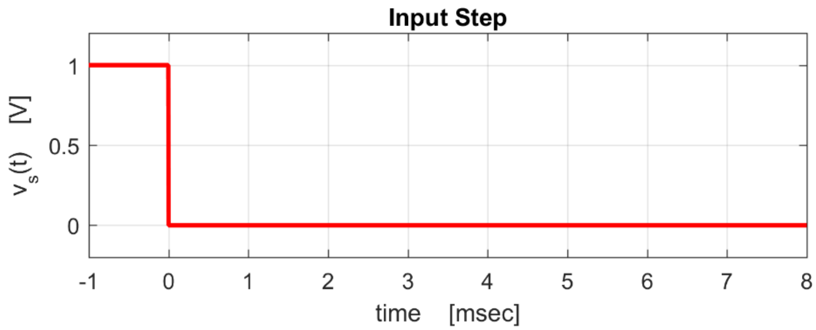
RC Circuit Response – Example

- RC circuit driven with negative-going step

$$v_s(t) = \begin{cases} 1\,V & t < 0 \\ 0\,V & t \geq 0 \end{cases}$$

- For $t < 0$
  - $v_s(t) = 1\,V$
  - $v_o(t) = 1\,V$
- At $t = 0$
  - $v_s(0) = 0\,V$
  - $v_o(t)$ cannot change instantaneously
  - $V_i = v_o(0) = 1\,V$
- As $t \to \infty$
  - $v_s(t) = 0\,V \to$ DC
  - Capacitor → open circuit
  - $i(t \to \infty) = 0\,A$
  - $v_o(t \to \infty) = 0\,V$
  - $V_f = 0\,V$





RC Circuit Response – Example

- Time constant

$$\tau = RC = 1\ k\Omega \cdot 1\ \mu F$$

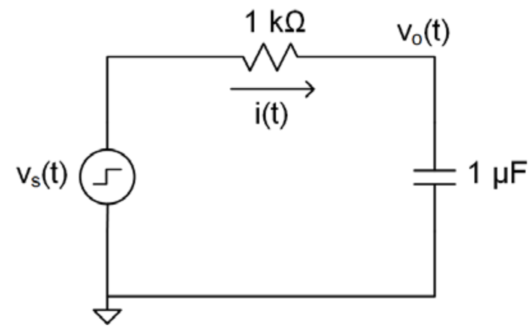
$$\tau = 1\ msec$$

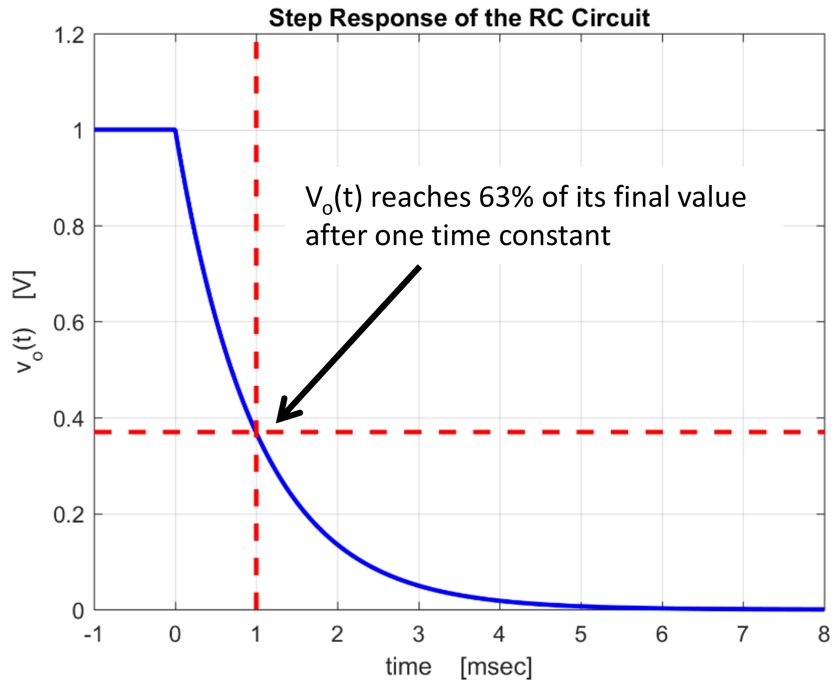
- Voltage step response

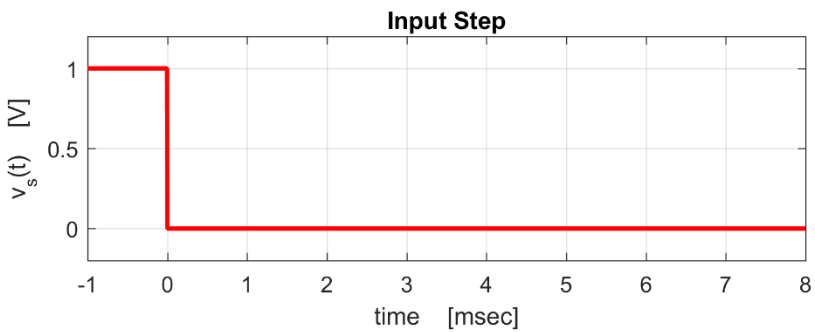
$$v_o(t) = V_f + \left(V_i - V_f\right)e^{-\frac{t}{\tau}}$$

$$v_o(t) = 0\ V + (1\ V - 0\ V)e^{-\frac{t}{\tau}}$$

$$v_o(t) = 1\ V \cdot e^{-\frac{t}{1\ msec}}$$







Current Step Response

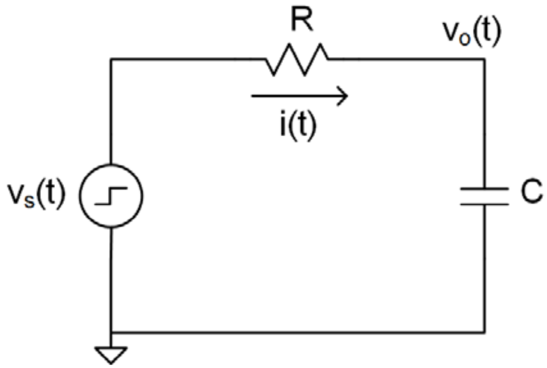
- Now, consider the ***current*** through an RC circuit driven by a positive-going $0\ V \dots 1\ V$ step
  - At $t = 0$:
    - $v_s(0) = 1\ V$
    - $v_o(0) = 0\ V$
    - Voltage across resistor:

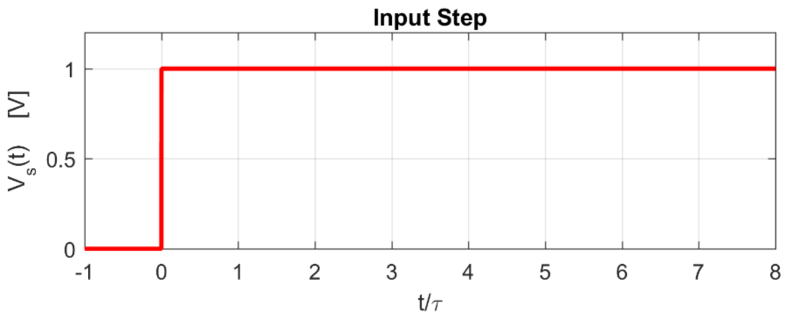
$$v_s(0) - v_o(0) = 1\ V$$

    - Current through resistor:

$$I_i = i(0) = \frac{v_s(0) - v_o(0)}{R}$$

$$I_i = \frac{1\ V}{R}$$

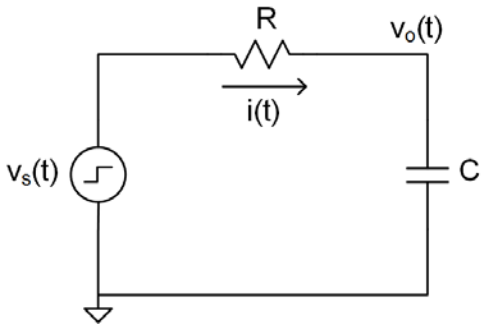




Current Step Response

- As $t \rightarrow \infty$:
  - Capacitor → open circuit
  - Current → 0

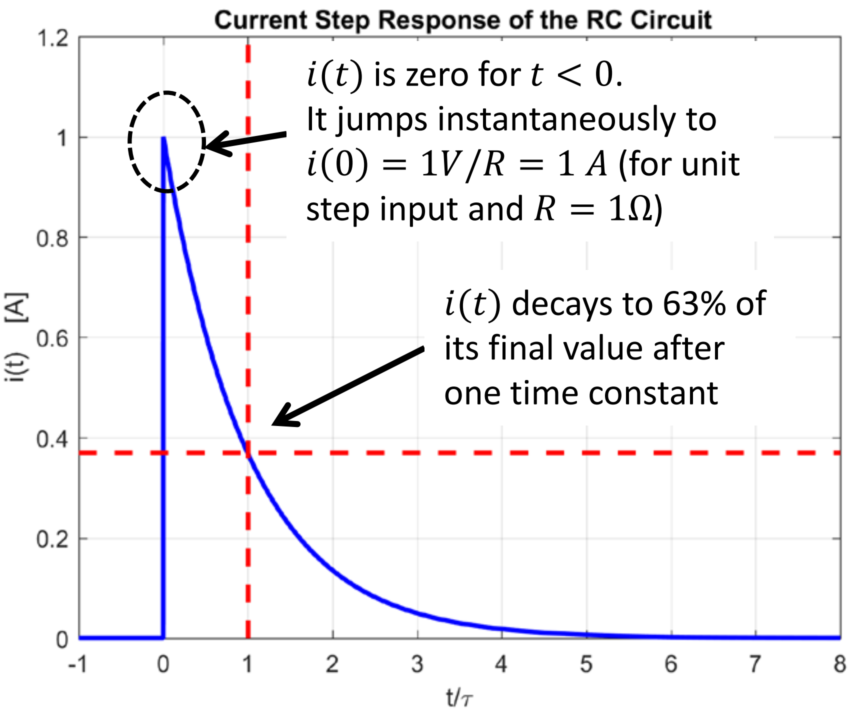
    $I_f = 0$



- The current step response:

$$i(t) = I_f + \left(I_i - I_f\right)e^{-\frac{t}{\tau}}$$

$$i(t) = \frac{1\,V}{R}e^{-\frac{t}{\tau}}$$

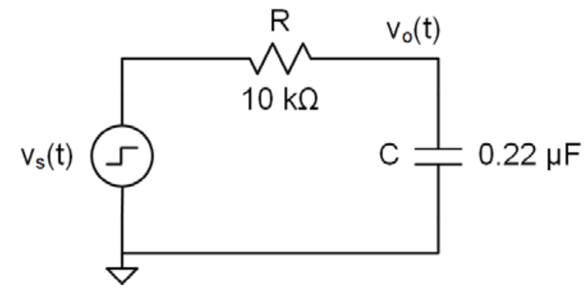


Example Problems

Determine $v_o(t)$ for $t \geq 0$.

The input source is:

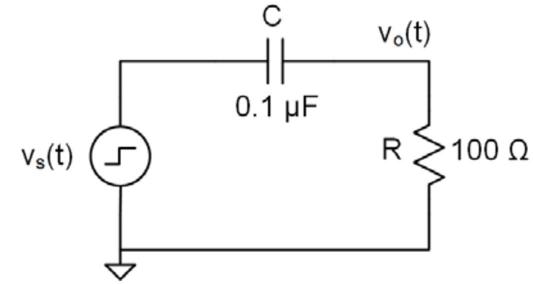
$$v_s(t) = 2\,V \cdot u(t) - 1\,V$$



Determine $v_o(t)$ for $t \geq 0$.

The input source is:

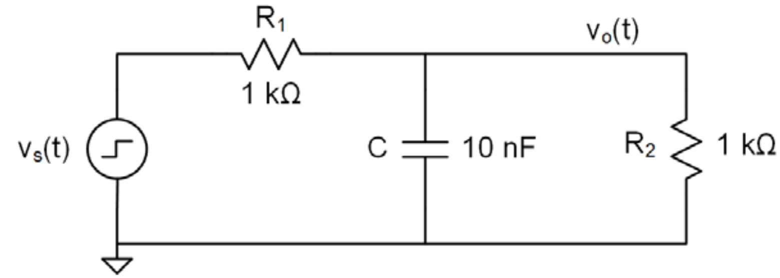
$$v_s(t) = 1\,V \cdot u(t)$$



Determine $v_o(t)$ for $t \geq 0$.

The input source is:

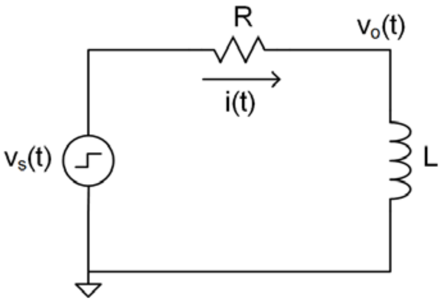
$$v_s(t) = 1\,V \cdot u(t)$$

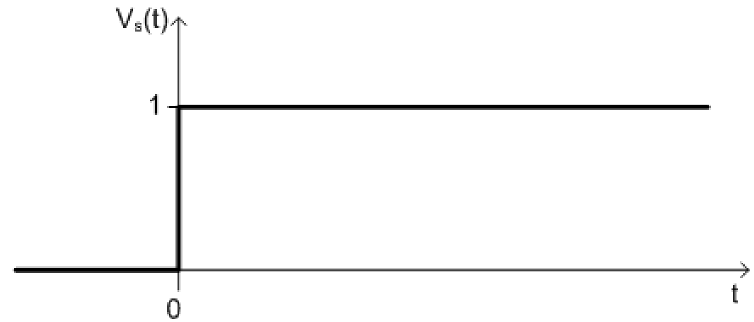


RL Circuits

RL Circuit – Step Response

- For the RL circuit, we'll first look at the *current* step response



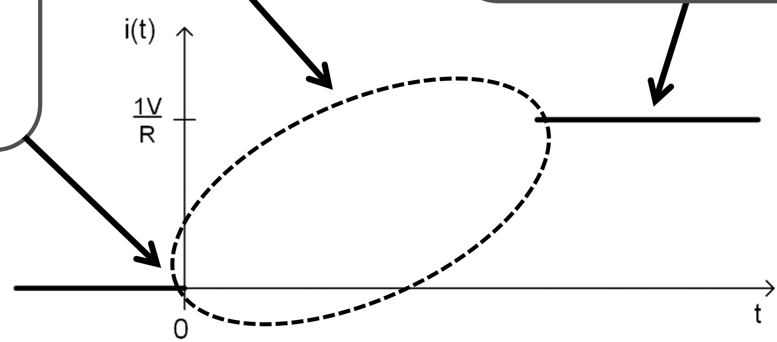


How does $i(t)$ get from I_i to I_f?

- For $t \gg 0$,
  - $v_s(t) = 1\,V \rightarrow$ DC
  - $L \rightarrow$ short circuit
  - $i(t) \rightarrow 1\,V/R = I_f$

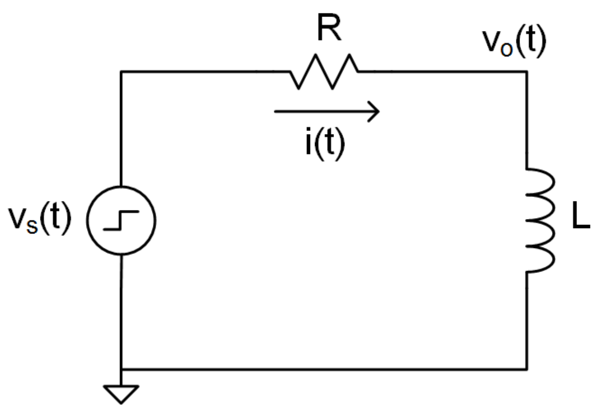
- At $t = 0$,
  - Current cannot change instantaneously
  - $i(0) = 0\,V = I_i$

- For $t < 0$,
  - $v_s(t) = 0\,V$
  - $i(t) = 0\,A$



RL Circuit – Step Response

- To determine the step response, apply KVL around the circuit



$$v_s(t) - i(t)R - L\frac{di}{dt} = 0$$

- Here, we have a differential equation for $i(t)$

$$\frac{di}{dt} + \frac{R}{L}i(t) = \frac{1}{L}v_s(t)$$

- This is in the exact same form as the voltage ODE for the RC circuit
  - Same general solution applies

$$i(t) = I_f + \left(I_i - I_f\right)e^{-\frac{t}{\tau}}$$

  - Where, now, the ***time constant*** is

$$\boxed{\tau = \frac{L}{R}}$$

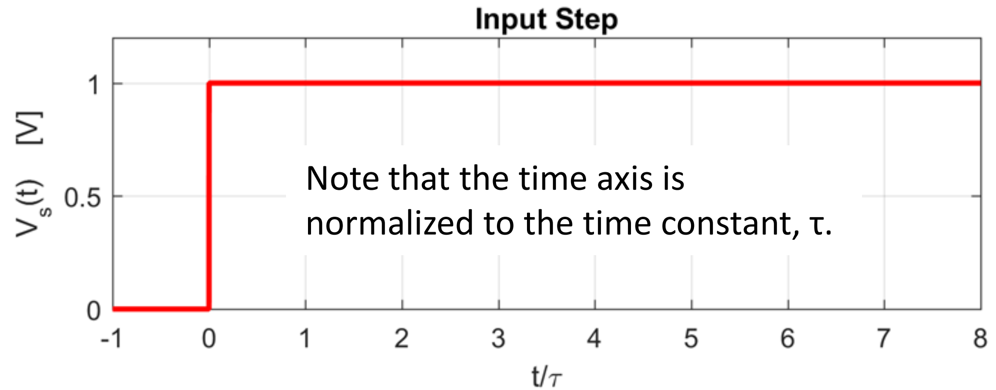
RL Circuit – Step Response

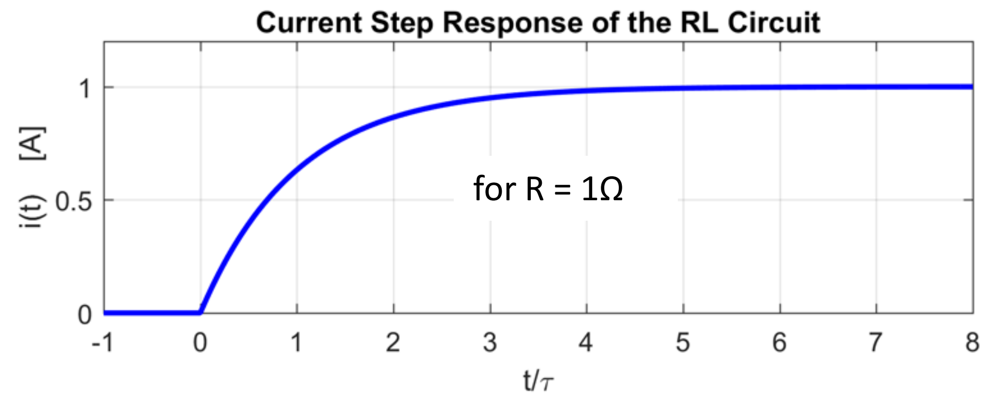
- Current step response:

$$i(t) = I_f + \left(I_i - I_f\right)e^{-\frac{t}{\tau}}$$

$$i(t) = \frac{1\,V}{R} - \frac{1\,V}{R}e^{-\frac{t}{\tau}}$$

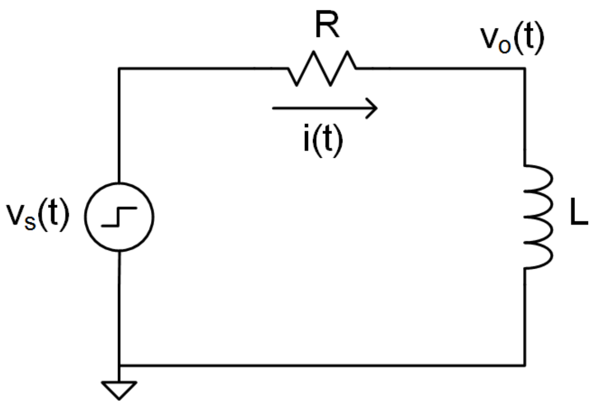
$$i(t) = \frac{1\,V}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$





RL Circuit – Step Response

- Now consider the voltage step response
- Determine ***initial*** and ***final values***:
  - For $t < 0$
    - $v_s(t) = 0\ V$
    - $i(t) = 0\ A$
    - $v_o(t) = 0\ V$
  - At $t = 0$
    - $v_s(0) = 1\ V$
    - Current through the inductor cannot change instantaneously, so $i(0) = 0\ A$
    - No voltage drop across the resistor, so $v_o(0) = 1\ V$
    - $V_i = 1\ V$
  - As $t \to \infty$
    - $v_s(t) \to$ DC
    - Inductor → short circuit, so $v_o(t \to \infty) = 0\ V$
    - $V_f = 0\ V$



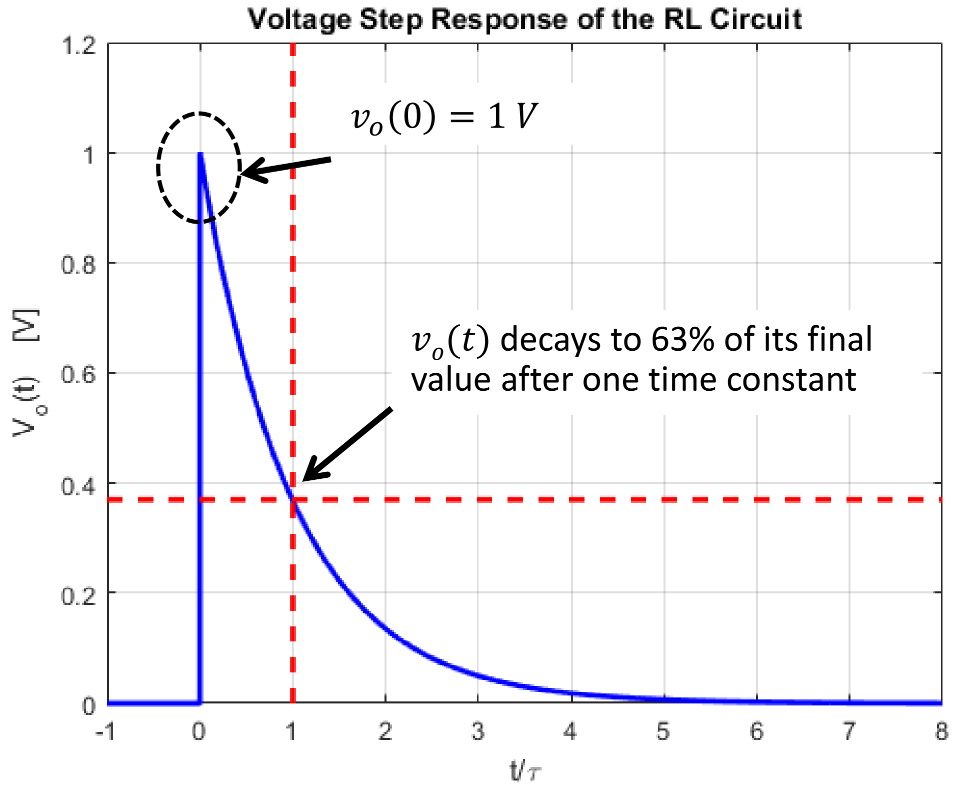
RL Circuit – Step Response

- Voltage step response:

$$v_o(t) = V_f + (V_i - V_f)e^{-\frac{t}{\tau}}$$

$$v_o(t) = 0\,V + (1\,V - 0\,V)e^{-\frac{t}{\tau}}$$

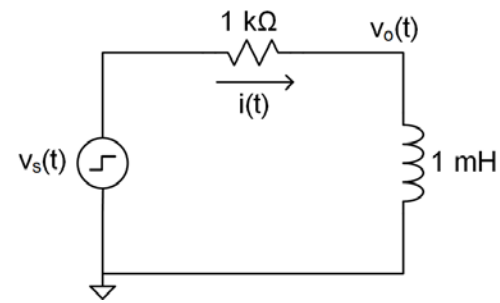
$$v_o(t) = 1\,V e^{-\frac{t}{\tau}}$$



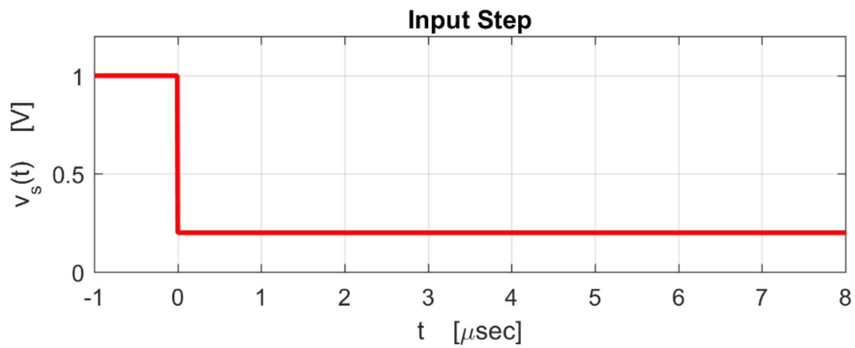
RL Circuit Response – Example

- RL circuit driven with negative-going step

$$v_s(t) = \begin{cases} 1\ V & t < 0 \\ 0.2\ V & t \geq 0 \end{cases}$$



- For $t < 0$
  - $v_s(t) = 1\ V$ (DC)
  - Inductor is a short circuit (DC)
  - $v_o(t) = 0\ V$
  - $i(t) = 1\ V/1\ k\Omega = 1\ mA$



- At $t = 0$
  - $v_s(0) = 0.2\ V$
  - $i(t)$ cannot change instantaneously
    - $i(0) = 1\ mA$
    - $v_o(0) = v_s(0) - 1\ mA \cdot 1\ k\Omega = -0.8\ V$
    - $V_i = -0.8\ V$

- As $t \to \infty$
  - $v_s(t) = 0.2\ V \to$ DC
  - Inductor → short circuit
  - $v_o(t \to \infty) = 0\ V$
  - $V_f = 0\ V$

RL Circuit Response – Example

- Time constant

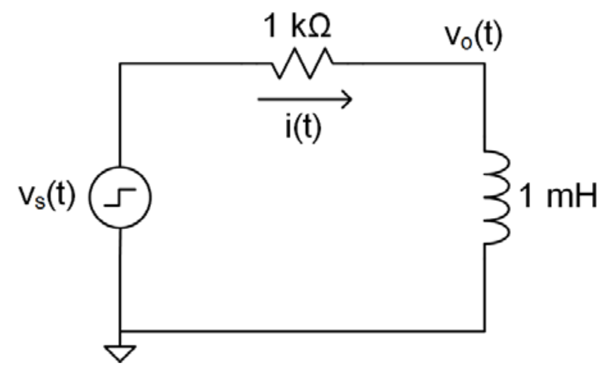
$$\tau = \frac{L}{R} = 1\,\frac{mH}{1\;k\Omega}$$
$$\tau = 1\;\mu sec$$

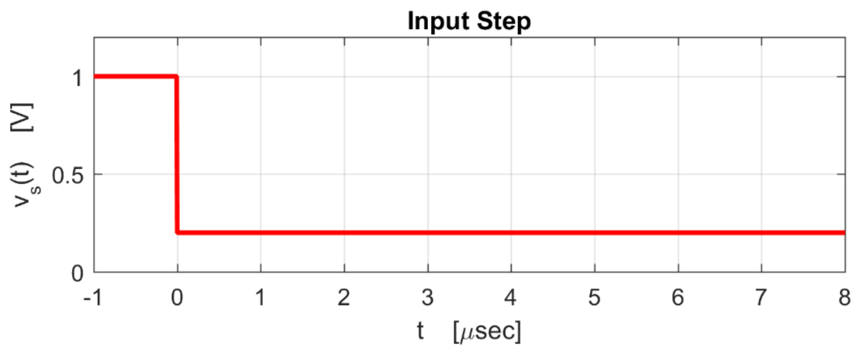
- Voltage step response

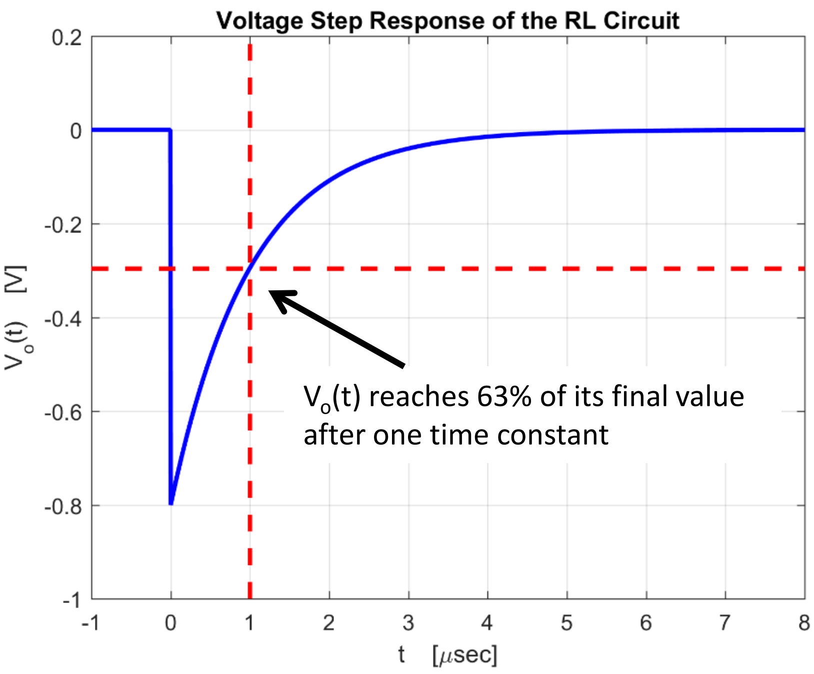
$$v_o(t) = V_f + \left(V_i - V_f\right)e^{-\frac{t}{\tau}}$$

$$v_o(t) = 0\;V + (-0.8\;V - 0\;V)e^{-\frac{t}{\tau}}$$

$$v_o(t) = -0.8\;V \cdot e^{-\frac{t}{1\;\mu sec}}$$





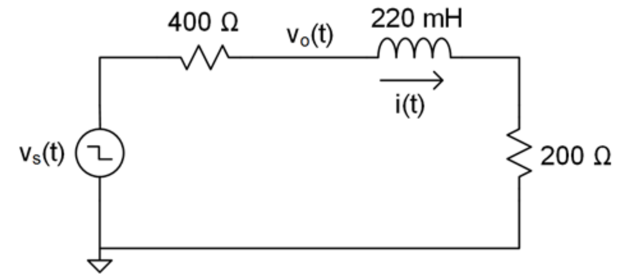


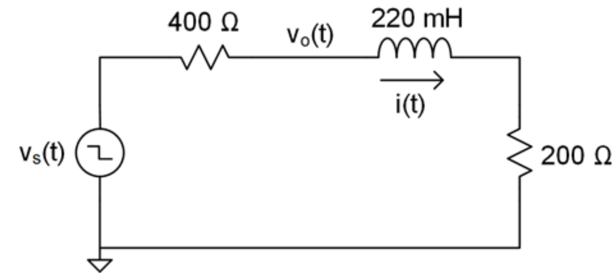
Example Problems

Determine $v_o(t)$ and $i(t)$ for $t \geq 0$.

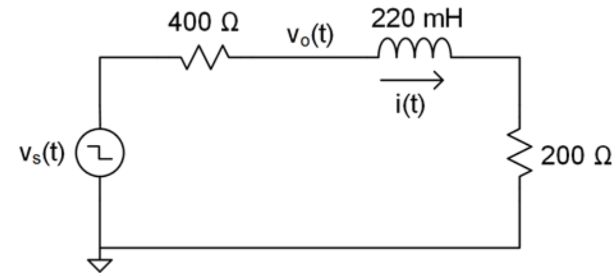
The input source is:

$$v_s(t) = -2\,V \cdot u(t) + 4\,V$$

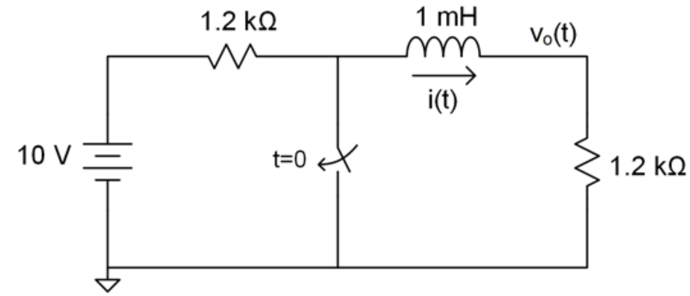


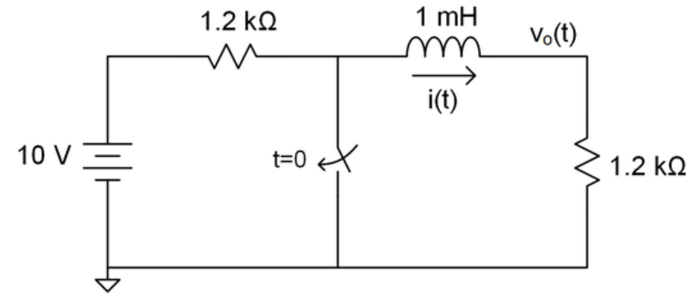


400 Ω
vo(t)
220 mH
i(t)
vs(t)
200 Ω


400 Ω
vo(t)
220 mH
i(t)
vs(t)
200 Ω

Determine $v_o(t)$ and $i(t)$ for $t \geq 0$.




1.2 kΩ
1 mH
v_o(t)
i(t)
10 V
t=0
1.2 kΩ

Determine $v_o(t)$ and $i(t)$ for $t \geq 0$.