SECTION 3:
SECOND-ORDER FILTERS
Introduction
Second-Order Circuits

- **Order** of a circuit (or system of any kind)
  - Number of independent energy-storage elements
  - Order of the differential equation describing the system

- **Second-order circuits**
  - Two energy-storage elements
  - Described by second-order differential equations

- We will primarily be concerned with second-order RLC circuits
  - Circuits with a resistor, an inductor, and a capacitor
Second-Order Circuits

- In this and the following section of notes, we will look at second-order RLC circuits from two distinct perspectives:

  - **Section 3**
    - Second-order *filters*
    - *Frequency-domain* behavior

  - **Section 4**
    - Second-order *transient response*
    - *Time-domain* behavior
Second-Order Filters
Second-Order Filters

- **First-order filters**
  - Roll-off rate: \(20 \text{ dB/decade}\)
  - This roll-off rate determines **selectivity**
    - Spacing of pass band and stop band
    - Spacing of passed frequencies and stopped or filtered frequencies

- **Second-order filters**
  - Roll-off rate: \(40 \text{ dB/decade}\)

- In general:
  - Roll-off = \(N \cdot 20 \text{ dB/dec}\), where \(N\) is the filter order
Resonance

- **Resonance**
  - Tendency of a system to oscillate at certain frequencies – *resonant frequencies* – often with larger amplitude than any input
  - Phenomenon that occurs in all types of dynamic systems (mechanical, electrical, fluid, etc.)

- Examples of resonant mechanical systems:
  - Mass on a spring
  - Pendulum, playground swing
  - Tacoma Narrows Bridge
**Electrical Resonance**

- **Electrical resonance**
  - Cancellation of *reactances*, resulting in purely resistive network impedance
  - Occurs at *resonant frequencies*
  - Second- and higher-order circuits

- **Reactances cancel** – sum to zero ohms
  - Inductive reactance is positive
  - Capacitive reactance is negative

- Voltages/currents in the circuit may be much larger than source voltages/currents

- We’ll take a look at resonance in two classes of circuits:
  - *Series* resonant circuits
  - *Parallel* resonant circuits
Series Resonant Circuits
Series Resonant RLC Circuit

- Series RLC circuit
  - Second-order – one capacitor, one inductor
  - Circuit will exhibit resonance

**Impedance of the network:**

\[
Z_{in}(\omega) = R + \frac{1}{j\omega C} + j\omega L = R + j \left( \omega L - \frac{1}{\omega C} \right)
\]

At the resonant frequency, \(\omega_0\) or \(f_0\):

\[
X_L + X_C = 0 \rightarrow X_L = -X_C
\]

\[
\omega_0 L = \frac{1}{\omega_0 C} \rightarrow \omega_0^2 = \frac{1}{LC}
\]

so

\[
\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad f_0 = \frac{1}{2\pi\sqrt{LC}}
\]

and

\[
Z_{in}(\omega_0) = R
\]
**Quality factor,** \( Q_s \)

- Ratio of inductive reactance *at the resonant frequency* to resistance

\[
Q_s = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R}
\]

- At resonance, inductive and capacitive reactances (magnitudes) are equal, so

\[
Q_s = \frac{1}{\omega_0 RC} = \frac{1}{2\pi f_0 RC}
\]

- The ratio of voltage magnitude across the inductor or capacitor to the voltage across the whole RLC network *at resonance*

- A measure of the *sharpness* of the resonance
Series RLC Circuit – $Z_{in}$ vs. $Q_s$

- At $f = f_0$
  - $|Z_{in}| = R$
  - $\angle Z_{in} = 0^\circ$

- $Q$ determines sharpness of the resonance
  - Higher $Q$ yields faster transition from capacitive, through resistive, to inductive regions

- To increase $Q$:
  - Increase $L$
  - Reduce $R$ and/or $C$
Series RLC Circuit – $Z_{in}$

**Understanding the impedance of a series resonant circuit**

Capacitor impedance goes up as $f$ goes down.

$\mathbf{f = f_0:}$
- $Z_{in} = R$
- $Z_{in}$ is real
- $\angle Z_{in} = 0^\circ$

---

Inductor impedance goes up as $f$ goes up.

$\mathbf{f \gg f_0:}$
- $\angle Z_{in} = +90^\circ$
- $Z_{in}$ looks inductive

---

Capacitor impedance goes up as $f$ goes down.

$\mathbf{f \ll f_0:}$
- $\angle Z_{in} = -90^\circ$
- $Z_{in}$ looks capacitive
At $\omega_0$, $Z_{in} = R$, so the current phasor is

$$I = \frac{V_s}{R} = \frac{V_s}{R} \angle 0^\circ$$

- **Capacitor voltage** at resonance:

$$V_C = \frac{I}{j\omega_0 C} = \frac{V_s \angle 0^\circ}{\omega_0 RC \angle 90^\circ} = \frac{V_s}{\omega_0 RC} \angle -90^\circ$$

- Recalling the expression for quality factor of a series resonant circuit, we have

$$V_C = Q_s \cdot V_s \angle -90^\circ$$

- The voltage across the capacitor is the source voltage multiplied by the quality factor and phase shifted by $-90^\circ$
Series RLC Circuit – Voltages and Currents

- The **inductor voltage** at resonance:

\[ V_L = I \cdot j\omega_0 = \frac{V_s \angle 0^\circ \cdot \omega_0 L \angle 90^\circ}{R} \]

\[ V_L = \frac{V_s \cdot \omega_0 L \angle 90^\circ}{R} \]

- Again, substituting in the expression for quality factor gives

\[ V_L = Q_s \cdot V_s \angle + 90^\circ \]

- The voltage across the inductor is the source voltage multiplied by the quality factor and phase shifted by +90°

- Capacitor and inductor voltage at resonance:
  - Equal magnitude
  - 180° out of phase – opposite sign – they cancel
Now assign component values

- The resonant frequency is
  \[ \omega_0 = \frac{1}{\sqrt{LC}} = 100 \frac{krad}{sec} \]
- The quality factor is
  \[ Q_s = \frac{\omega_0 L}{R} = \frac{100 \frac{krad}{sec} \cdot 1 mH}{10 \Omega} = 10 \]
- The current phasor at the resonant frequency is
  \[ I = \frac{V_s}{R} = \frac{1 V \angle 0^\circ}{10 \Omega} = 100 \angle 0^\circ mA \]
The capacitor voltage at the resonant frequency is
\[ V_C = \frac{I}{j\omega_0 C} = \frac{V_s}{\omega_0 RC} \angle -90^\circ = Q_s \cdot V_s \angle -90^\circ \]

\[ V_C = 10 \cdot 1 \, V \angle -90^\circ \]

\[ V_C = 10 \, V \angle -90^\circ \]

The inductor voltage at the resonant frequency:
\[ V_L = I \cdot j\omega_0 L = \frac{V_s \cdot \omega_0 L \angle 90^\circ}{R} = Q_s \cdot V_s \angle +90^\circ \]

\[ V_L = 10 \cdot 1 \, V \angle +90^\circ \]

\[ V_L = 10 \, V \angle +90^\circ \]
Series RLC Circuit — Voltages and Currents

- $|V_S| = 1 \text{ V}$
- $\angle I = 0^\circ$
- $|V_C| = |V_L| = Q_s |V_S| = 10 \text{ V}$
- $|V_C|$ and $|V_L|$ are $180^\circ$ out of phase
  - They cancel
  - KVL is satisfied
Parallel Resonant Circuits
Parallel Resonant RLC Circuit

- Parallel RLC circuit
  - Second-order – one capacitor, one inductor
  - Circuit will exhibit resonance

**Impedance of the network:**

\[ Z_{in}(\omega) = \left[ \frac{1}{R} + j\omega C + \frac{1}{j\omega L} \right]^{-1} = \left[ \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \right]^{-1} \]

At the resonant frequency, \( \omega_0 \) or \( f_0 \):

\[ B_C + B_L = 0 \rightarrow B_C = -B_L \]

\[ \omega_0 C = \frac{1}{\omega_0 L} \rightarrow \omega_0^2 = \frac{1}{LC} \]

so

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad f_0 = \frac{1}{2\pi\sqrt{LC}} \]

and

\[ Z_{in}(\omega_0) = R \]
**Parallel RLC Circuit – Quality Factor**

- **Quality factor, \( Q_p \)**
  - Ratio of inductive susceptance *at the resonant frequency* to conductance

\[
Q_p = \frac{1/\omega_0 L}{1/R} = \frac{R}{\omega_0 L} = \frac{R}{2\pi f_0 L}
\]

- At resonance, inductive and capacitive susceptances (magnitudes) are equal, so

\[
Q_p = \omega_0 RC = 2\pi f_0 RC
\]

- The ratio of current magnitude through the inductor or capacitor to the current through the whole RLC network *at resonance*

- A measure of the *sharpness* of the resonance
Parallel RLC Circuit – $Z_{in}$ vs. $Q_p$

- At $f = f_0$
  - $|Z_{in}| = R$
  - $\angle Z_{in} = 0^\circ$
- Q determines sharpness of the resonance
  - Higher Q yields faster transition from inductive, through resistive, to capacitive regions
- To increase Q:
  - Reduce L
  - Increase R and/or C
Parallel RLC Circuit – $Z_{in}$

**Understanding the impedance of a series resonant circuit**

$f = f_0$:
- $Z_{in} = R$
- $Z_{in}$ is real

Inductor tends toward a short as $f \to 0$

$f < f_0$:
- $\angle Z_{in} = +90^\circ$
- $Z_{in}$ looks inductive

$f = f_0$:
- $Z_{in}$ is real
- $\angle Z_{in} = 0^\circ$

$f \gg f_0$:
- $\angle Z_{in} = -90^\circ$
- $Z_{in}$ looks capacitive

Capacitor tends toward a short as $f \to \infty$
Parallel RLC Circuit — Voltages and Currents

- Sinusoidal current source, $I_s$
- At resonance, $Z_{in} = R$, so the voltage across the network is:
  $$V_o = I_s R = I_s \angle 0^\circ \cdot R$$

- Current through the **capacitor** at resonance:
  $$I_C = V_o \cdot j\omega_0 C = I_s R \cdot \omega_0 C \angle 90^\circ$$

- Recalling the expression for quality factor of the parallel resonant circuit, we have
  $$I_C = Q_p \cdot I_s \angle 90^\circ$$

- The current through the capacitor is the source current multiplied by the quality factor and phase shifted by 90°
The inductor current at resonance:

\[ I_L = \frac{V_o}{j\omega_0 L} = \frac{I_s R}{\omega_0 L} \angle -90^\circ \]

Again, substituting in the expression for quality factor gives

\[ I_L = Q_p \cdot I_s \angle -90^\circ \]

The current through the inductor is the source current multiplied by the quality factor and phase shifted by \(-90^\circ\).

Capacitor and inductor current at resonance:
- Equal magnitude
- \(180^\circ\) out of phase – opposite sign – they cancel
Now, assign component values

- The resonant frequency is
  \[ \omega_0 = \frac{1}{\sqrt{LC}} = 1 \frac{Mrad}{sec} \]

- The quality factor is
  \[ Q_p = \frac{R}{\omega_0 L} = \frac{100 \Omega}{1 \frac{Mrad}{sec} \cdot 1 \mu H} = 100 \]

- The phasor for the voltage across the network at the resonant frequency is
  \[ V_o = I_s R = 100 mA \angle 0^\circ \cdot 100 \Omega = 10 \angle 0^\circ V \]
The capacitor current at the resonant frequency is

\[ I_C = V_0 \cdot j \omega_0 C = I_s \cdot \omega_0 R C \angle 90^\circ \]

\[ I_C = Q_p \cdot I_s \angle 90^\circ = 100 \cdot 100 \text{ mA} \angle 90^\circ \]

\[ I_C = 10 \text{ A} \angle 90^\circ \]

The inductor current at the resonant frequency:

\[ I_L = \frac{V_0}{j \omega_0 L} = \frac{I_s R}{\omega_0 L} \angle -90^\circ = Q_p \cdot I_s \angle -90^\circ \]

\[ I_L = 100 \cdot 100 \text{ mA} \angle -90^\circ \]

\[ I_L = 10 \text{ A} \angle -90^\circ \]
Parallel RLC Circuit – Voltages and Currents

- $|I_S| = 1 \text{ V}$
- $\angle V_o = 0^\circ$
- $|I_C| = |I_L| = Q_p |I_S| = 10 \text{ A}$
- $|I_C|$ and $|I_L|$ are $180^\circ$ out of phase
- They cancel
- KCL is satisfied
Example Problems
Determine the voltage across the capacitor, $V_o$, at the resonant frequency.
Determine $V_o$, $I$, $I_C$, and $I_L$ at the resonant frequency.
Determine $R$, such that $|V_o| = 100|V_s|$ at the resonant frequency.
Second-Order Filters
Derive the frequency response functions of second-order filters by treating the circuits as voltage dividers.

\[ H(\omega) = \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)} \]

Now, \( Z_1 \) and \( Z_2 \) can be either a single R, L, or C, or a series or parallel combination of any two components for \( Z_1 \) or \( Z_2 \):

- Series combination:

- Parallel combination:
Second-Order Band Pass Filter
One option for a second-order band pass filter:

- The frequency response function:

\[ H(\omega) = \frac{Z_2}{Z_1 + Z_2} \]

where

\[ Z_1 = R \quad \text{and} \quad Z_2 = \left[ j\omega C + \frac{1}{j\omega L} \right]^{-1} = \frac{j\omega L}{1 + (j\omega)^2 LC} \]

so

\[ H(\omega) = \frac{j\omega L}{1 + (j\omega)^2 LC} \]

\[ = \frac{j\omega L}{R + \frac{j\omega L}{1 + (j\omega)^2 LC}} \]

\[ = \frac{j\omega L}{(j\omega)^2 RLC + j\omega L + R} \]

\[ H(\omega) = \frac{j\omega/RC}{(j\omega)^2 + j\omega/RC + 1/LC} \]
Second-Order Band Pass Filter

- Consider the filter’s behavior at three limiting cases for frequency

  - $f \to 0$:
    - $C \to$ open
    - $L \to$ short
    - $v_o$ shorted to ground
    - Gain $\to 0$

  - $f = f_0$:
    - $B_C, B_L$ cancel
    - $L||C \to$ open
    - $v_o = v_i$
    - Gain $\to 1$

  - $f \to \infty$:
    - $C \to$ short
    - $L \to$ open
    - $v_o$ shorted to ground
    - Gain $\to 0$
A second option for a second-order band pass filter:

Now, the impedances are:

\[ Z_1 = j \omega L + \frac{1}{j \omega C} = \frac{(j \omega)^2 LC + 1}{j \omega C} \]

\[ Z_2 = R \]

The frequency response function:

\[
H(\omega) = \frac{R}{R + \frac{(j \omega)^2 LC + 1}{j \omega C}} = \frac{j \omega RC}{(j \omega)^2 LC + j \omega RC + 1}
\]

\[
H(\omega) = \frac{j \omega R/L}{(j \omega)^2 + j \omega R/L + 1/LC}
\]
Consider the filter’s behavior at three limiting cases for frequency:

- **$f \rightarrow 0$:**
  - $C \rightarrow \text{open}$
  - $L \rightarrow \text{short}$
  - Current $\rightarrow 0$
  - $v_o \rightarrow 0$
  - Gain $\rightarrow 0$

- **$f = f_0$:**
  - $X_C, X_L$ cancel
  - $L, C \rightarrow \text{short}$
  - $v_o = v_i$
  - Gain $\rightarrow 1$

- **$f \rightarrow \infty$:**
  - $L \rightarrow \text{open}$
  - $C \rightarrow \text{short}$
  - Current $\rightarrow 0$
  - $v_o \rightarrow 0$
  - Gain $\rightarrow 0$
Each of the two BPF variations has the same resonant frequency:

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

They have different frequency response functions and quality factors:

\[ Q = \omega_0 RC = \frac{R}{\omega_0 L} \]
\[ Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} \]

Each frequency response function can be expressed in terms of \( \omega_0 \) and \( Q \):

\[ H(\omega) = \frac{j\omega/RC}{(j\omega)^2 + j\omega/RC + 1/LC} \]
\[ H(\omega) = \frac{j\omega R/L}{(j\omega)^2 + j\omega R/L + 1/LC} \]
Second-Order Band Pass Filter

- Same frequency response for each band pass filter

- $Q$ determines the sharpness of the resonance
  - Higher $Q$ provides higher selectivity
    - Narrower pass band
    - Steeper transition to the stop bands
Bandwidth of a low pass filter is the 3 dB frequency

A band pass filter has two 3 dB frequencies

**Bandwidth** is the difference between the two 3 dB frequencies

\[ BW = f_U - f_L \]

Bandwidth is inversely proportional to Q

\[ BW = \frac{f_0}{Q} \]

\[ BW = f_U - f_L = \frac{f_0}{Q} = 200 \text{ kHz} \]
Need a band pass filter to isolate a broadcast TV channel
- Carrier frequency: 180MHz
- Bandwidth of the filter: 6MHz
- Thévenin equivalent resistance of signal source: 75Ω

Use a parallel LC network
- A *tank circuit*
2\textsuperscript{nd}-Order BP Filter – Example

- Center frequency of the filter is:
  
  \[ f_0 = \frac{1}{2\pi \sqrt{LC}} = 180 \text{ MHz} \]

- Specified bandwidth dictates the required \( Q \) value
  
  \[ Q = \frac{f_0}{BW} = \frac{180 \text{ MHz}}{6 \text{ MHz}} = 30 \]

- Calculate the required inductance (and/or capacitance) using the values of \( R_s \), \( Q \), and \( f_0 \):
  
  \[ L = \frac{R_s}{\omega_0 Q} = \frac{75 \Omega}{2\pi \cdot 180 \text{ MHz} \cdot 30} = 2.2 \text{ nH} \]

- Use the center frequency to determine the required capacitance
  
  \[ C = \frac{1}{L\omega_0^2} = \frac{1}{2.2 \text{ nH}(2\pi \cdot 180 \text{ MHz})^2} = 355 \text{ pF} \]
2\textsuperscript{nd}-Order BP Filter – Example
Second-Order Band Stop Filter
One option for a second-order **band stop**, or **notch**, filter:

- The frequency response function:

\[ H(\omega) = \frac{Z_2}{Z_1 + Z_2} \]

where

\[ Z_1 = R \quad \text{and} \quad Z_2 = j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + 1}{j\omega C} \]

so

\[ H(\omega) = \frac{(j\omega)^2 LC + 1}{j\omega C} = \frac{(j\omega)^2 LC + 1}{(j\omega)^2 LC + j\omega RC + 1} \]

\[ H(\omega) = \frac{(j\omega)^2 + 1/LC}{(j\omega)^2 + j\omega R/L + 1/LC} \]
Consider the filter’s behavior at three limiting cases for frequency:

- **$f \to 0$:**
  - $C \to$ open
  - $L \to$ short
  - Current $\to 0$
  - $v_o \to v_i$
  - Gain $\to 1$

- **$f = f_0$:**
  - $X_C, X_L$ cancel
  - $L, C \to$ short
  - $v_o$ shorted to ground
  - Gain $\to 0$

- **$f \to \infty$:**
  - $C \to$ short
  - $L \to$ open
  - Current $\to 0$
  - $v_o \to v_i$
  - Gain $\to 1$
Second-order band stop filter

- Resonant (center) frequency:
  \[ f_0 = \frac{1}{2\pi\sqrt{LC}} \]

- Quality factor:
  \[ Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} \]

- Frequency response function:
  \[ H(\omega) = \frac{(j\omega)^2 + 1/LC}{(j\omega)^2 + j\omega R/L + 1/LC} \]

- General form, in terms of \( \omega_0 \) and \( Q \):
  \[ H(\omega) = \frac{(j\omega)^2 + \omega_0^2}{(j\omega)^2 + \frac{\omega_0^2}{Q} j\omega + \omega_0^2} \]
Second-Order Band Stop Filter

- All second-order notch filters provide same response as a function of $Q$ and $\omega_0$
- $Q$ determines the sharpness of the response
- **Higher Q** provides higher selectivity
  - Narrower stop band
  - Steeper transition to the pass bands
Like the band pass filter, the **band stop filter** has two 3 dB frequencies

- **Bandwidth** is the difference between the two 3 dB frequencies
  
  \[ BW = f_U - f_L \]

- Bandwidth is inversely proportional to Q
  
  \[ BW = \frac{f_0}{Q} \]

\[
BW = f_U - f_L = \frac{f_0}{Q} = 500 \text{ kHz}
\]
Consider the following scenario:

- Measuring transient pressure fluctuations inside an enclosed chamber
- Pressure transducer monitored by a data acquisition system
- Measured signal is small – all frequency content lies in the 1KHz – 15KHz range
- Also interested in the average (DC) pressure value
  - AC coupling (HP filter) is not an option
  - Need to keep DC as well as 1KHz – 15KHz
- Measurements are extremely noisy
  - Signal is completely buried in 60Hz power line noise
- Design a notch filter to reject any 60Hz power line noise
Filter design considerations

- Center frequency: 60 Hz
- Attenuate signal of interest as little as possible
  - Set upper 3 dB frequency one decade below the lower end of the signal range (1 kHz)
- Sensor output resistance: 100 Ω
- DAQ system input resistance: 1 MΩ
2\textsuperscript{nd}-Order Notch Filter – Example

- Upper 3 dB frequency is one decade below 1 kHz
  \[ f_U = 100 \text{ Hz} \]

- Simplify by assuming that the 3 dB frequencies are evenly spaced about \( f_0 \)
  \[ BW = 2(f_U - f_0) = 80 \text{ Hz} \]

- Required Q is then
  \[ Q = \frac{f_0}{BW} = \frac{60 \text{ Hz}}{80 \text{ Hz}} = 0.75 \]

- Sensor output resistance can serve as the filter resistor

- DAQ input resistance of 1 M\(\Omega\) is large enough to be neglected
2\textsuperscript{nd}-Order Notch Filter – Example

- Determine $L$ and $C$ values to satisfy $f_0$ and $Q$ requirements
  - The required inductance:
    \[
    L = \frac{Q \cdot R}{\omega_0} = \frac{0.75 \cdot 100 \ \Omega}{2\pi \cdot 60 \ Hz} = 198 \ mH
    \]
  - Calculate $C$ to place the center frequency at 60 Hz
    \[
    C = \frac{1}{\omega_0^2 L} = \frac{1}{(2\pi \cdot 60 \ Hz)^2 \cdot 198 \ mH} = 35.5 \ \mu F
    \]

- A couple things worth noting:
  - Some iteration selecting standard-value components would be required
  - Accuracy and stability of sensor output resistance would need to be verified
2nd-Order Notch Filter – Example

2nd-Order Bandstop Filter with $f_0 = 60$Hz, $Q = 0.75$

Magnitude

Phase

Pressure Sensor

Notch Filter

198mH
33.5μF

$V_o$
To
Data
Acq.
Second-order low pass filter:

The frequency response function:

\[ H(\omega) = \frac{Z_2}{Z_1 + Z_2} \]

where

\[ Z_1 = R + j\omega L \quad \text{and} \quad Z_2 = \frac{1}{j\omega C} \]

so

\[ H(\omega) = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{(j\omega)^2 LC + j\omega RC + 1} \]

\[ H(\omega) = \frac{1/LC}{(j\omega)^2 + j\omega R/L + 1/LC} \]
2\textsuperscript{nd}-Order LPF – General-Form Frequency Response

- **Second-order low pass filter**
  - Resonant frequency:
    
    \[ f_0 = \frac{1}{2\pi\sqrt{LC}} \]

  - Quality factor:
    
    \[ Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} \]

  - Frequency response function:
    
    \[ H(\omega) = \frac{1/LC}{(j\omega)^2 + j\omega R/L + 1/LC} \]

  - General form, in terms of \( \omega_0 \) and \( Q \):
    
    \[ H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + \omega_0^2/Q j\omega + \omega_0^2} \]
Consider the filter’s behavior at three limiting cases for frequency:

- **$f \to 0$:**
  - $L \to$ short
  - $C \to$ open
  - Current $\to 0$
  - $v_o \to v_i$
  - Gain $\to 1$

- **$f = f_0$:**
  - Behavior at resonance is a bit trickier here

- **$f \to \infty$:**
  - $L \to$ open
  - $C \to$ short
  - $v_o$ shorted to ground
  - Gain $\to 0$
Input impedance at resonance: $Z_{in}(\omega_0) = R$

The series LC section is essentially a short
- But, neither the inductor nor the capacitor, individually, are shorts
- And, output is taken across the capacitor
  - Recall that at resonance, capacitor and inductor voltages can exceed the input voltage

$Z_{in} = R$, so

\[ I = \frac{V_i}{R} \]

Output voltage phasor is:

\[ V_o = \frac{I}{j\omega_0 C} \]

\[ V_o = \frac{V_i}{\omega_0 RC} \angle -90^\circ \]

which may be larger than $V_i$
Second-Order Low Pass Filter

- Second-order roll-off rate: \[ 40 \frac{dB}{dec} \]

- \( Q \) determines:
  - Amount of peaking
  - Rate of phase transition

- No peaking at all for \( Q \leq \frac{1}{\sqrt{2}} = 0.707 \)

- Phase at \( \omega_0 \): \(-90^\circ\)
2nd-Order Low Pass Filter – 3dB Bandwidth

- 3 dB bandwidth is a function of Q
  - Increases with Q
- Peaking occurs for $Q > 0.707$
- For $Q = 0.707$
  - \textit{Maximally-flat response}
  - \textit{Butterworth response}
  - The 3 dB frequency is equal to the resonant frequency
  \[ f_c = f_0 \]
Second-Order High Pass Filter
Second-Order High Pass Filter

- Second-order high pass filter:
  - The frequency response function:

\[
H(\omega) = \frac{Z_2}{Z_1 + Z_2}
\]

where

\[
Z_1 = R + \frac{1}{j\omega C}
\]

and

\[
Z_2 = j\omega L
\]

so

\[
H(\omega) = \frac{j\omega L}{R + \frac{1}{j\omega C} + j\omega L} = \frac{(j\omega)^2 LC}{(j\omega)^2 LC + j\omega RC + 1}
\]

\[
H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + j\omega R/L + 1/LC}
\]
2\textsuperscript{nd}-Order HPF – General-Form Frequency Response

- **Second-order high pass filter**
  - Resonant frequency:
    \[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]
  - Quality factor:
    \[ Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} \]
  - Frequency response function:
    \[ H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + j\omega R/L + 1/LC} \]
  - General form, in terms of \( \omega_0 \) and \( Q \):
    \[ H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + \frac{\omega_0^2}{Q} j\omega + \omega_0^2} \]
Consider the filter’s behavior at three limiting cases for frequency

- \( f \to 0 \):
  - \( L \to \) short
  - \( C \to \) open
  - \( v_o \) shorted to ground
  - Gain \( \to 0 \)

- \( f = f_0 \):
  - Behavior at resonance is, once again, a bit more complicated

- \( f \to \infty \):
  - \( L \to \) open
  - \( C \to \) short
  - Current \( \to 0 \)
  - \( v_o \) \( \to v_i \)
  - Gain \( \to 1 \)
Second-Order HPF at Resonance

- Input impedance at resonance: \( Z_{in}(\omega_0) = R \)
- The series LC section is essentially a short
  - But, neither the inductor nor the capacitor, individually, are shorts
  - And, output is taken across the inductor
    - Recall that at resonance, capacitor and inductor voltages can exceed the input voltage

\[ Z_{in} = R, \text{ so } I = \frac{V_i}{R} \]

Output voltage phasor is:
\[ V_o = I \cdot j\omega_0 L \]
\[ V_o = \frac{V_i}{R} \omega_0 L \angle 90^\circ \]
which may be larger than \( V_i \)
Second-Order High Pass Filter

- Second-order roll-off rate:
  \[ 40 \frac{dB}{dec} \]

- \( Q \) determines:
  - Amount of peaking
  - Rate of phase transition

- No peaking at all for
  \[ Q \leq \frac{1}{\sqrt{2}} = 0.707 \]

- Phase at \( \omega_0 \): \(+90^\circ\)
2nd-Order High Pass Filter – 3dB Bandwidth

- Corner frequency is a function of Q
  - Decreases with increasing Q
- Peaking occurs for $Q > 0.707$
- For $Q = 0.707$
  - Maximally-flat response
  - Butterworth response
  - The 3 dB frequency is equal to the resonant frequency
    $$f_c = f_0$$
Damping Ratio - $\zeta$

- We’ve been using **quality factor** to describe second-order filter response
  - A measure of the sharpness of the resonance
  - For band pass/stop filters, Q tells us about **bandwidth**
  - For low/high pass filters, Q tells us about **peaking**

- Another way to describe the same characteristic: **damping ratio**, $\zeta$
  - Damping ratio is inversely proportional to Q:
    \[
    \zeta = \frac{1}{2Q}
    \]
  - A measure of the amount of **damping** in a circuit/system
  - Higher $\zeta$ implies a less resonant system
    - Less peaking
    - Wider bandwidth for band pass/stop filters
2\textsuperscript{nd}-Order Low Pass Response vs. $\zeta$

- As $\zeta$ goes down,
  - Less damping
  - More peaking
- No peaking at all for $\zeta \geq 0.707$
- For $\zeta = 0.707$
  - \textit{Maximally-flat response}
  - \textit{Butterworth response}
  - The 3 dB frequency is equal to the resonant frequency $f_c = f_0$
General-Form Freq. Response Functions in Terms of $\zeta$ 

**Low pass in terms of $Q$:**

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

**Low pass in terms of $\zeta$:**

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\zeta \omega_0 j\omega + \omega_0^2}$$

**High pass in terms of $Q$:**

$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

**High pass in terms of $\zeta$:**

$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + 2\zeta \omega_0 j\omega + \omega_0^2}$$
Audio Filter Demo
Analog Discovery Instrument

- 2-chan. Scope
  - 14-bit, 100MSa/s
  - 5MHz bandwidth
- 2-chan. function generator
  - 14-bit, 100MSa/s
  - 5MHz bandwidth
- 2-chan. spectrum analyzer
- Network analyzer
- Voltmeter
- ±5V power supplies
- 16-chan. logic analyzer
- 16-chan. digital pattern generator
- USB connectivity
Analog Discovery – Audio Demo

- Demo board plugs in to Analog Discovery module
- Summation of multiple tones
- Optional filtering of audio signal
- 3.5 mm audio output jack
Example Problems
The higher-frequency signal is unwanted noise. Design a second-order Butterworth LPF to attenuate the higher-frequency component by 40 dB.

What is the SNR at the output of the filter?