SECTION 4: SECOND-ORDER TRANSIENT RESPONSE
Introduction
In this and the previous section of notes, we consider second-order RLC circuits from two distinct perspectives:

- **Frequency-domain**
  - Second-order, RLC filters

- **Time-domain**
  - Second-order, RLC step response
Transient Response of Second-Order Circuits
Second-Order Transient Response

- In ENGR 201 we looked at the transient response of first-order RC and RL circuits
  - Applied KVL
    - Governing differential equation
  - Solved the ODE
    - Expression for the step response

- For **second-order circuits**, process is the same:
  - **Apply KVL**
    - Second-order ODE
  - **Solve the ODE**
    - Second-order step response
Step Response of RLC Circuit

- Determine the response of the following RLC circuit
  - Source is a voltage step: $v_s(t) = 1V \cdot u(t)$
  - Output is the voltage across the capacitor

- Apply KVL around the loop
  $$v_s(t) - i(t)R - L \frac{di}{dt} - v_o(t) = 0$$

- Want an ODE in terms of $v_o(t)$
  - Need to eliminate $i(t)$
  - Can express $i(t)$ in terms of the output voltage:
    $$i(t) = C \frac{dv_o}{dt} \quad \text{so,} \quad \frac{di}{dt} = C \frac{d^2v_o}{dt^2}$$
Step Response of RLC Circuit

Substituting in the expression for current, the KVL equation becomes

$$v_s(t) - C \frac{dv_o}{dt} R - LC \frac{d^2 v_o}{dt^2} - v_o(t) = 0$$

Rearranging gives the governing second-order ODE:

$$\frac{d^2 v_o}{dt^2} + \frac{R}{L} \frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = \frac{1}{LC} v_s(t)$$

A second-order, linear, non-homogeneous, ordinary differential equation

Non-homogeneous, so solve in two parts

1) Find the complementary solution to the homogeneous equation
2) Find the particular solution for the step input

General solution will be the sum of the two individual solutions:

$$v_o(t) = v_{oc}(t) + v_{op}(t)$$
Complementary Solution
The homogeneous equation is obtained by setting the forcing function (input) to zero

\[ \frac{d^2 v_o}{dt^2} + \frac{R}{L} \frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = 0 \]

For an ODE of this form, we assume a solution of the form

\[ v_{oc}(t) = e^{st} \]

Where \( s \) is an unknown complex value. Then

\[ \frac{dv_{oc}}{dt} = se^{st} \quad \text{and} \quad \frac{d^2 v_{oc}}{dt^2} = s^2 e^{st} \]

Substituting back into the homogeneous ODE yields the characteristic equation

\[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \]
Complementary Solution – $\nu_{oc}(t)$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

- The characteristic equation can be rewritten as
  $$s^2 + 2\alpha s + \omega_0^2 = 0$$
  or
  $$s^2 + 2\zeta \omega_o s + \omega_0^2 = 0$$

- The roots of the characteristic equation (also called poles) tell us about the:
  - Form of the complementary solution
  - Nature of the response

- These roots (or poles) are
  $$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$
Complementary Solution – $\nu_{oc}(t)$

- We’ve said we can write the **characteristic equation** as
  \[ s^2 + 2\alpha s + \omega_0^2 = 0 \]
  or
  \[ s^2 + 2\zeta \omega_0 s + \omega_0^2 = 0 \]

- The **damping ratio**, $\zeta$, can be defined as
  \[ \zeta = \frac{\alpha}{\omega_0} \]

- A few key points:
  - $\omega_0$ is the resonant frequency
  - $\zeta$ characterizes the nature (sharpness) of the resonance
  - Both are related to the roots of the characteristic equation
Complementary Solution – \( v_{oc}(t) \)

- Complementary solution has the same form as that of a first-order circuit:
  \[ v_{oc}(t) = e^{st} \]
  - \( s \) is the roots of the characteristic equation
    - Now two values – identical or distinct
    - May be complex

- Form of the solution depends on the values of \( s \)
  - Can be characterized in terms of the value of \( \zeta \):
    - \( \zeta > 1 \) – over-damped case:
      \[ v_{oc}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \]
    - \( \zeta = 1 \) – critically-damped case:
      \[ v_{oc}(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} \]
    - \( \zeta < 1 \) – under-damped case:
      \[ v_{oc}(t) = K_1 e^{-\alpha t} \cos(\omega_d t) + K_2 e^{-\alpha t} \sin(\omega_d t) \]
Over-Damped RLC Circuit – $\zeta > 1$

- Roots of the characteristic equation are
  $$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

  - These are related to the damping ratio as
    $$\zeta = \frac{\alpha}{\omega_0}$$

- If $\zeta > 1$, then
  - $\alpha > \omega_0$
  - $\alpha^2 - \omega_0^2 > 0$ – i.e., the discriminant is positive
  - $s_1$ and $s_2$ are real and distinct

- Complimentary solution has the following form
  $$v_{oc}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

- Recall that $\zeta > 1$ (actually, $\zeta \geq 0.707$) corresponded to no peaking in the frequency domain
Critically-Damped RLC Circuit – $\zeta = 1$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} , \quad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

- If $\zeta = 1$, then
  - $\alpha = \omega_0$
  - $\alpha^2 - \omega_0^2 = 0$ – i.e., the discriminant is **zero**
  - $s_1$ and $s_2$ are **real and identical**
- Complimentary solution has the following form
  $$v_{oc}(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$
- This is the lowest value of $\zeta$ for which the step response is **monotonic**
  - Constantly increasing
  - **No overshoot**
Under-Damped RLC Circuit – $\zeta < 1$

- If $\zeta < 1$, then
  - $\alpha < \omega_0$
  - $\alpha^2 - \omega_0^2 < 0$ – i.e., the discriminant is **negative**
  - $s_1$ and $s_2$ are a **complex conjugate pair**

- Complimentary solution has the following form

$$\nu_{oc}(t) = K_1 e^{-\alpha t} \cos(\omega_d t) + K_2 e^{-\alpha t} \sin(\omega_d t)$$

- $\omega_d$ is the **damped natural frequency**

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0 \sqrt{1 - \zeta^2}$$

- Response now contains **damped sinusoidal** components
  - Will exhibit **overshoot**
  - Possible **ringing**
Damping Cases – Geometric Interpretation

- Roots of characteristic equation (system poles) are, in general, complex
  - Can plot them in the complex plane
  - Pole locations tell us a lot about the nature of the response
    - Speed – risetime, settling time
    - Overshoot, ringing

Case 1:
\[ \zeta > 1 \text{ – overdamped} \]

Case 2:
\[ \zeta = 1 \text{ – critically-damped} \]

Case 3:
\[ \zeta < 1 \text{ – underdamped} \]
Under-Damped Case – $\alpha$, $\zeta$, and $\omega_0$

- Under-damped case – $\zeta < 1$
- Roots are a complex-conjugate pair
  
  $s_{1,2} = -\alpha \pm j\omega_d$

- $\alpha$ is the real part
- $\omega_d$ is the imaginary part
- The **magnitude** of the root is
  
  $\omega_0 = \sqrt{\alpha^2 + \omega_d^2}$

- Angle between imaginary axis and vector to the poles is related to damping
  
  $\zeta = \frac{\alpha}{\omega_0} = \sin(\theta)$
Particular Solution
Particular solution – $v_{op}(t)$

- General solution for the RLC step response is the sum of the complementary and particular solutions

  $$v_o(t) = v_{oc}(t) + v_{op}(t)$$

- We now have the complementary solution with two unknown constants, $K_1$ and $K_2$
  - Constants to be determined later through application of initial conditions

- Next, determine the particular solution, $v_{op}(t)$
  - For a circuit driven by a step input, this is simply the circuit’s steady-state response

  $$v_{op}(t) = v_o(t \to \infty)$$
Particular solution – $v_{op}(t)$

- Particular solution is the circuit’s steady-state response
  - Inductor → short
  - Capacitor → open

$$v_{op}(t) = v_o(t \to \infty) = v_s(t > 0)$$

- For a unit step input, the particular solution is

$$v_{op}(t) = 1 \text{ V}$$
Example Problems
Derive the governing differential equation for the following circuit.
Determine:
- Damping ratio
- Damping case
- Characteristic equation
- Poles
Determine:
- Initial conditions
- Particular solution

For
\[ v_s(t) = -1V \cdot u(t) + 2V \]
Derive the governing differential equation for the following circuit.
Over-Damped Circuit Response
RLC Step Response – Example 1

- Determine $v_o(t)$
- Input is a unit voltage step
  
  $v_s(t) = 1V \cdot u(t)$

- First, apply KVL

  $$v_s(t) - i(t)R - L \frac{di}{dt} - v_o(t) = 0$$

- Eliminate $i(t)$ using the i-v relationship for the capacitor

  $$i(t) = C \frac{dv_o}{dt} \quad \text{and} \quad \frac{di}{dt} = C \frac{d^2v_o}{dt^2}$$

- This gives the second-order ODE in terms of $v_o(t)$, which can then be rearranged to standard form

  $$v_s(t) - RC \frac{dv_o}{dt} - LC \frac{d^2v_o}{dt^2} - v_o(t) = 0 \quad \rightarrow \quad \frac{d^2v_o}{dt^2} + \frac{R}{L} \frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = \frac{1}{LC} v_s(t)$$
RLC Step Response – Example 1

- Find the complementary solution, $v_{oc}(t)$
- The homogeneous equation
  \[
  \frac{d^2v_o}{dt^2} + \frac{R}{L} \frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = 0
  \]
- The characteristic equation
  \[
  s^2 + \frac{R}{L} s + \frac{1}{LC} = 0
  \]
  - This can be rewritten as
    \[
    s^2 + 2\alpha s + \omega_0^2 = 0
    \]
    where
    \[
    \alpha = \frac{R}{2L} = \frac{40 \ \Omega}{2 \cdot 10 \ \mu H} = 2 \times 10^6 \ \text{rad/sec}
    \]
    and
    \[
    \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \ \mu H \cdot 100 \ nF}} = 1 \times 10^6 \ \text{rad/sec}
    \]
The damping ratio is

\[ \zeta = \frac{\alpha}{\omega_0} = \frac{2 \times 10^6}{1 \times 10^6} = 2 \]

\(\zeta > 1\), so the circuit is over-damped

Solution is of the form

\[ v_{oc}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \]

\(s_1\) and \(s_2\) are the roots of the characteristic equation

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -268 \times 10^3 \frac{rad}{sec} \]

\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -3.73 \times 10^6 \frac{rad}{sec} \]

The complementary solution is

\[ v_{oc}(t) = K_1 e^{-268 \times 10^3 t} + K_2 e^{-3.73 \times 10^6 t} \]
The particular solution is the circuit’s steady-state solution.

Steady-state equivalent circuit:
- Capacitor → open
- Inductor → short

So, the particular solution is

\[ v_{op}(t) = 1 \, V \]

The general solution:

\[ v_o(t) = v_{oc}(t) + v_{op}(t) \]

\[ v_o(t) = K_1 e^{-268 \times 10^3 t} + K_2 e^{-3.73 \times 10^6 t} + 1 \, V \]

Next, we’ll apply initial conditions to determine the unknown coefficients, \( K_1 \) and \( K_2 \).
Initial Conditions:

For $t < 0$
- $v_s(t < 0) = 0$
- $i(t < 0) = 0$
- $v_o(t < 0) = 0$

At $t = 0$
- $v_s(0) = 1 \, \text{V}$
- Capacitor voltage cannot change instantaneously
  - $v_o(0) = v_o(t < 0) = 0 \, \text{V}$
- Inductor current cannot change instantaneously
  - $i(0) = i(t < 0) = 0 \, \text{A}$
- And, current is related to the output voltage, so
  - $\frac{dv_o}{dt} \bigg|_{t=0} = \dot{v}_o(0) = 0$
The two initial conditions are:

\[ v_o(0) = 0 \quad (1) \]
\[ \dot{v}_o(0) = 0 \quad (2) \]

Use the initial conditions to determine \( K_1 \) and \( K_2 \)

- Applying the first initial condition
  \[ v_o(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + 1 \text{ V} \]
  \[ v_o(0) = K_1 + K_2 + 1 \text{ V} = 0 \]
  \[ K_2 = -K_1 - 1 \text{ V} \quad (3) \]

- Applying the second initial condition
  \[ \dot{v}_o(t) = s_1 K_1 e^{s_1 t} + s_2 K_2 e^{s_2 t} \]
  \[ \dot{v}_o(0) = s_1 K_1 + s_2 K_2 = 0 \quad (4) \]
RLC Step Response – Example 1

- Substituting (3) into (4)

\[
s_1K_1 - s_2(K_1 + 1 \text{ V}) = 0
\]

\[
K_1(s_1 - s_2) = s_2 \cdot 1 \text{ V}
\]

\[
K_1 = \frac{s_2}{s_1 - s_2} \cdot 1 \text{ V} = -1.08 \text{ V}
\]

- Substituting the value of \( K_1 \) back into (3)

\[
K_2 = -K_1 - 1 \text{ V} = 0.08 \text{ V}
\]

- The step response for this over-damped RLC circuit is

\[
v_o(t) = -1.08 \text{ V} \cdot e^{s_1 t} + 0.08 \text{ V} \cdot e^{s_2 t} + 1 \text{ V}
\]

\[
v_o(t) = -1.08 \text{ V} \cdot e^{-268 \times 10^3 t} + 0.08 \text{ V} \cdot e^{-3.73 \times 10^6 t} + 1 \text{ V}
\]
RLC Step Response – Example 1

\[ v_o(t) = -1.08 \, V \cdot e^{-268 \times 10^3 t} + 0.08 \, V \cdot e^{-3.73 \times 10^6 t} + 1 \, V \]

- Similar to first-order response
  - Sum of two decaying exponentials
  - Monotonic increase to final value
- Initial slope differs from first-order response
  - Increases after \( t = 0 \)
Critically-Damped Circuit Response
Now consider the same circuit with decreased resistance.

To determine the form of the response, first determine the **damping ratio**, $\zeta$

$$\zeta = \frac{\alpha}{\omega_0}$$

where

$$\alpha = \frac{R}{2L} = \frac{20 \ \Omega}{2 \cdot 10 \ \mu H} = 1 \times 10^6 \ \frac{\text{rad}}{\text{sec}}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \ \mu H \cdot 100 \ \text{nf}}} = 1 \times 10^6 \ \frac{\text{rad}}{\text{sec}}$$

The damping ratio is $\zeta = 1$, and the circuit is **critically-damped**

The **complementary solution** will be of the following form:

$$v_{oc}(t) = K_1 e^{s_{1,2}t} + K_2 t e^{s_{1,2}t}$$
The critically-damped circuit will have two real, identical poles

\[ s_{1,2} = -\alpha = -1 \times 10^6 \frac{rad}{sec} \]

The complementary solution is

\[ v_{oc}(t) = K_1 e^{-1 \times 10^6 t} + K_2 t e^{-1 \times 10^6 t} \]

The particular solution is still given by the steady-state response, and has not changed

\[ v_{op}(t) = v_o(t \rightarrow \infty) = v_s(t > 0) = 1 \text{ V} \]

The general solution is the sum of the complementary and particular solutions

\[ v_o(t) = K_1 e^{-1 \times 10^6 t} + K_2 t e^{-1 \times 10^6 t} + 1 \text{ V} \]
Next, determine the unknown coefficients by applying initial conditions

Following the same reasoning as in the previous example, initial conditions are the same

\[ v_o(0) = 0 \quad \text{and} \quad \dot{v}_o(0) = 0 \]

Applying the first initial condition

\[ v_o(0) = K_1 + 1 \, V = 0 \quad \rightarrow \quad K_1 = -1 \, V \]

Applying the second initial condition

\[ \dot{v}_o(t) = K_1 s_{1,2} e^{s_{1,2} t} + K_2 (t s_{1,2} e^{s_{1,2} t} + e^{s_{1,2} t}) \]

\[ \dot{v}_o(0) = K_1 s_{1,2} + K_2 = 0 \quad \rightarrow \quad K_2 = -s_{1,2} K_1 = -1 \times 10^6 \, V \]

The step response of this critically-damped circuit:

\[ v_o(t) = -e^{-1 \times 10^6 t} - 1 \times 10^6 t e^{-1 \times 10^6 t} + 1 \, V \]
RLC Step Response – Example 2

\[ v_o(t) = -e^{-1 \times 10^6 t} - 1 \times 10^6 t e^{-1 \times 10^6 t} + 1 \text{ V} \]

- Similar to over-damped response
- Faster risetime
  - Dominant, slow pole replaced by higher-frequency double pole
- Again, increasing initial slope differs from first-order response
- Response never exceeds its final value
Under-Damped Circuit Response
Again decrease the resistance

First, determine the **damping ratio**, $\zeta$

$$\zeta = \frac{\alpha}{\omega_0}$$

where, now

$$\alpha = \frac{R}{2L} = \frac{10 \, \Omega}{2 \cdot 10 \, \mu H} = 500 \times 10^3 \frac{rad}{sec}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \, \mu H \cdot 100 \, nF}} = 1 \times 10^6 \frac{rad}{sec}$$

The damping ratio is $\zeta = 0.5$, and the circuit is **under-damped**

The **complementary solution** will be of the following form:

$$v_{oc}(t) = K_1 e^{-\alpha t} \cos(\omega_d t) + K_2 e^{-\alpha t} \sin(\omega_d t)$$
RLC Step Response – Example 3

\[ v_{oc}(t) = K_1 e^{-at} \cos(\omega_d t) + K_2 e^{-at} \sin(\omega_d t) \]

- The **damped natural frequency** is

\[ \omega_d = \omega_0 \sqrt{1 - \zeta^2} = \sqrt{\omega_0^2 - \alpha^2} = 866 \times 10^3 \text{ rad/sec} \]

- The **complementary solution** is

\[ v_{oc}(t) = K_1 e^{-500 \times 10^3 t} \cos(866 \times 10^3 t) + K_2 e^{-500 \times 10^3 t} \sin(866 \times 10^3 t) \]

- Once again, the **particular solution** is

\[ v_{op}(t) = v_o(t \rightarrow \infty) = v_s(t > 0) = 1 \text{ V} \]

- The **general solution** is the sum of the complementary and particular solutions

\[ v_o(t) = K_1 e^{-500 \times 10^3 t} \cos(866 \times 10^3 t) + K_2 e^{-500 \times 10^3 t} \sin(866 \times 10^3 t) + 1 \text{ V} \]
RLC Step Response – Example 3

\[ v_o(t) = K_1 e^{-\alpha t} \cos(\omega_d t) + K_2 e^{-\alpha t} \sin(\omega_d t) + 1 V \]

Next, determine the unknown coefficients by applying initial conditions
\[ v_o(0) = 0 \quad \text{and} \quad \dot{v}_o(0) = 0 \]

Applying the first initial condition
\[ v_o(0) = K_1 + 1 V = 0 \]
\[ K_1 = -1 V \]

Applying the second initial condition
\[ \dot{v}_o(t) = K_1 [ -\omega_d e^{-\alpha t} \sin(\omega_d t) - \alpha e^{-\alpha t} \cos(\omega_d t) ] + K_2 [ \omega_d e^{-\alpha t} \cos(\omega_d t) - \alpha e^{-\alpha t} \sin(\omega_d t) ] \]
\[ \dot{v}_o(0) = -K_1 \alpha + K_2 \omega_d = 0 \]
\[ K_2 = K_1 \frac{\alpha}{\omega_d} = - \frac{500 \times 10^3}{866 \times 10^3} = -0.58 \]
The step response for this under-damped RLC circuit is

\[ v_o(t) = -e^{-500 \times 10^3 t} \cos(866 \times 10^3 t) \]
\[ -0.58e^{-500 \times 10^3 t} \sin(866 \times 10^3 t) + 1 \text{ V} \]

- Damped oscillatory components
  - Overshoot
  - Possible ringing

- Exponential damping
  - Oscillatory components decay to zero
  - Rate of decay determined by \( \alpha \), real part of poles
RLC Step Response – Example 3

- **Overshoot**
  - Response exceeds its final value

- **Ringing**
  - Response oscillate about its final value
  - Not much ringing in this example

- **Damping ratio**
  - Overshoot and ringing are inversely proportional to $\zeta$
Step Response Characteristics
Risetime

- The time it takes a response to transition between two set levels

- Typically 10% and 90% of full swing

- Occasionally 20% and 80%

- Very rough approximation:
  \[ t_r \approx \frac{1.8}{\omega_0} \]
Step Response – Overshoot

- **Overshoot**
  - Response’s excursion beyond its final value
  - Expressed as a percentage of the full-scale swing
  - Inversely proportional to damping ratio

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<th>%OS</th>
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<td>10</td>
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<td>0.7</td>
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$%OS = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \cdot 100\%$
Step Response – Settling Time

- **Settling time**
  - The time it takes a response to settle (finally) to within some percentage of the final value
  - Typically ±1%, ±2%, or ±5%
  - Inversely proportional to the real part of the circuit’s poles (roots of the characteristic equation)
  - For ±1% settling time:
    $$t_s \approx \frac{4.6}{\alpha} = \frac{4.6}{\zeta \omega_0}$$
Determine:
- R and L, such that
  - \( O.S. = 10\% \)
  - \( t_s \approx 2\mu \text{sec} \)
- System poles
- \( v_o(t) \) for \( t \geq 0 \)
Determine the minimum resistance for 0% overshoot.