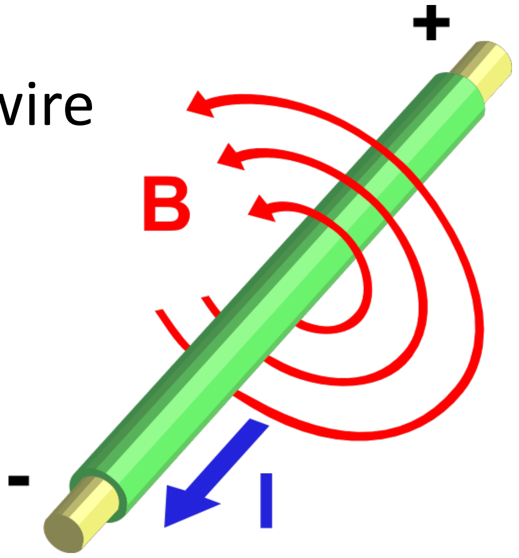
SECTION 5: MAGNETICALLY-COUPLED CIRCUITS

ENGR 202 – Electrical Fundamentals II

Electromagnetic Fundamentals

Ampere's Law

- Electrical current flowing through a wire generates a magnetic field encircling that wire
- Direction of field given by right-hand rule
  - Thumb points in direction of current
  - Fingers curl in direction of field



- ***Ampere's law***

$$\oint \boldsymbol{H} \cdot dl = \frac{1}{\mu} \oint \boldsymbol{B} \cdot dl = I$$

  - $\boldsymbol{H}$ is the magnetic field intensity, $\boldsymbol{B}$ is the magnetic flux density, μ is permeability, and I is current
- Ampere's law says:
  - *Integrating the magnetic field around a closed contour gives the total current enclosed by that contour*

Faraday's Law

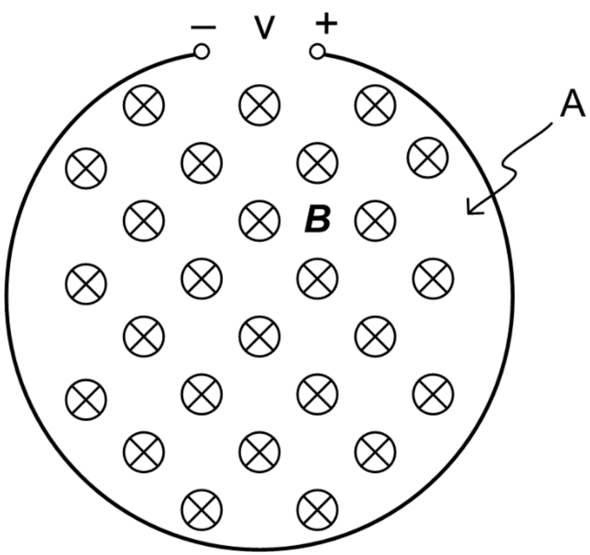
- A time-varying magnetic flux through a loop or coil of wire will produce a voltage across that loop or coil
- ***Faraday's law*** gives the voltage produced across an N**-**turn coil

$$v(t) = -N\frac{d\phi}{dt}$$

  - ϕ is the magnetic flux penetrating the coil:

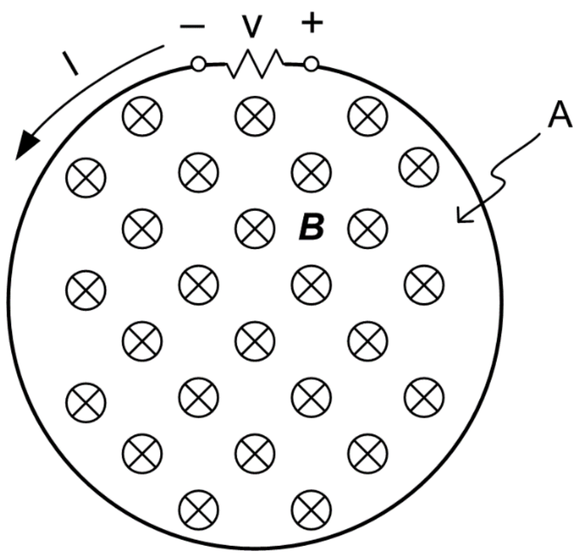
$$\phi = B \cdot A$$

where A is the cross-sectional area of the coil



Lenz's Law

$$v(t) = -N\frac{d\phi}{dt}$$

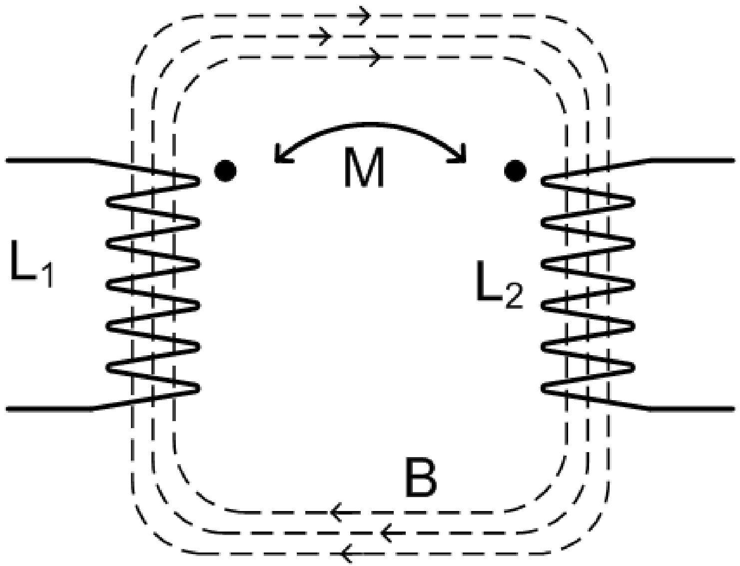


- The negative sign in Faraday's law gives the voltage polarity
  - Close the loop with an external resistance
  - Current flows and generates a magnetic field
  - Magnetic field opposes the original change in magnetic flux
- This is ***Lenz's law***
- Often see Faraday's law written without the negative sign

Mutual Inductance

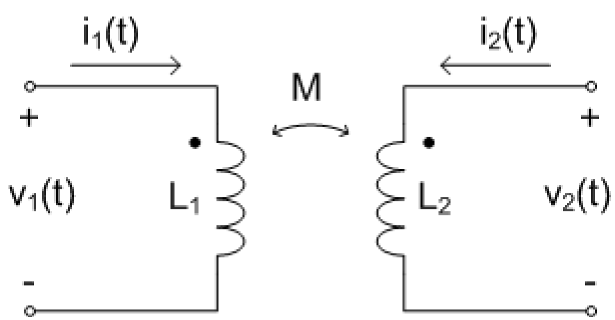
Mutual Inductance

- When current flows through an inductor, energy is stored in a magnetic field
- Magnetic field, B, may penetrate another coil
- If the field is time-varying, it will induce a voltage across the second coil
  - ***Faraday's Law***



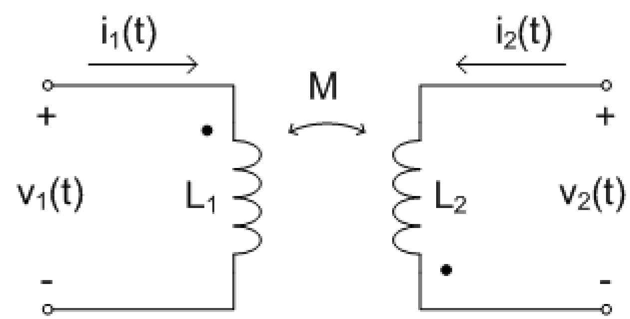
Mutual Inductance

- ***Mutual inductance***
  - Property of electric circuits in which a time-varying current in one inductor results in a voltage across a second inductor
  - Due to flux linkage between the two inductors
  - Denoted as **M**
  - Units: **Henries (H)**
  - ***Dot convention*** determines polarity of induced voltages:



$$v_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2(t) = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

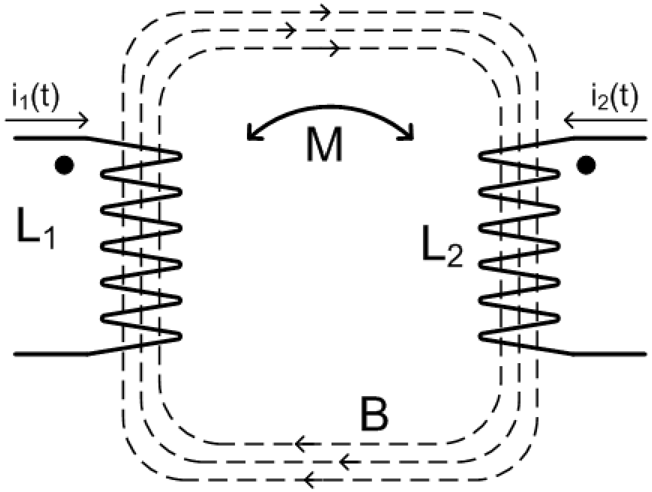


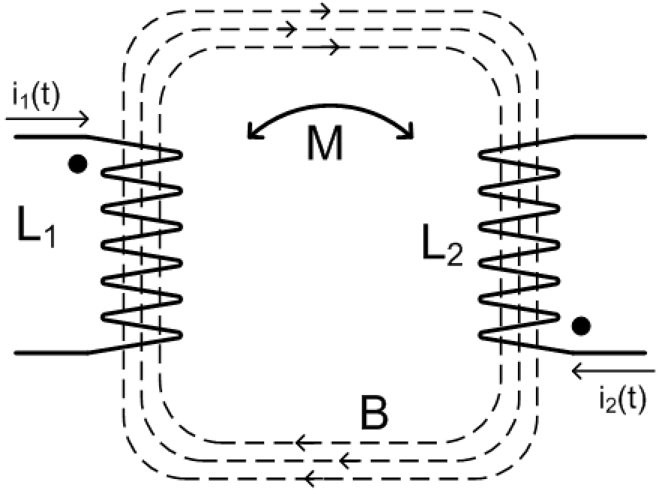
$$v_1(t) = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2(t) = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

Mutual Inductance – Dot Convention

- ***Dot convention***
  - Current entering each dotted terminal produces magnetic flux in the same direction





- In both cases above, i_1 and i_2 in the directions indicated produce the magnetic flux, B, in same direction

Mutual Inductance – Coupling Coefficient

- ***Coupling coefficient***, k
  - A measure of the amount of magnetic coupling between two inductors
- $0 \leq k \leq 1$
  - $k = 0$: completely un-coupled inductors
  - $k = 1$: perfect coupling – all magnetic flux generated by one inductor penetrates the coil of the other inductor
- Relationship to ***mutual inductance***, M:

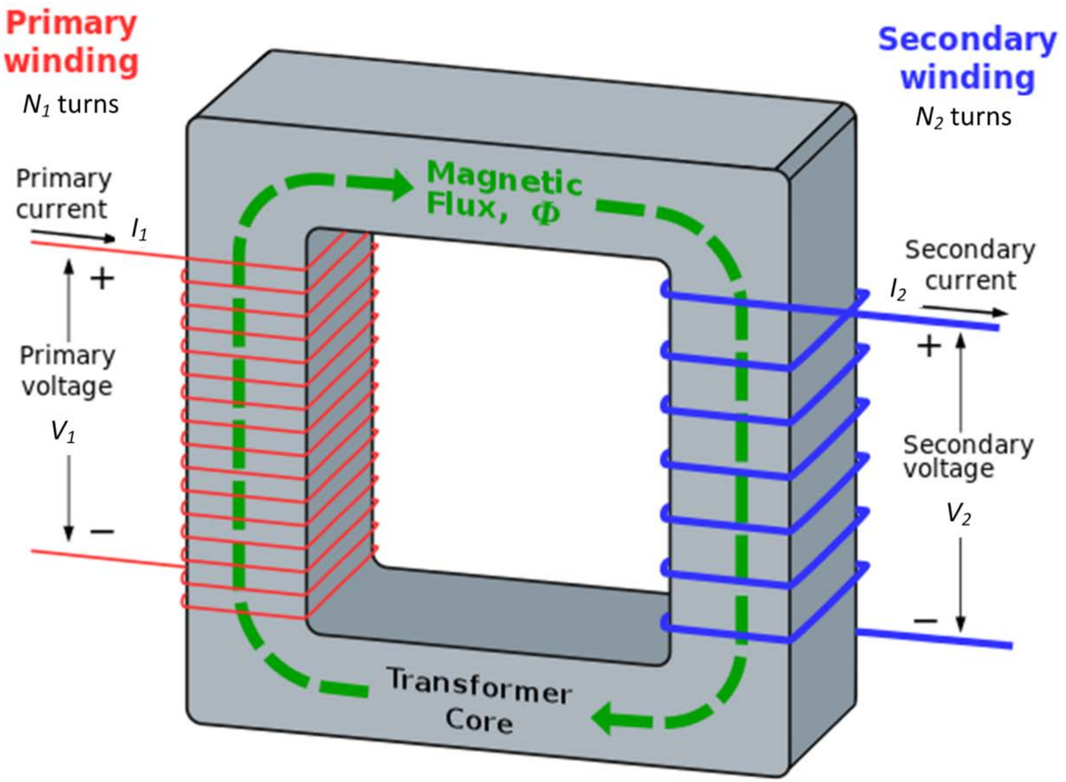
$$M = k\sqrt{L_1 L_2}$$

Ideal Transformers

Ideal Transformers

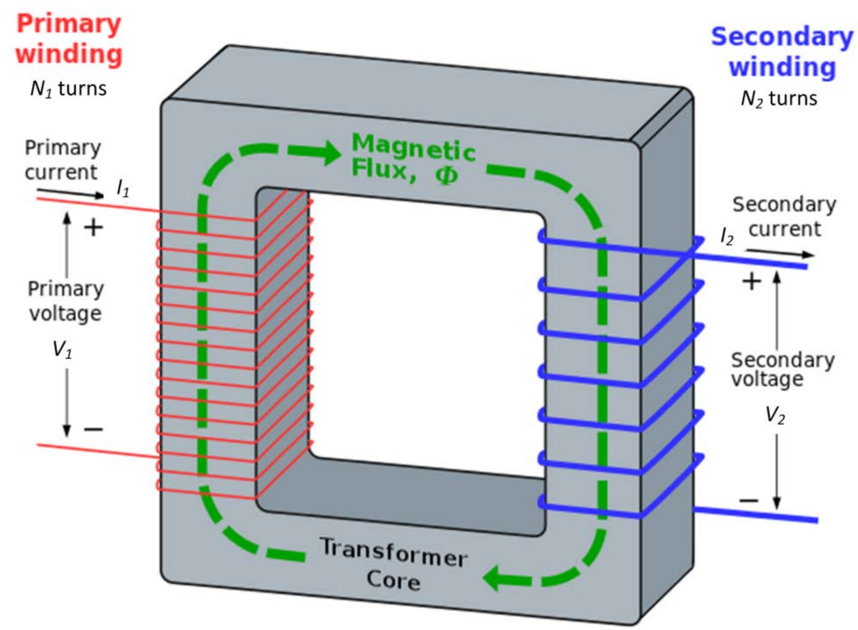
- An ideal transformer consists of two coils of wire wound around a magnetic core

- Used for ***stepping voltages up or down***
  - Step-up transformer
  - Step-down transformer



Ideal Transformers

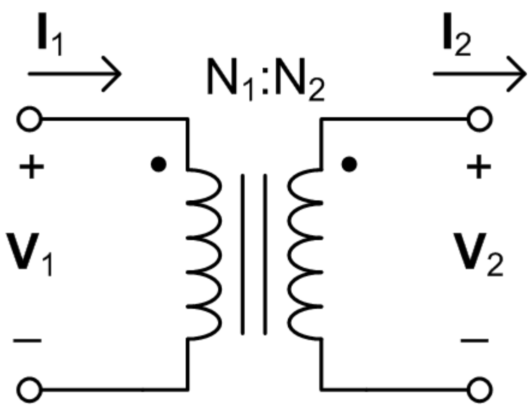
- Current flow in the primary winding generates a magnetic flux in the core
  - ***Ampere's law***
- Flux in the core penetrates the secondary winding



- If that flux is time-varying, a voltage is induced across the secondary
  - ***Faraday's law***

Ideal Transformers

- For ***ideal transformers***, we assume the following:
  1. No losses in the windings or in the core
  2. Perfect magnetic coupling between windings
     - $k = 1$
     - ***All*** flux generated in one winding penetrates the other winding



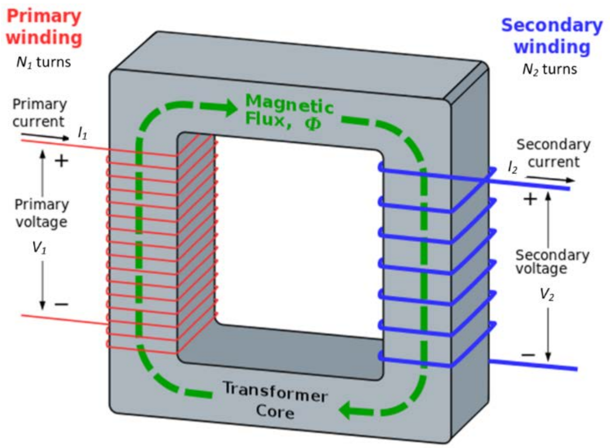
- Dots indicate polarity
  - Current enters one dotted terminal, current leaves the other
  - Positive voltage at one dotted terminal, positive voltage at the other

Ideal Transformers – Current Relationships

- Ampere's law relates the current at each winding to the magnetic field in the core

$$\oint \boldsymbol{H} \cdot dl = I$$

- H-field is due to the ***total current*** at each winding
  - $N_1 I_1$ at the primary winding
  - $N_2 I_2$ at the secondary winding



- The same H-Field links both windings, so

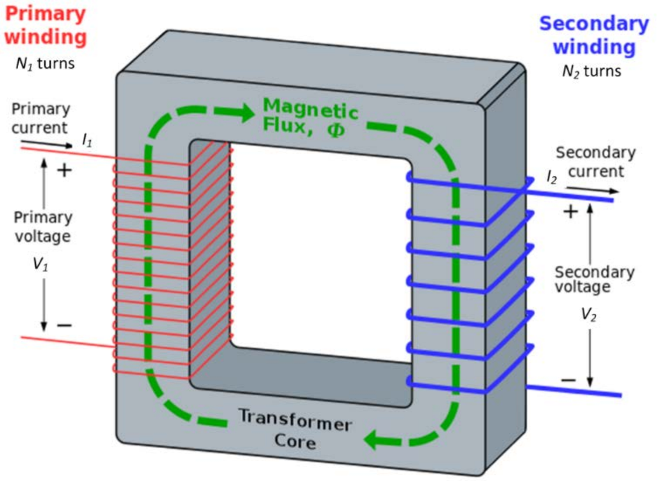
$$\oint \boldsymbol{H} \cdot dl = N_1 I_1 = N_2 I_2$$

- Ratio of currents at each winding is inversely proportional to the ratio of turns at those windings

$$\frac{\mathbf{I_1}}{\mathbf{I_2}} = \frac{N_2}{N_1}$$

Ideal Transformers – Current Relationships

$$\frac{\mathbf{I_1}}{\mathbf{I_2}} = \frac{N_2}{N_1}$$



- Current at each winding:

$$\mathbf{I_1} = \frac{N_2}{N_1}\mathbf{I_2} \quad \text{and} \quad \mathbf{I_2} = \frac{N_1}{N_2}\mathbf{I_1}$$

- ***Turns ratio***
  - Ratio of the number of turns on the primary winding to the number of turns on the secondary winding

$$a_t = \frac{N_1}{N_2}$$

- Using the turns ratio, the current relationships are

$$\mathbf{I_1} = \frac{1}{a_t}\mathbf{I_2} \quad \text{and} \quad \mathbf{I_2} = a_t\mathbf{I_1}$$

Ideal Transformers – Voltage Relationships

- Faraday's law relates the voltage at each winding to the flux through that winding

$$\mathbf{V_1} = -N_1 \frac{d\mathbf{\Phi}}{\mathrm{dt}} \quad \text{and} \quad \mathbf{V_2} = -N_2 \frac{d\mathbf{\Phi}}{dt}$$

- Dividing $\mathbf{V_1}$ by $\mathbf{V_2}$ gives

$$\frac{\mathbf{V_1}}{\mathbf{V_2}} = \frac{N_1}{N_2}$$

So,

$$\boxed{\mathbf{V_1} = a_t \mathbf{V_2} \quad \text{and} \quad \mathbf{V_2} = \frac{1}{a_t} \mathbf{V_1}}$$

Ideal Transformers

- To summarize current and voltage relationships:

$$\mathbf{I_2} = a_t \mathbf{I_1}$$

$$\mathbf{V_2} = \frac{1}{a_t} \mathbf{V_1}$$

- ***Step-up transformer***
  - $a_t < 1, \; N_1 < N_2$
  - Voltage increases from primary to secondary
  - Current decreases
- ***Step-down transformer***
  - $a_t > 1, \; N_1 > N_2$
  - Voltage decreases from primary to secondary
  - Current increases

Ideal Transformers - Power

- The instantaneous power entering the primary side of the transformer is

$$p_1(t) = v_1(t) i_1(t)$$

- And, the instantaneous power delivered out of the secondary side is

$$p_2(t) = v_2(t) i_2(t)$$

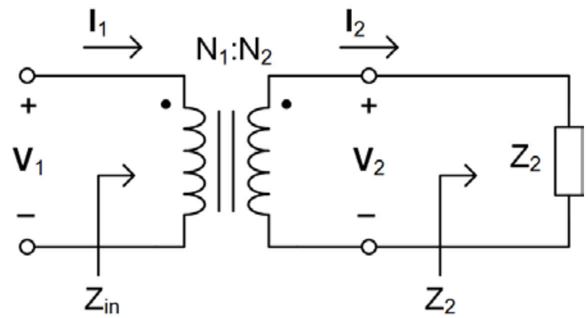
- Using the transformer voltage and current relationships,

$$p_1(t) = a_t v_2(t) \frac{1}{a_t} i_2(t) = v_2(t) i_2(t)$$

$$p_1(t) = p_2(t)$$

- Power is conserved in an ideal transformer
  - As expected, since we've assumed there are no losses

Ideal Transformers - Impedance



- By definition, the impedance seen looking into the primary side of a transformer is

$$Z_{in} = \frac{\mathbf{V_1}}{\mathbf{I_1}}$$

- For an impedance, Z_2, at the secondary side:

$$Z_2 = \frac{\mathbf{V_2}}{\mathbf{I_2}}$$

- Using the I/V relationships, we get

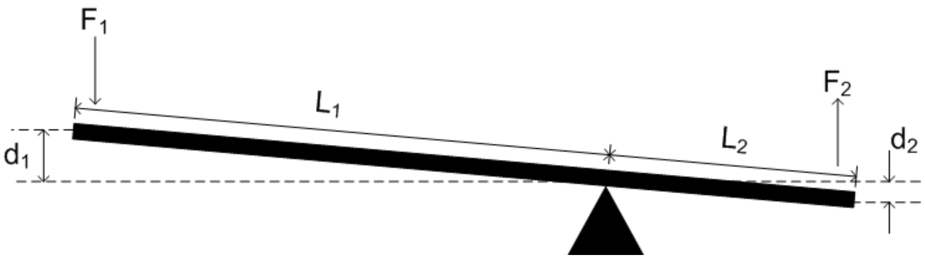
$$Z_{in} = \frac{a_t \mathbf{V_2}}{1/a_t \mathbf{I_2}} = a_t^2 \frac{\mathbf{V_2}}{\mathbf{I_2}} = a_t^2 Z_2$$

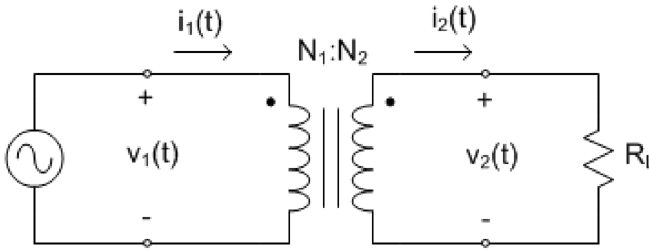
$$\boxed{Z_{in} = a_t^2 Z_2 = Z_2'}$$

- Load impedance is seen at the primary side multiplied by the turns ratio squared
  - The ***reflected load impedance***

Ideal Transformers & Levers

- Transformers are analogous to ***levers***





Lever

$$F_2 = \frac{L_1}{L_2} F_1$$

$$d_2 = \frac{L_2}{L_1} d_1$$

$$F_2 d_2 = F_1 d_1$$

Transformer

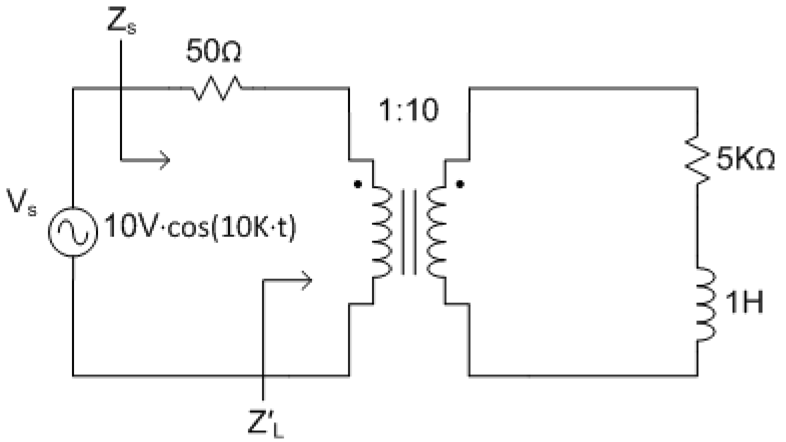
$$\mathbf{V_2} = \frac{N_2}{N_1} \mathbf{V_1}$$

$$\mathbf{I_2} = \frac{N_1}{N_2} \mathbf{I_1}$$

$$\mathbf{I_2 V_2} = \mathbf{I_1 V_1}$$

Ideal Transformer – Example

- Determine:
  - Primary-side voltage and current
  - Secondary-side voltage and current

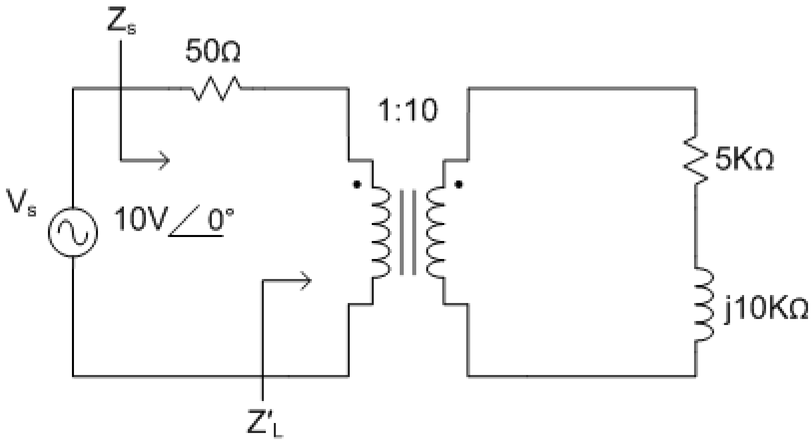


- First, convert to the phasor domain
- Primary-side input impedance:

$$Z_L' = \left(\frac{1}{10}\right)^2 (5 + j10)k\Omega$$

$$Z_L' = 50 + j100\ \Omega$$

$$Z_L' = 118\angle 63.4°\ \Omega$$

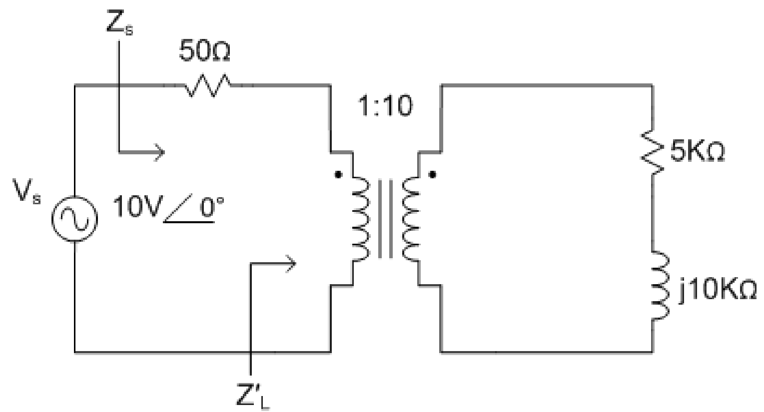


Ideal Transformer – Example

- The impedance seen by the source is

$$Z_s = 50\ \Omega + Z_L'$$

$$Z_s = 100 + j100\ \Omega$$



- Primary side current

$$\mathbf{I_1} = \frac{\mathbf{V_s}}{Z_s} = \frac{10\ \angle 0°}{100 + j100\ \Omega}$$

$$\boxed{\mathbf{I_1} = 70.7\angle - 45°\ mA}$$

- Apply voltage division to determine the voltage at the primary side

$$\mathbf{V_1} = \mathbf{V_s}\frac{Z_L'}{50\ \Omega + Z_L'} = 10\angle 0°\ V\ \frac{50 + j100\ \Omega}{100 + j100\ \Omega}$$

$$\boxed{\mathbf{V_1} = 7.9\angle 18.4°\ V}$$

Ideal Transformer – Example

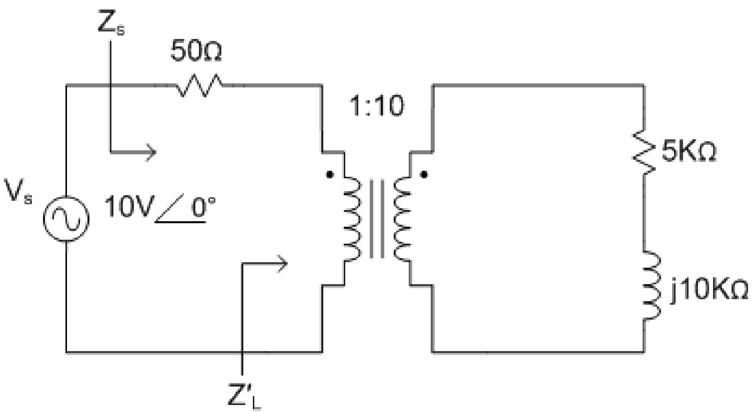
- Use the turns ratio to scale voltage and current to the secondary side

$$\mathbf{I_2} = a_t \mathbf{I_1} = \frac{1}{10} 70.7\angle -45°\ mA$$

$$\boxed{\mathbf{I_2} = 7.1\angle -45°\ mA}$$

$$\mathbf{V_2} = \frac{1}{a_t}\mathbf{V_1} = 10 \cdot 7.9\angle 18.4°\ V$$

$$\boxed{\mathbf{V_2} = 79\angle 18.4°\ V}$$



- Converting back to time-domain expressions:

$$v_1(t) = 7.9\ V cos(10k \cdot t + 18.4°) \qquad v_2(t) = 79\ V \cos(10k \cdot t + 18.4°)$$

$$i_1(t) = 70.7\ mA cos(10k \cdot t - 45°) \qquad i_2(t) = 7.1\ mA \cos(10k \cdot t - 45°)$$

Example Problems

Determine:

- Z_L' and Z_S'
- Primary- and Secondary-side equivalent circuits

$v_S(t) = 1V \cdot \cos(2\pi \cdot 10kHz \cdot t)$