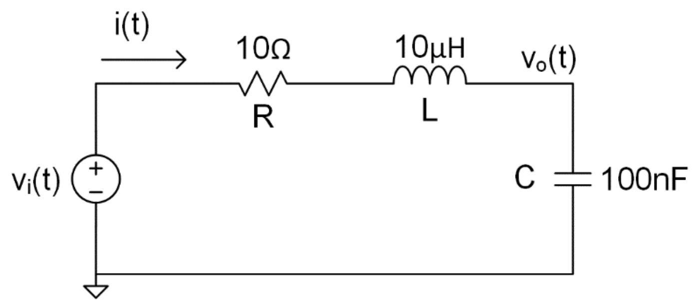
SECTION 1: INTRODUCTION

ENGR 203 – Electrical Fundamentals III

Introduction

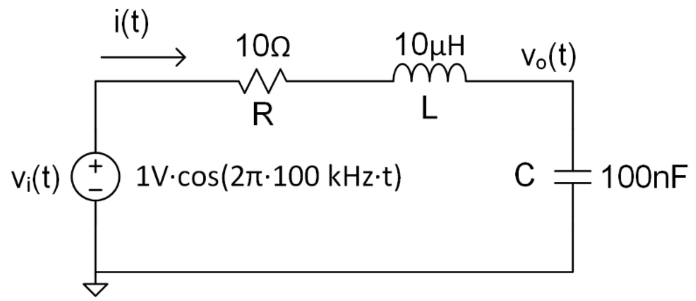
Introduction



- In ***ENGR 202***, we learned how to analyze circuits like this one in several ways:
  - *Sinusoidal steady-state (phasor) analysis*
  - *Frequency-response analysis*
  - *Step-response analysis*
- In ***ENGR 203***, we will learn some new mathematical tools that we can apply to these same types of analyses
- First, we will briefly review each of the above analysis techniques

Review of Circuit Analysis Tools

Sinusoidal Steady-State Analysis



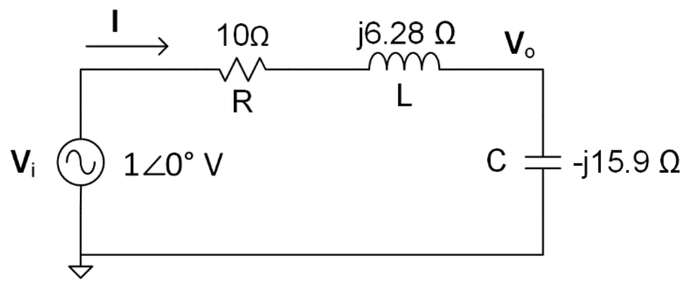
- For a sinusoidal input of a certain frequency, ***phasor analysis*** allows us to determine output quantities
  - Also sinusoidal, at the same frequency
- First, convert the circuit to the phasor domain:

$$\mathbf{V}_i = 1\angle 0° \, V$$

$$Z_L = j\omega L = j \cdot 2\pi \cdot 100\ kHz \cdot 10\ \mu H = j6.28\ \Omega$$

$$Z_C = \frac{1}{j\omega C} = -\mathrm{j}\frac{1}{2\pi \cdot 100\ kHz \cdot 100\ \mathrm{nF}} = -\mathrm{j}15.9\ \Omega$$

Sinusoidal Steady-State Analysis



- With the circuit in the phasor domain, we can apply any of the usual circuit analysis tools
  - Here, we can treat the circuit as a voltage divider

$$\mathbf{V}_o = \mathbf{V}_i \frac{-j15.9\Omega}{10 + j(6.28 - 15.9)\ \Omega} = 1\angle 0°\ V \frac{-j15.9\ \Omega}{10 - \mathrm{j}9.62\ \Omega}$$

$$\mathbf{V}_o = 1.15\angle - 46.1°\ V$$

- Finally, transform back to the time domain

$$v_o(t) = 1.15\ V \cdot \cos(2\pi \cdot 100\ kHz \cdot t - 46.1°)$$

Frequency Response Analysis

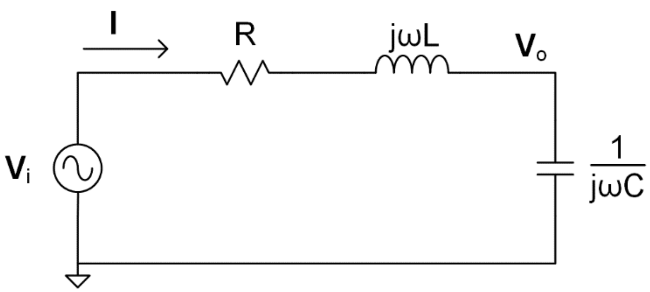
- Phasor analysis gets us the steady-state response of the circuit to a sinusoidal input at ***one specific frequency***
  - The output will be at the same frequency but, in general, different amplitude and phase

- We also saw how to determine the circuit's ***frequency response*** function and generate its ***Bode plot***
  - Response to sinusoidal input of varying frequency
  - ***Gain*** – ratio of output amplitude to input amplitude
  - ***Phase*** – phase shift from input to output

$$H(j\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} \quad \rightarrow \quad |H(j\omega)|\,, \qquad \angle H(j\omega)$$

Frequency Response Analysis

- Express impedances as functions of frequency
- Treat the circuit as a voltage divider



$$H(j\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$$

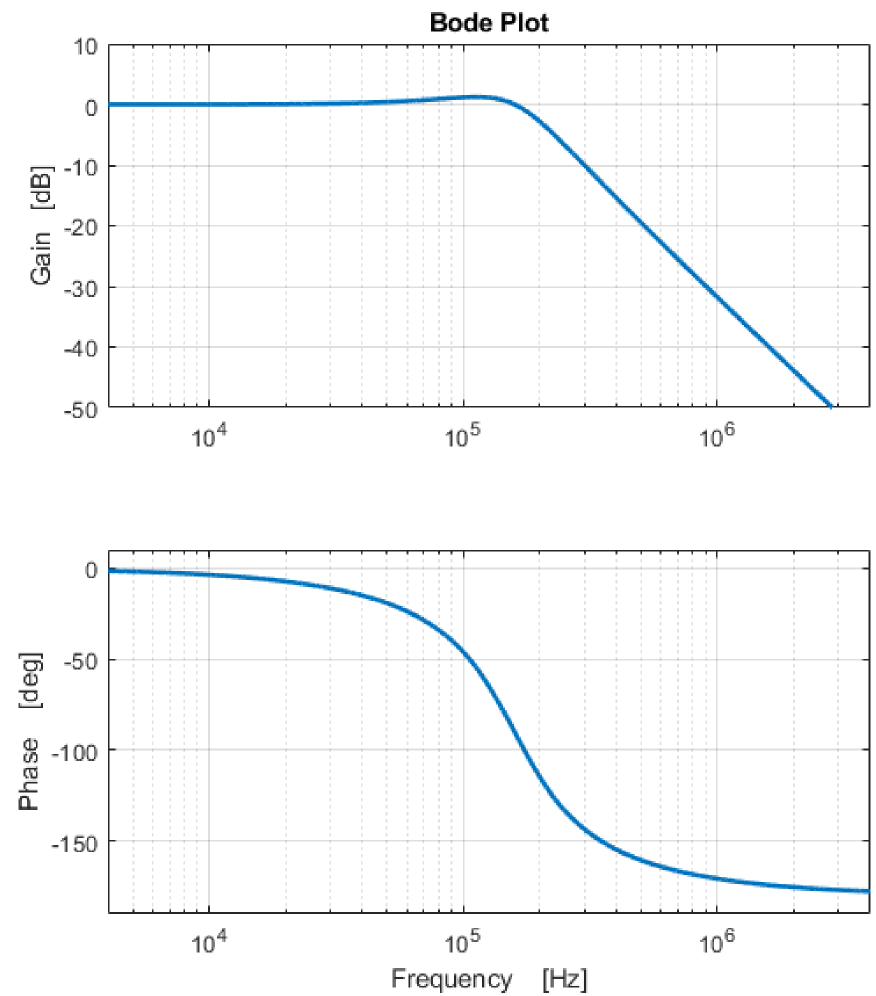
- The resulting frequency response function is

$$H(j\omega) = \frac{\frac{1}{LC}}{(j\omega)^2 + \frac{R}{L} j\omega + \frac{1}{LC}}$$

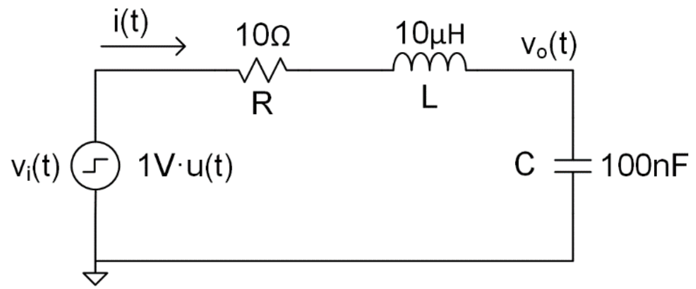
Frequency Response Analysis

$$H(j\omega) = \frac{\frac{1}{LC}}{(j\omega)^2 + \frac{R}{L}j\omega + \frac{1}{LC}}$$

- This type of analysis is particularly useful for ***filters***
  - Frequency-varying gain
  - Pass some frequencies, attenuate others



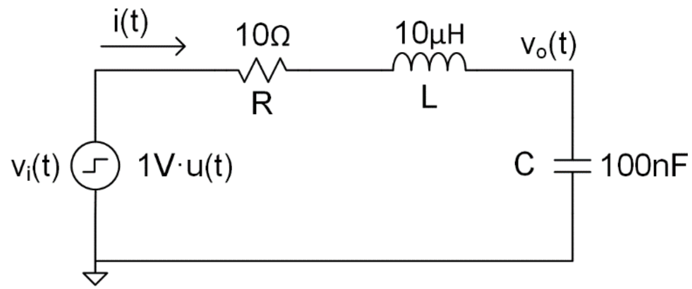
Step Response Analysis



- Finally, in ENGR 202, we learned how to determine the circuit's step response
- The first step was to derive the governing differential equation
- Applying KVL around the loop yields

$$v_i(t) - i(t)R - L\frac{di}{dt} - v_o(t) = 0$$

Step Response Analysis



- After eliminating $i(t)$ and rearranging, we have the governing equation for the circuit

$$\frac{d^2 v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = \frac{1}{LC} v_i(t)$$

- A second-order, linear, non-homogeneous, ordinary differential equation
- Non-homogeneous, so we solve it in two parts
  1) Find the ***complementary solution*** to the homogeneous equation
  2) Find the ***particular solution*** for the step input
- General solution is the sum of the two individual solutions:

$$v_o(t) = v_{oc}(t) + v_{op}(t)$$

Step Response Analysis

- For the ***complementary solution***, we assume a solution of the form

$$v_{oc}(t) = e^{st}$$

where s is an unknown complex value

- Substituting this into the homogeneous equation allows us to generate the characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

which we can also write as

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

where

ζ is the damping ratio

ω_0 is the natural frequency

Step Response Analysis

- The damping ratio, and the roots of the characteristic equation, determine the form of the complementary solution
  - Roots are called the circuit ***poles***

- $\zeta > 1$ – ***over-damped***
  - Two real, distinct poles, s_1 and s_2

$$v_{oc}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

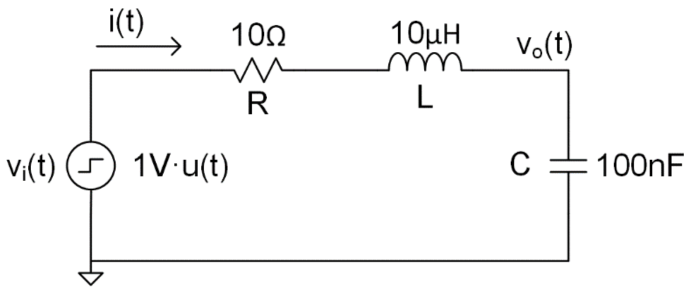
- $\zeta = 1$ – ***critically-damped***
  - Two real, identical poles, $s_1 = s_2$

$$v_{oc}(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$

- $\zeta < 1$ – ***Under-damped***
  - Complex-conjugate pair of poles, $s_{1,2} = -\alpha \pm j\omega_d$

$$v_{oc}(t) = K_1 e^{-\alpha t} \cos(\omega_d t) + K_2 e^{-\alpha t} \sin(\omega_d t)$$

Step Response Analysis



- The second piece of the general solution is the ***particular solution***
  - For a step input, this is simply the steady-state output

$$v_{op}(t) = v_o(t \to \infty) = 1\,V$$

- For our circuit, the characteristic equation is

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 1E6s + 1E12 = s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

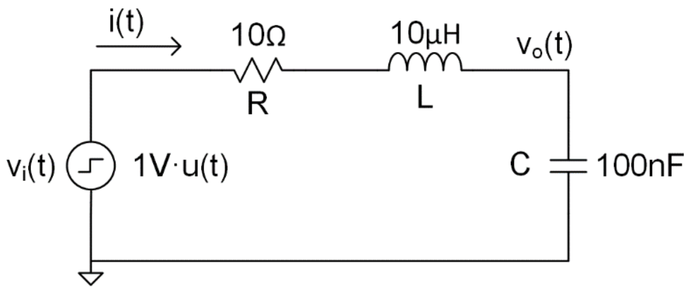
  - From which we find

$$\omega_0 = 1E6\,rad/sec$$

$$\alpha = 500E3\,rad/sec$$

$$\zeta = 0.5$$

Step Response Analysis



- $\zeta = 0.5$, so the circuit is under-damped
  - Complementary solution is a sum of damped sinusoids
  - General solution has the following form

$$v_o(t) = K_1 e^{-\alpha t}\cos(\omega_d t) + K_2 e^{-\alpha t}\sin(\omega_d t) + v_o(t \to \infty)$$

- The damped natural frequency is

$$\omega_d = \omega_0\sqrt{1-\zeta^2} = 866E3\, rad/sec$$

  - General solution is

$$v_o(t) = K_1 e^{-500E3t}\cos(866E3t) + K_2 e^{-500E3t}\sin(866E3t) + 1\,V$$

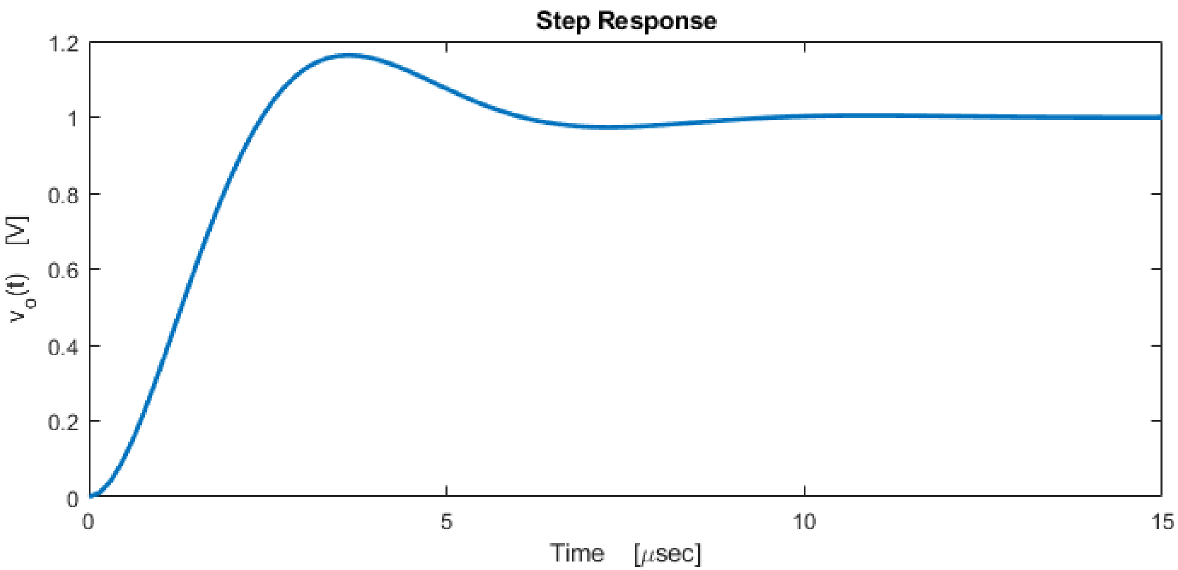
Step Response Analysis

- The last step in determining the step response is to determine the unknown coefficients, K_1 and K_2
  - Apply the initial conditions:

$$v_o(0) = 0 \text{ and } \dot{v}_o(0) = 0$$

- The circuit's response to a 1 V step is

$$v_o(t) = -1V\, e^{-500E3t} \cos(866E3t) - 0.58V\, e^{-500E3t} \sin(866E3t) + 1V$$



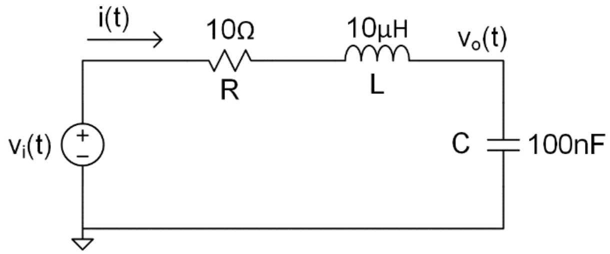
ENGR 203 Course Preview

ENGR 203 Preview

- In this course, we will learn a new mathematical tool that will simplify and unify our circuit analyses
  - The ***Laplace transform***
- We will use the Laplace transform to:
  1) Solve a circuit's governing ODE to determine its transient response to any type of input
  2) Determine a circuit's time-domain response to any input without needing to derive the governing ODE
  3) Determine a circuit's frequency response and Bode plot
- We will now step through a preview of the application of Laplace transforms to various circuit analyses

Solving the Governing ODE with Laplace Transforms

Solving the Differential Equation



- First, we will see how we will use Laplace transforms to solve a circuit's governing ODE to determine its response to any type of input
- The governing ODE for our example circuit is

$$\frac{d^2v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

- Our first step will be to apply the Laplace transform to the governing equation

$$s^2V_o(s) - sv_o(0) - \dot{v}_o(0) + \frac{R}{L}[sV_o(s) - v_o(0)] + \frac{1}{LC}V_o(s) = \frac{1}{LC}V_i(s)$$

- Assuming the same unit step input, and zero initial conditions, this simplifies to

$$s^2V_o(s) + \frac{R}{L}sV_o(s) + \frac{1}{LC}V_o(s) = \frac{1}{LC}\frac{1}{s}$$

Solving the Differential Equation

$$s^2 V_o(s) + \frac{R}{L} s V_o(s) + \frac{1}{LC} V_o(s) = \frac{1}{LC}\frac{1}{s}$$

- Our differential equation has transformed to an algebraic equation
  - Solve algebraically for the Laplace transform of the output, $V_o(s)$

$$V_o(s)\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right] = \frac{1}{LC}\frac{1}{s}$$

$$V_o(s) = \frac{\frac{1}{LC}}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$V_o(s) = \frac{1E12}{s(s^2 + 1E6s + 1E12)}$$

Solving the Differential Equation

$$V_o(s) = \frac{1E12}{s(s^2 + 1E6s + 1E12)}$$

- This is the Laplace transform of the thing we are looking for, the output voltage, $v_o(t)$

$$V_o(s) = \mathcal{L}\{v_o(t)\}$$

- To get $v_o(t)$, we ***inverse Laplace transform*** $V_o(s)$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$$

- To inverse transform, we will make use of ***partial fraction expansion***

Solving the Differential Equation

- The partial fraction expansion looks like

$$V_o(s) = \frac{1E12}{s(s^2 + 1E6s + 1E12)} = \frac{r_1}{s} + \frac{r_2(s + 500E3) + r_3 866E3}{(s + 500E3) + (866E3)^2}$$

- Solve for the unknown ***residues***, r_1, r_2, and r_3, giving the output voltage in the Laplace domain

$$V_o(s) = \frac{1}{s} - \frac{(s + 500E3)}{(s + 500E3) + (866E3)^2} - 0.58\frac{866E3}{(s + 500E3) + (866E3)^2}$$

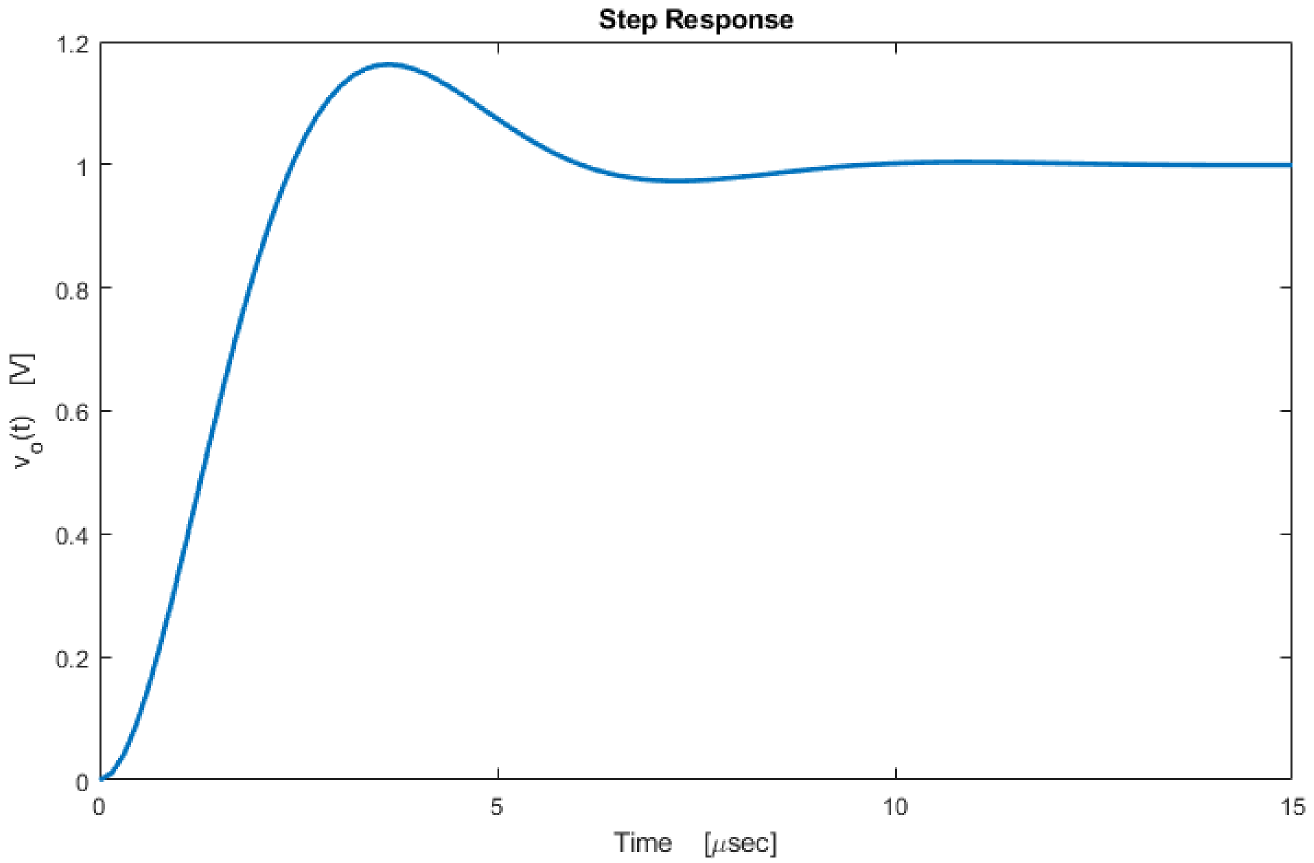
- This is now a sum of Laplace domain functions whose inverse transforms are known
  - Inverse transform by inspection

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$$

$$v_o(t) = 1V - 1Ve^{-500E3t}\cos(866E3t) - 0.58Ve^{-500E3t}\sin(866E3t)$$

Solving the Differential Equation

$$v_o(t) = 1V - 1Ve^{-500E3t}\cos(866E3t) - 0.58Ve^{-500E3t}\sin(866E3t)$$

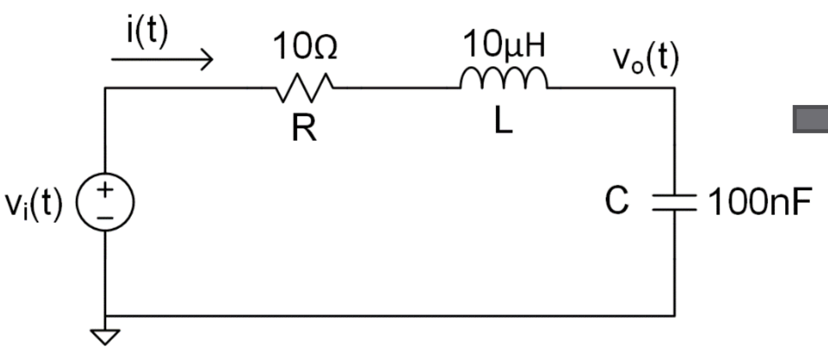


Laplace Domain Circuit Analysis

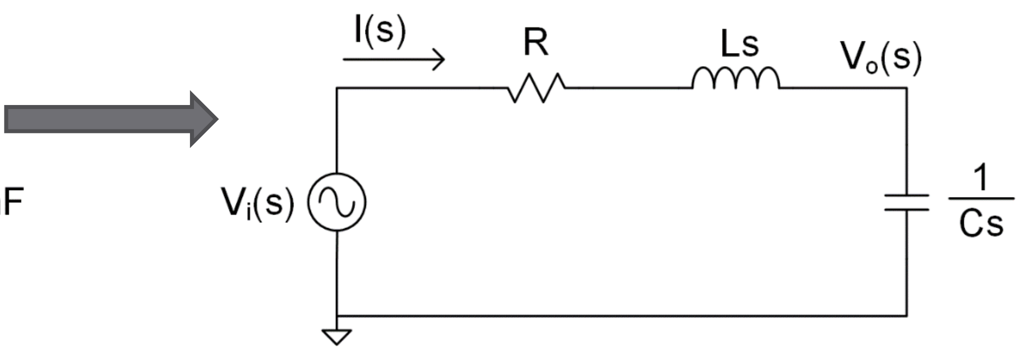
Laplace Domain Circuit Analysis

- Previous analysis required derivation of the governing differential equation
- Can often skip that step
  - Transform the *circuit* to the Laplace domain
  - Apply usual circuit analysis tools
- Similar to transforming to the phasor domain

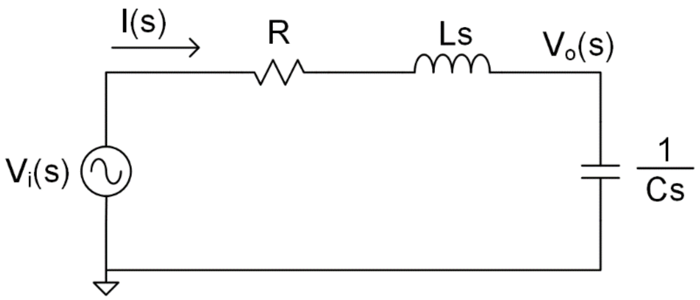
Time Domain:



Laplace Domain:



Laplace Domain Circuit Analysis



- With the circuit in the Laplace domain, we can apply any of the usual circuit analysis tools
  - Here, we are looking for $V_o(s)$, so treat it as a voltage divider

$$V_o(s) = V_i(s)\frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = V_i(s)\frac{1}{LCs^2 + RCs + 1}$$

$$V_o(s) = V_i(s)\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Laplace Domain Circuit Analysis

- For the given component values and a step input the output in the Laplace domain is

$$V_o(s) = \frac{1}{s}\frac{1E12}{s^2 + 1E6s + 1E12}$$

- The same result arrived at by transforming the governing ODE
  - Solve the same way: inverse Laplace transform via partial fraction expansion

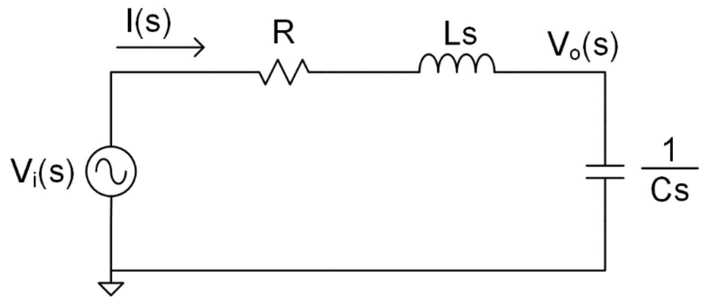
$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$$

- Note that this did not involve derivation of the differential equation in the time domain

Laplace Domain Frequency-Response Analysis

Frequency Response via Laplace Transform

- Finally, we will see how we can apply Laplace transforms to determine a circuit's ***frequency response*** and ***Bode plot***
- First, convert the circuit to the Laplace domain



- As before, apply the usual circuit analysis techniques to determine the output

$$V_o(s) = V_i(s) \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Frequency Response via Laplace Transform

$$V_o(s) = V_i(s)\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

- From here, we get the circuit's ***transfer function***

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

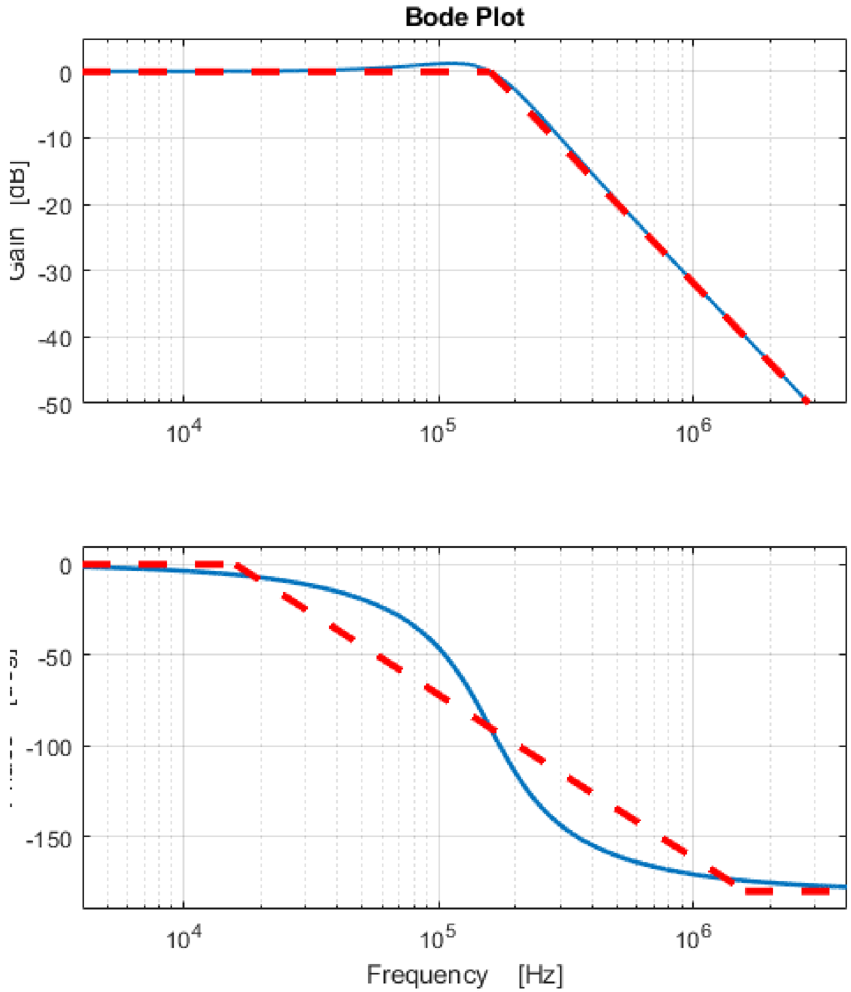
- From the transfer function, we get the ***frequency response function*** by substituting in $j\omega$ for s

$$H(s) \xrightarrow{s \rightarrow j\omega} H(j\omega) = \frac{\frac{1}{LC}}{(j\omega)^2 + \frac{R}{L}(j\omega) + \frac{1}{LC}}$$

Frequency Response Analysis

$$H(j\omega) = \frac{\frac{1}{LC}}{(j\omega)^2 + \frac{R}{L}(j\omega) + \frac{1}{LC}}$$

- We will learn to generate ***Bode plot*** from the frequency response function or transfer function
  - Hand-sketching a straight-line approximation
  - Plotting in numerical tools like MATLAB or Python



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Fourier Analysis

Fourier Analysis

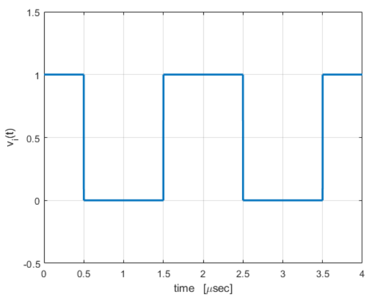
- We will learn other mathematical tools to enable further circuit analysis
  - ***Fourier series***
  - ***Fourier transform***

- In ENGR 202 we introduced the ***frequency spectrum***
  - Frequency content of a signal
  - How will a signal be affected by a circuit with a particular frequency response?

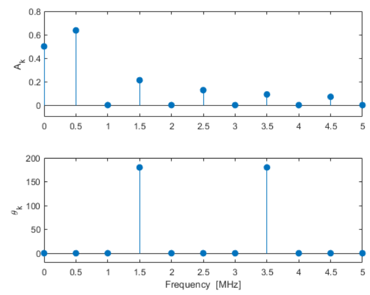
- Fourier series/transform allow us to mathematically determine frequency spectra
  - Spectrum of input, along with circuit's frequency response, allow us to determine spectrum of output

Fourier Analysis

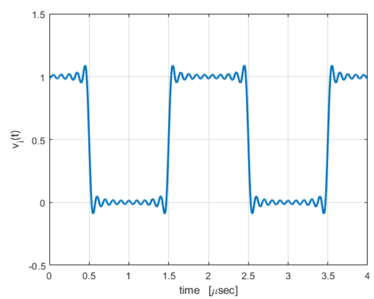
Input signal



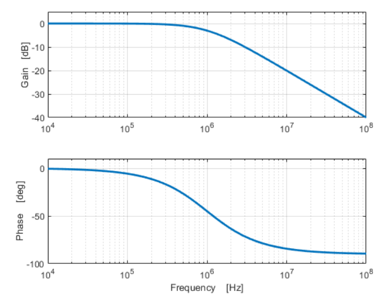
Input signal
Fourier series



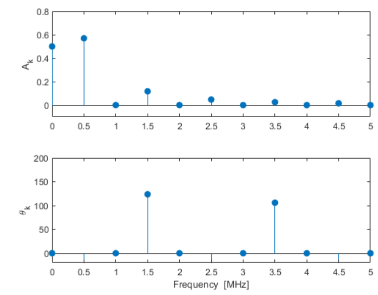
Input signal
approximation



Circuit frequency
response



Output signal
Fourier series



Output signal
approximation