SECTION 2: LAPLACE TRANSFORMS

ENGR 203 – Electrical Fundamentals III

Introduction – Transforms

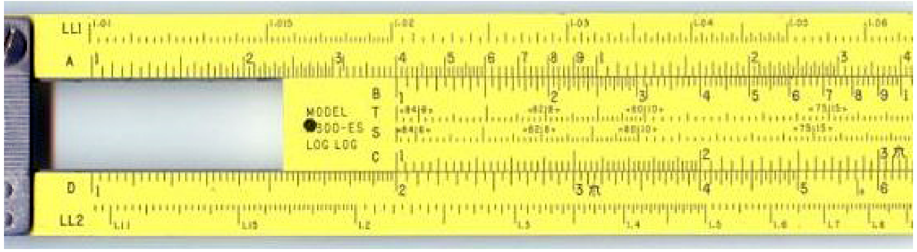
This section of notes contains an introduction to Laplace transforms. This may mostly be a review of material covered in your differential equations course.

Transforms

- **What is a transform?**
  - A ***mapping*** of a mathematical function from one ***domain*** to another
  - A change in ***perspective*** not a change of the function
- **Why use transforms?**
  - Some mathematical problems are difficult to solve in their natural domain
    - Transform to and solve in a new domain, where the problem is simplified
    - Transform back to the original domain
  - Trade off the extra effort of transforming/inverse-transforming for simplification of the solution procedure

Transform Example – Slide Rules

- Slide rules make use of a logarithmic transform



- Multiplication/division of large numbers is difficult
  - Transform the numbers to the logarithmic domain
  - Add/subtract (easy) in the log domain to multiply/divide (difficult) in the linear domain
  - Apply the inverse transform to get back to the original domain
- Extra effort is required, but the problem is simplified

Laplace Transforms

Laplace Transforms

- An ***integral transform*** mapping functions from the ***time domain*** to the ***Laplace domain*** or ***s-domain***

$$g(t) \overset{\mathcal{L}}{\leftrightarrow} G(s)$$

  - Time-domain functions are functions of time, t

$$g(t)$$

  - Laplace-domain functions are functions of $\boldsymbol{s}$

$$G(s)$$

  - s is a complex variable

$$s = \sigma + j\omega$$

Laplace Transforms – Motivation

- We'll use Laplace transforms to ***solve differential equations***
  - ***Differential equations*** in the ***time domain***
    - difficult to solve
  - Apply the Laplace transform
    - Transform to ***the s-domain***
  - ***Differential equations*** become ***algebraic equations***
    - easy to solve
  - Transform the s-domain solution back to the time domain
- Transforming back and forth requires extra effort, but the solution is greatly simplified

Laplace Transform

- **Laplace Transform**:

$$\mathcal{L}\{g(t)\} = G(s) = \int_0^\infty g(t)e^{-st}dt \qquad (1)$$

- ***Unilateral*** or ***one-sided*** transform
  - Lower limit of integration is $t = 0$
  - Assumed that the time domain function is zero for all negative time, i.e.

$$g(t) = 0, \qquad t < 0$$

Laplace Transform Properties

In the following section of notes, we'll derive a few important properties of the Laplace transform.

Laplace Transform – Linearity

- Say we have two time-domain functions:

$$g_1(t) \text{ and } g_2(t)$$

- Applying the transform definition, (1)

$$\mathcal{L}\{\alpha g_1(t) + \beta g_2(t)\} = \int_0^\infty \big(\alpha g_1(t) + \beta g_2(t)\big)e^{-st}dt$$

$$= \int_0^\infty \alpha g_1(t)e^{-st}dt + \int_0^\infty \beta g_2(t)e^{-st}dt$$

$$= \alpha \int_0^\infty g_1(t)e^{-st}dt + \beta \int_0^\infty g_2(t)e^{-st}dt$$

$$= \alpha \cdot \mathcal{L}\{g_1(t)\} + \beta \cdot \mathcal{L}\{g_2(t)\}$$

$$\boxed{\mathcal{L}\{\alpha g_1(t) + \beta g_2(t)\} = \alpha G_1(s) + \beta G_2(s)} \qquad (2)$$

- The Laplace transform is a ***linear operation***

Laplace Transform of a Derivative

- Of particular interest, given that we want to use Laplace transform to solve differential equations

$$\mathcal{L}\{\dot{g}(t)\} = \int_0^\infty \dot{g}(t)e^{-st}dt$$

- Use ***integration by parts*** to evaluate

$$\int u dv = uv - \int v du$$

- Let

$$u = e^{-st} \qquad \text{and} \qquad dv = \dot{g}(t)dt$$

then

$$du = -se^{-st}dt \qquad \text{and} \qquad v = g(t)$$

Laplace Transform of a Derivative

$$\mathcal{L}\{\dot{g}(t)\} = e^{-st}g(t)\Big|_0^\infty - \int_0^\infty g(t)(-se^{-st})dt$$

$$= 0 - g(0) + s\int_0^\infty g(t)e^{-st}dt = -g(0) + s\mathcal{L}\{g(t)\}$$

- The Laplace transform of the derivative of a function is the Laplace transform of that function multiplied by s minus the initial value of that function

$$\boxed{\mathcal{L}\{\dot{g}(t)\} = sG(s) - g(0)} \qquad (3)$$

Higher-Order Derivatives

- The Laplace transform of a ***second derivative*** is

$$\mathcal{L}\{\ddot{g}(t)\} = s^2 G(s) - s g(0) - \dot{g}(0) \tag{4}$$

- In general, the Laplace transform of the $\boldsymbol{n^{th}}$ ***derivative*** of a function is given by

$$\mathcal{L}\left\{g^{(n)}\right\} = s^n G(s) - s^{n-1} g(0) - s^{n-2} \dot{g}(0) - \cdots - g^{(n-1)}(0) \tag{5}$$

Laplace Transform of an Integral

- The Laplace Transform of a ***definite integral*** of a function is given by

$$\mathcal{L}\left\{\int_0^t g(\tau)d\tau\right\} = \frac{1}{s}G(s) \qquad (6)$$

- ***Differentiation*** in the time domain corresponds to ***multiplication by s*** in the Laplace domain
- ***Integration*** in the time domain corresponds to ***division by s*** in the Laplace domain

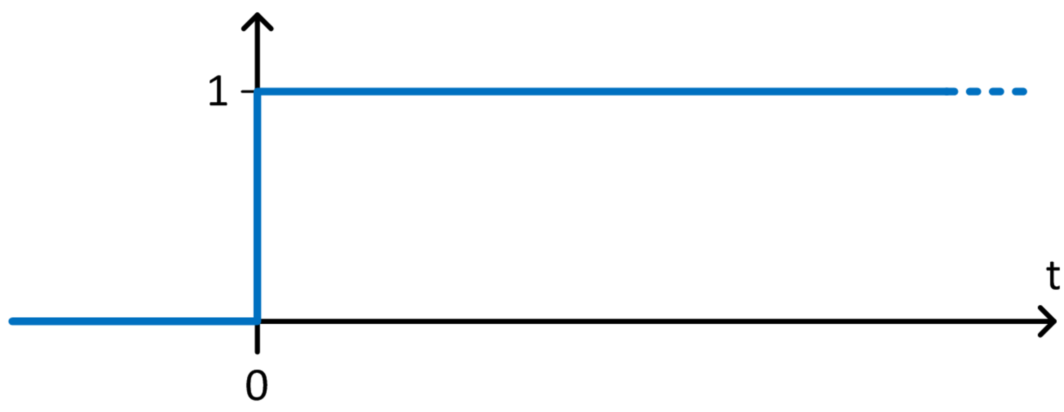
Laplace Transforms of Common Functions

Next, we'll derive the Laplace transform of some common mathematical functions

Unit Step Function

- A useful and common way of characterizing a linear system is with its ***step response***
  - The system's response (output) to a unit step input
- The ***unit step function*** or ***Heaviside step function***:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Unit Step Function – Laplace Transform

- Using the definition of the Laplace transform

$$\mathcal{L}\{u(t)\} = \int_0^{\infty} u(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^{\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

- The Laplace transform of the unit step

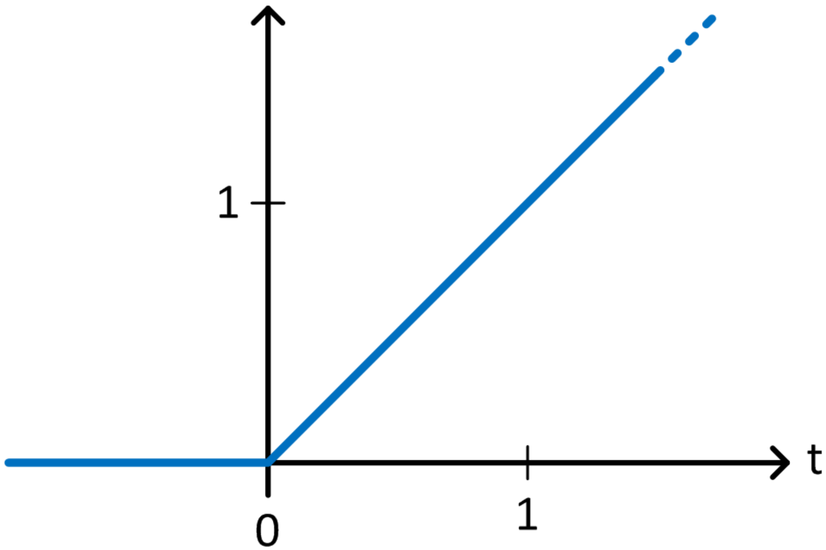
$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}} \qquad (7)$$

- Note that the unilateral Laplace transform assumes that the signal being transformed is zero for $t < 0$
  - Equivalent to multiplying any signal by a unit step

Unit Ramp Function

- The unit ramp function is a useful input signal for evaluating how well a system tracks a constantly-increasing input
- The ***unit ramp function***:

$$g(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$



Unit Ramp Function – Laplace Transform

- Could easily evaluate the transform integral
  - Requires integration by parts
- Alternatively, recognize the relationship between the unit ramp and the unit step
  - ***Unit ramp is the integral of the unit step***
- Apply the integration property, (6)

$$\mathcal{L}\{t\} = \mathcal{L}\left\{\int_0^t u(\tau)d\tau\right\} = \frac{1}{s}\cdot\frac{1}{s}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}} \qquad (8)$$

Exponential – Laplace Transform

$$g(t) = e^{-at}$$

- Exponentials are common components of the responses of dynamic systems

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt$$

$$= -\frac{e^{-(s+a)t}}{s+a}\bigg|_0^\infty = 0 - \left(-\frac{1}{s+a}\right)$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}} \qquad (9)$$

Sinusoidal functions

- Another class of commonly occurring signals, when dealing with dynamic systems, is ***sinusoidal signals*** – both $\sin(\omega t)$ and $\cos(\omega t)$

$$g(t) = \sin(\omega t)$$

- Recall ***Euler's formula***

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- From which it follows that

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Sinusoidal functions

$$\mathcal{L}\{\sin(\omega t)\} = \frac{1}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt$$

$$= \frac{1}{2j}\int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right)dt$$

$$= \frac{1}{2j}\int_0^\infty e^{-(s-j\omega)t}dt - \frac{1}{2j}\int_0^\infty e^{-(s+j\omega)t}dt$$

$$= \frac{1}{2j}\frac{\left(e^{-(s-j\omega)t}\right)}{-(s-j\omega)}\Bigg|_0^\infty - \frac{1}{2j}\frac{\left(e^{-(s+j\omega)t}\right)}{-(s+j\omega)}\Bigg|_0^\infty$$

$$= \frac{1}{2j}\left[0 + \frac{1}{s-j\omega}\right] - \frac{1}{2j}\left[0 + \frac{1}{s+j\omega}\right] = \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2}$$

$$\boxed{\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}} \qquad (10)$$

Sinusoidal functions

- It can similarly be shown that

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \tag{11}$$

- Note that for neither $\sin(\omega t)$ nor $\cos(\omega t)$ is the function equal to zero for $t < 0$ as the Laplace transform assumes
- Really, what we've derived is

$$\mathcal{L}\{u(t) \cdot \sin(\omega t)\} \quad \text{and} \quad \mathcal{L}\{u(t) \cdot \cos(\omega t)\}$$

More Properties and Theorems

Multiplication by an Exponential, e^{-at}

- We've seen that $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$
- What if another function is multiplied by the decaying exponential term?

$$\mathcal{L}\{g(t)e^{-at}\} = \int_0^\infty g(t)e^{-at}e^{-st}dt = \int_0^\infty g(t)e^{-(s+a)t}dt$$

- This is just the Laplace transform of $g(t)$ with s replaced by $(s+a)$

$$\boxed{\mathcal{L}\{g(t)e^{-at}\} = G(s+a)} \qquad (12)$$

Decaying Sinusoids

- The Laplace transform of a sinusoid is

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

- And, multiplication by an decaying exponential, e^{-at}, results in a substitution of $(s + a)$ for s, so

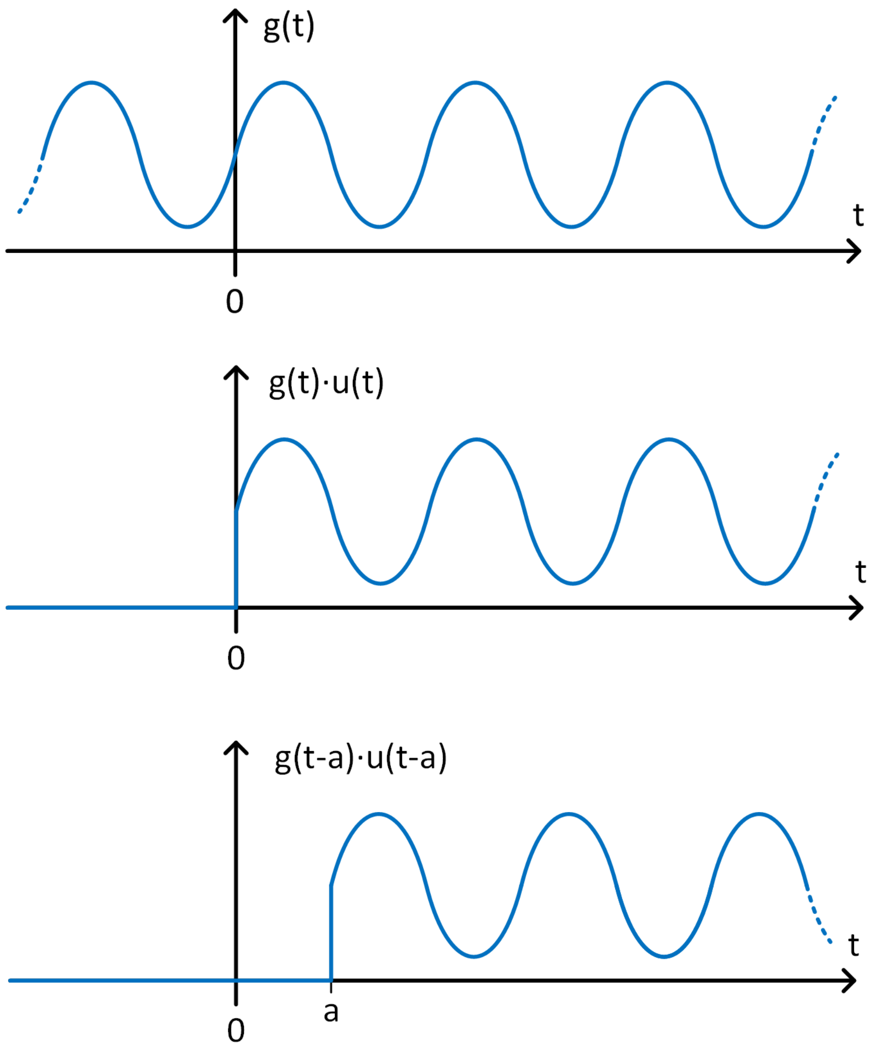
$$\mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

and

$$\mathcal{L}\{e^{-at}\cos(\omega t)\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

Time Shifting

- Consider a time-domain function, $g(t)$
- To Laplace transform $g(t)$ we've assumed $g(t) = 0$ for $t < 0$, or equivalently multiplied by $u(t)$
- To shift $g(t)$ by an amount, a, in time, we must also multiply by a shifted step function, $u(t-a)$



Time Shifting – Laplace Transform

- The transform of the shifted function is given by

$$\mathcal{L}\{g(t-a)\cdot u(t-a)\} = \int_a^\infty g(t-a)e^{-st}dt$$

- Performing a change of variables, let

$$\tau = (t-a) \text{ and } d\tau = dt$$

- The transform becomes

$$\mathcal{L}\{g(\tau)\cdot u(\tau)\} = \int_0^\infty g(\tau)e^{-s(\tau+a)}d\tau = \int_0^\infty g(\tau)e^{-as}e^{-s\tau}d\tau = e^{-as}\int_0^\infty g(\tau)e^{-s\tau}d\tau$$

- A shift by a in the time domain corresponds to multiplication by e^{-as} in the Laplace domain

$$\boxed{\mathcal{L}\{g(t-a)\cdot u(t-a)\} = e^{-as}G(s)} \qquad (13)$$

Multiplication by time, t

- The Laplace transform of a function multiplied by time:

$$\boxed{\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds}F(s)} \qquad (14)$$

- Consider a unit ramp function:

$$\mathcal{L}\{t\} = \mathcal{L}\{t \cdot u(t)\} = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

- Or a parabola:

$$\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

- In general

$$\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}}$$

Initial and Final Value Theorems

- **Initial Value Theorem**
  - Can determine the initial value of a time-domain signal or function from its Laplace transform

$$g(0) = \lim_{s \to \infty} sG(s) \qquad (15)$$

- **Final Value Theorem**
  - Can determine the steady-state value of a time-domain signal or function from its Laplace transform

$$g(\infty) = \lim_{s \to 0} sG(s) \qquad (16)$$

Convolution

- ***Convolution*** of two functions or signals is given by

$$g(t) * x(t) = \int_0^t g(\tau)x(t-\tau)d\tau$$

- Result is a function of time
  - $x(\tau)$ is ***flipped*** in time and ***shifted*** by t
  - Multiply the flipped/shifted signal and the other signal
  - Integrate the result from $\tau = 0 \ldots t$
- May seem like an odd, arbitrary function now, but we'll later see why it is very important
- ***Convolution in the time domain corresponds to multiplication in the Laplace domain***

$$\boxed{\mathcal{L}\{g(t) * x(t)\} = G(s)X(s)} \quad (17)$$

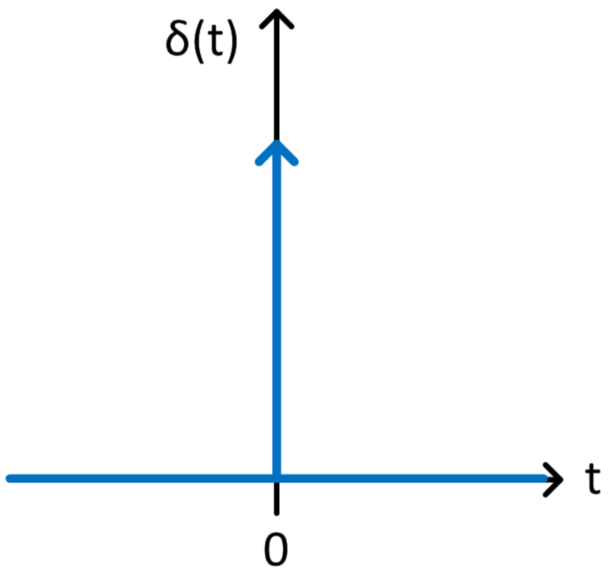
Impulse Function

- Another common way to describe a dynamic system is with its ***impulse response***
  - System output in response to an impulse function input
- ***Impulse function*** defined by

$$\delta(t) = 0, \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

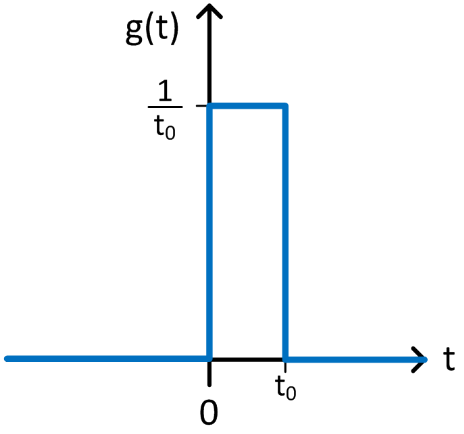
  - An infinitely tall, infinitely narrow pulse



Impulse Function – Laplace Transform

- To derive $\mathcal{L}\{\delta(t)\}$, consider the following function

$$g(t) = \begin{cases} \frac{1}{t_0}, & 0 \le t \le t_0 \\ 0, & t < 0 \text{ or } t > t_0 \end{cases}$$



- Can think of $g(t)$ as the sum of two step functions:

$$g(t) = \frac{1}{t_0}u(t) - \frac{1}{t_0}u(t - t_0)$$

- The transform of the first term is

$$\mathcal{L}\left\{\frac{1}{t_0}u(t)\right\} = \frac{1}{t_0 s}$$

- Using the time-shifting property, the second term transforms to

$$\mathcal{L}\left\{-\frac{1}{t_0}u(t - t_0)\right\} = -\frac{e^{-t_0 s}}{t_0 s}$$

Impulse Function – Laplace Transform

- In the limit, as $t_0 \to 0$, $g(t) \to \delta(t)$, so

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \frac{1 - e^{-t_0 s}}{t_0 s}$$

- Apply l'Hôpital's rule

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}\left(1 - e^{-t_0 s}\right)}{\frac{d}{dt_0}(t_0 s)} = \lim_{t_0 \to 0} \frac{s e^{-t_0 s}}{s} = \frac{s}{s}$$

- The Laplace transform of an impulse function is one

$$\boxed{\mathcal{L}\{\delta(t)\} = 1} \tag{18}$$

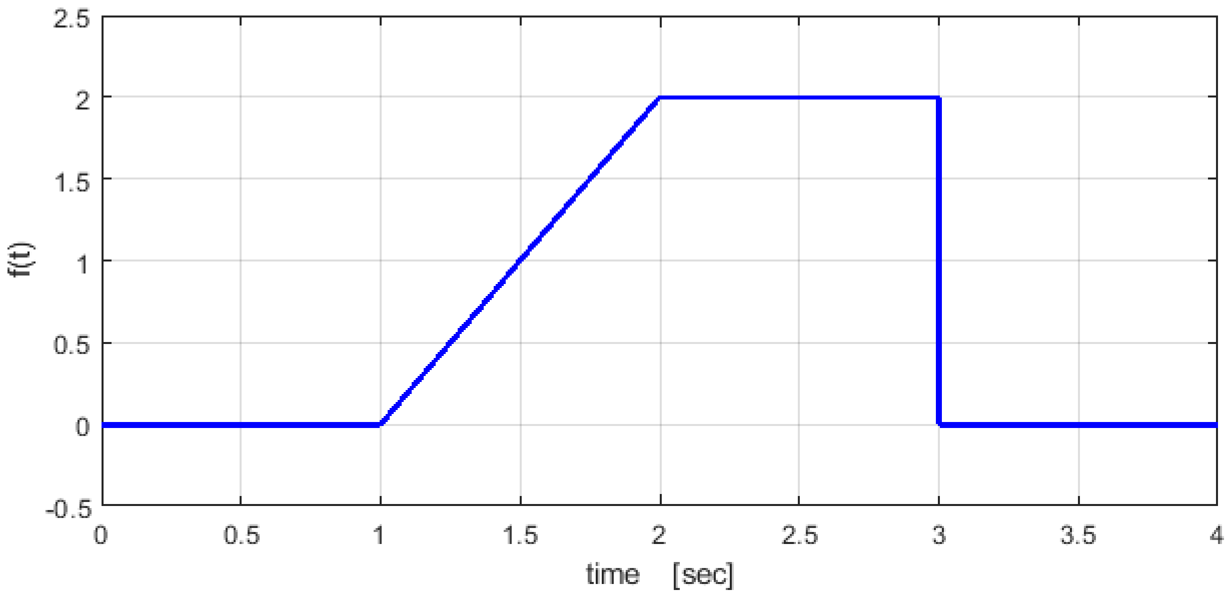
Common Laplace Transforms

$g(t)$	$G(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^m	$\frac{m!}{s^{m+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

$g(t)$	$G(s)$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\dot{g}(t)$	$sG(s)-g(0)$
$\ddot{g}(t)$	$s^2G(s)-sg(0)-\dot{g}(0)$
$\int_0^t g(\tau)d\tau$	$\frac{1}{s}G(s)$
$e^{-at}g(t)$	$G(s+a)$
$g(t-a)\cdot u(t-a)$	$e^{-as}G(s)$
$t\cdot g(t)$	$-\frac{d}{ds}G(s)$

Example – Piecewise Function Laplace Transform

- Determine the Laplace transform of a piecewise function:



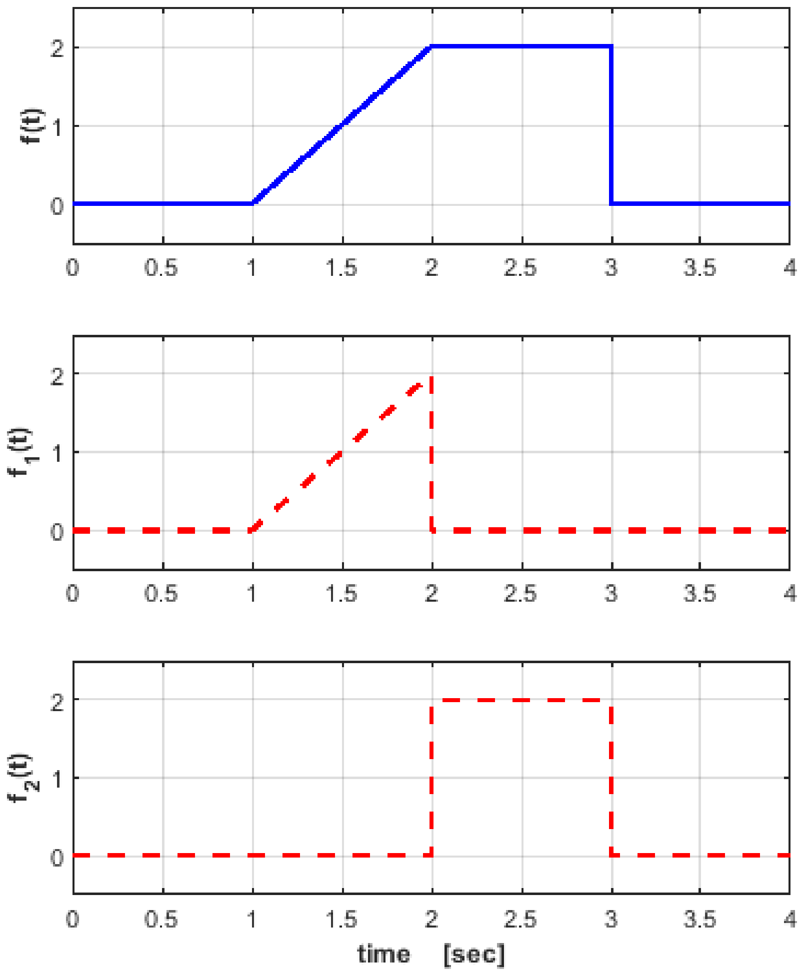
- A summation of functions with known transforms:
  - Ramp
  - Pulse – sum of positive and negative steps
- Transform is the sum of the individual, known transforms

Example – Piecewise Function Laplace Transform

- Treat the piecewise function as a sum of individual functions

$$f(t) = f_1(t) + f_2(t)$$

- $f_1(t)$
  - Time-shifted, gated ramp
- $f_2(t)$
  - Time-shifted pulse
    - Sum of staggered positive and negative steps



Example – Piecewise Function Laplace Transform

- $f_1(t)$: time-shifted, gated ramp
- Ramp w/ slope of 2:

$$r(t) = 2 \cdot t$$

- Time-shifted ramp:

$$r_s(t) = 2 \cdot (t - 1)$$

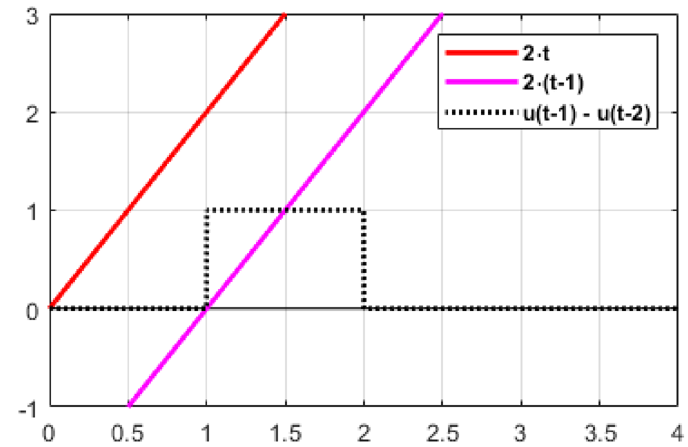
- Gating function
  - Unity-amplitude pulse:

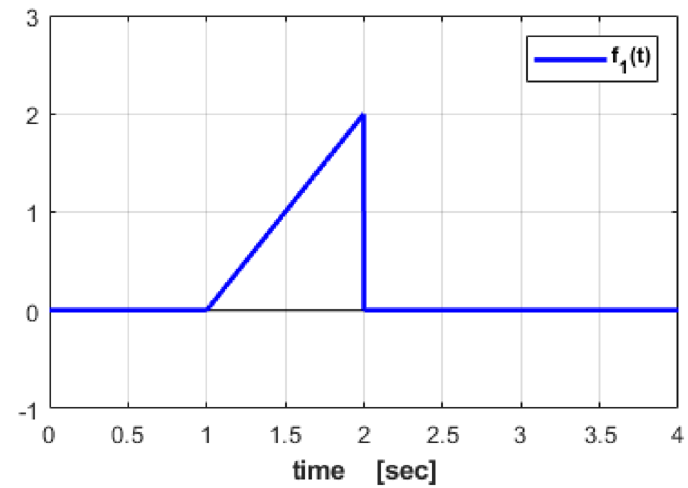
$$g(t) = u(t - 1) - u(t - 2)$$

- Gate the shifted ramp:

$$f_1(t) = r_s(t) \cdot g(t)$$

$$f_1(t) = 2 \cdot (t - 1) \cdot [u(t - 1) - u(t - 2)]$$





Example – Piecewise Function Laplace Transform

- $f_2(t)$: time-shifted pulse
  - Sum of staggered positive and negative steps
- Positive step delayed by 2 sec:

$$s_2(t) = 2 \cdot u(t-2)$$

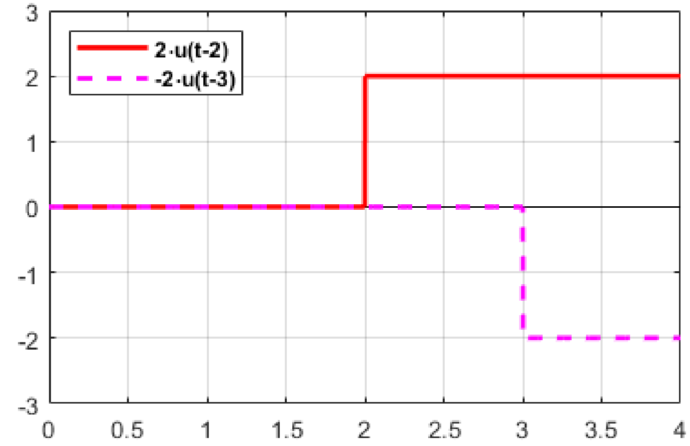
- Negative step delayed by 3 sec:

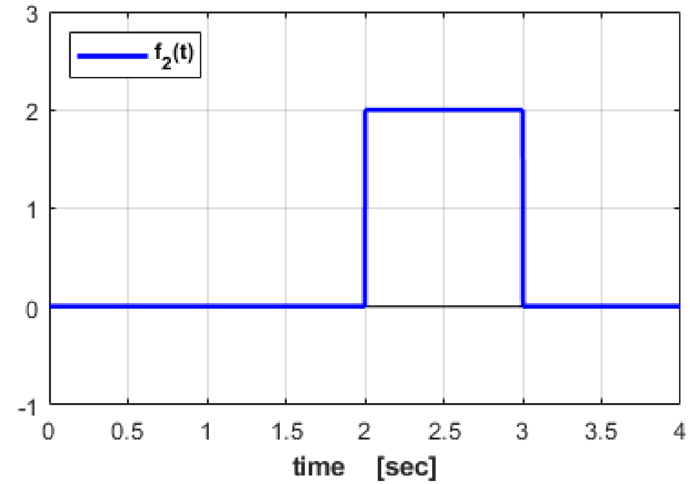
$$s_3(t) = -2 \cdot u(t-3)$$

- Time-shifted pulse

$$f_2(t) = s_2(t) + s_3(t)$$

$$f_2(t) = 2 \cdot u(t-2) - 2 \cdot u(t-3)$$





Example – Piecewise Function Laplace Transform

- Sum the two individual time-domain functions

$$f(t) = f_1(t) + f_2(t)$$

$$f(t) = 2 \cdot (t-1) \cdot [u(t-1) - u(t-2)] + 2 \cdot u(t-2) - 2 \cdot u(t-3)$$

$$\begin{aligned} f(t) = 2[(t-1) \cdot u(t-1)] \\ -2[t \cdot u(t-2)] \\ +4[u(t-2)] \\ -2[u(t-3)] \end{aligned}$$

- Transform the individual terms in $f(t)$

$$\begin{aligned} F(s) = \mathcal{L}\{2[(t-1) \cdot u(t-1)]\} \\ +\mathcal{L}\{-2[t \cdot u(t-2)]\} \\ +\mathcal{L}\{+4[u(t-2)]\} \\ +\mathcal{L}\{-2[u(t-3)]\} \end{aligned}$$

Example – Piecewise Function Laplace Transform

- First term is a time-shifted ramp function

$$\mathcal{L}\{2[(t-1)\cdot u(t-1)]\} = \frac{2e^{-s}}{s^2}$$

- The next term is a time-shifted step function multiplied by time

$$\mathcal{L}\{-2[t\cdot u(t-2)]\} = 2\frac{d}{ds}\left[\frac{e^{-2s}}{s}\right]$$

$$= -2\left[\frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}\right]$$

Example – Piecewise Function Laplace Transform

- The last two terms are time-shifted step functions

$$\mathcal{L}\{4 \cdot u(t-2) - 2 \cdot u(t-3)\} = \frac{4e^{-2s}}{s} - \frac{2e^{-3s}}{s}$$

- The piecewise function in the Laplace domain:

$$F(s) = \frac{2e^{-s}}{s^2} - 2\left[\frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}\right] + \frac{4e^{-2s}}{s} - \frac{2e^{-3s}}{s}$$

$$\boxed{F(s) = \frac{2e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{2e^{-3s}}{s}}$$