# SECTION 3: INVERSE LAPLACE TRANSFORMS

ENGR 203 – Electrical Fundamentals III

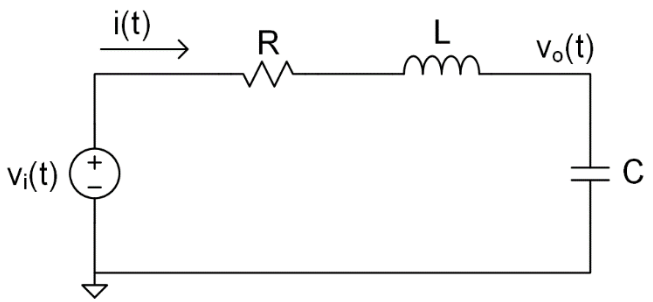
# Inverse Laplace Transform

We've just seen how time-domain functions can be transformed to the Laplace domain. Next, we'll look at how we can solve differential equations in the Laplace domain and transform back to the time domain.

# Laplace Transforms – Differential Equations



- Consider the simple RLC circuit from the introductory section of notes:




- The governing differential equation is

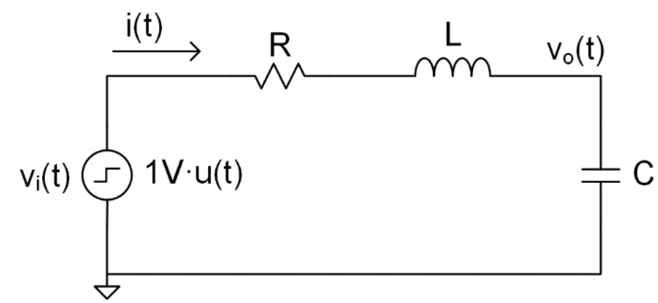
$$\frac{d^2v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

- Or, using dot notation

$$\ddot{v}_o(t) + \frac{R}{L}\dot{v}_o(t) + \frac{1}{LC}v_{o(t)} = \frac{1}{LC}v_i(t) \qquad (1)$$

# Laplace Transforms – Differential Equations






- We'll now use Laplace transforms to determine the ***step response*** of the system
- 1 V step input

$$v_i(t) = 1\,V \cdot u(t) = \begin{cases} 0\,V, & t < 0 \\ 1\,V, & t \geq 0 \end{cases} \tag{2}$$

- For the step response, we assume ***zero initial conditions***

$$v_o(0) = 0 \text{ and } \dot{v}_o(0) = 0 \tag{3}$$

- Using the derivative property of the Laplace transform, (1) becomes

$$s^2V_o(s) - sv_o(0) - \dot{v}_o(0) + \frac{R}{L}sV_o(s) - \frac{R}{L}v_o(0) + \frac{1}{LC}V_o(s) = \frac{1}{LC}V_i(s)$$

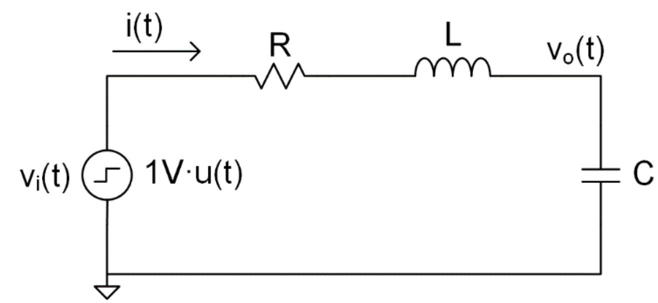
$$s^2V_o(s) + \frac{R}{L}sV_o(s) + \frac{1}{LC}V_o(s) = \frac{1}{LC}V_i(s) \tag{4}$$

# Laplace Transforms – Differential Equations



- The input is a step, so (4) becomes

$$s^2V_o(s)+\frac{R}{L}sV_o(s)+\frac{1}{LC}V_o(s)=\frac{1}{LC}\frac{1}{s} \qquad (5)$$




- Solving (5) for $V_o(s)$

$$V_o(s)\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)=\frac{1}{LC}\frac{1}{s}$$

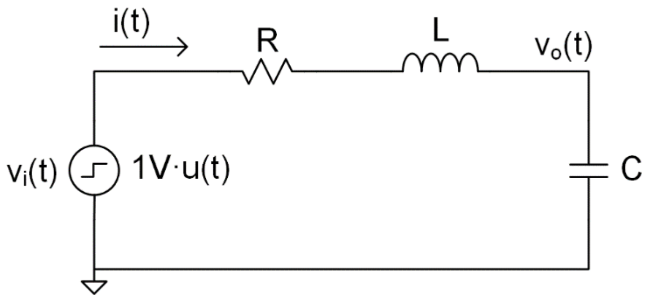
$$V_o(s)=\frac{1/LC}{s\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} \qquad (6)$$

- Equation (6) is the solution to the differential equation of (1), given the step input and I.C.'s
  - The system step response in the Laplace domain
  - Next, we need to transform back to the time domain

# Laplace Transforms – Differential Equations



$$V_o(s) = \frac{1/LC}{s\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} \qquad (6)$$




- The form of (6) is typical of Laplace transforms when dealing with linear systems
  - A ***rational polynomial*** in $s$
  - Here, the numerator is 0th-order

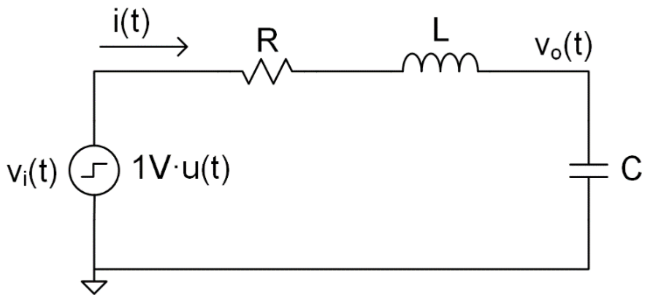
$$V_o(s) = \frac{B(s)}{A(s)}$$

- Roots of the numerator polynomial, $B(s)$, are called the ***zeros*** of the function
- Roots of the denominator polynomial, $A(s)$, are called the ***poles*** of the function

# Inverse Laplace Transforms



$$V_o(s) = \frac{1/LC}{s\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} \qquad (6)$$




- To get (6) back into the time domain, we need to perform an ***inverse Laplace transform***
  - An integral inverse transform exists, but we don't use it
  - Instead, we use ***partial fraction expansion***

- **Partial fraction expansion**
  - Idea is to express the Laplace transform solution, (6), as a sum of Laplace transform terms that appear in the table
  - Procedure depends on the type of roots of the denominator polynomial
    - Real and distinct
    - Repeated
    - Complex

# Inverse Laplace Transforms – Example 1

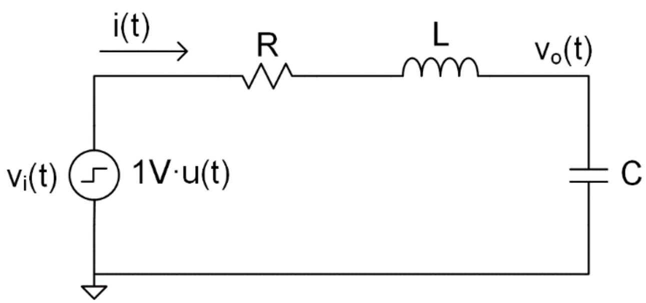


- Consider the following system parameters

$$R = 25\ \Omega$$

$$L = 10\ \mu H$$

$$C = 100\ nF$$




- Laplace transform of the step response becomes

$$V_o(s) = \frac{1E12}{s(s^2+2.5E6s+1E12)} \tag{7}$$

- Factoring the denominator

$$V_o(s) = \frac{1E12}{s(s+500E3)(s+2E6)} \tag{8}$$

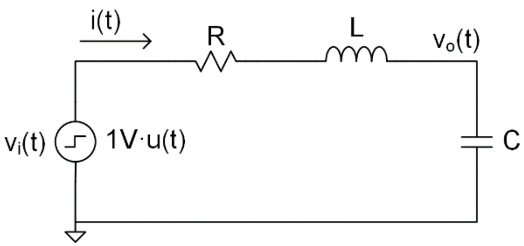
- In this case, the denominator polynomial has three ***real, distinct roots***:

$$s_1 = 0, \qquad s_2 = -500E3, \qquad s_3 = -2E6$$

# Inverse Laplace Transforms – Example 1



- Partial fraction expansion of (8) has the form




$$V_o(s) = \frac{1E12}{s(s+500E3)(s+2E6)} = \frac{r_1}{s} + \frac{r_2}{s+500E3} + \frac{r_3}{s+2E6} \quad (9)$$

  - The numerator coefficients, $r_1$, $r_2$, and $r_3$, are called ***residues***

- Can already see the form of the time-domain function
  - Sum of a ***constant*** and ***two decaying exponentials***

- To determine the residues, multiply both sides of (9) by the denominator of the left-hand side

$$1E12 = r_1(s + 500E3)(s + 2E6) + r_2 s(s + 2E6) + r_3 s(s + 500E3)$$

$$1E12 = r_1 s^2 + 2.5E6 r_1 s + 1E12 r_1 + r_2 s^2 + 2E6 r_2 s + r_3 s^2 + 500E3 r_3 s$$

- Collecting terms, we have

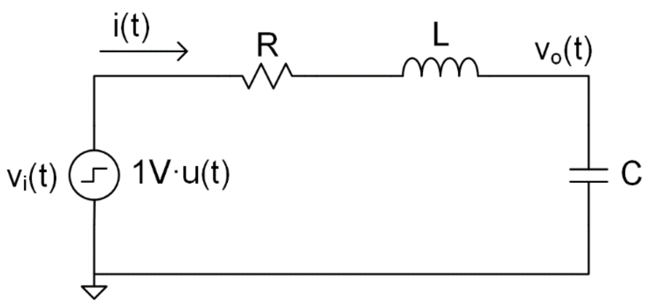
$$1E12 = s^2(r_1 + r_2 + r_3) + s(2.5E6 r_1 + 2E6 r_2 + 500E3 r_3) + 1E12 r_1 \quad (10)$$

# Inverse Laplace Transforms – Example 1



- Equating coefficients of powers of $s$ on both sides of (10) gives a system of three equations in three unknowns

$$s^2: \quad 0 = r_1 + r_2 + r_3$$
$$s^1: \quad 0 = 2.5E6r_1 + 2E6r_2 + 500E3r_3$$
$$s^0: \quad 1E12 = 1E12r_1$$




- Solving for the residues gives

$$r_1 = 1$$
$$r_2 = -1.333$$
$$r_3 = 0.333$$

- The Laplace transform of the step response is

$$V_o(s) = \frac{1}{s} - \frac{1.333}{s+500E3} + \frac{0.333}{s+2E6} \qquad (11)$$

- Equation (11) can now be transformed back to the time domain using the Laplace transform table

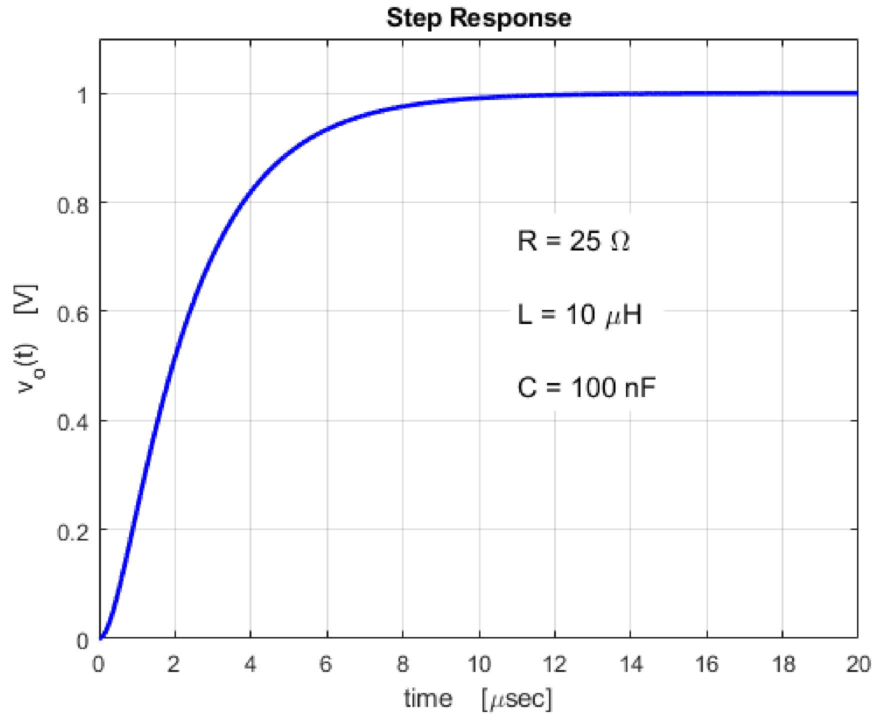
# Inverse Laplace Transforms – Example 1



- The time-domain step response of the system is the ***sum of a constant term and two decaying exponentials***:

$$v_o(t) = 1\,V - 1.333\,Ve^{-500E3t} + 0.333\,Ve^{-2E6t} \tag{12}$$

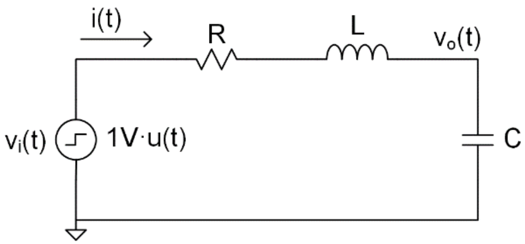
- Step response plotted in MATLAB
- Characteristic of a signal having ***only real poles***
  - No overshoot/ringing
- Steady-state voltage agrees with intuition



# Inverse Laplace Transforms – Example 1



- Go back to (7) and apply the ***initial value theorem***




$$v_o(0) = \lim_{s\to\infty} sV_o(s) = \lim_{s\to\infty} \frac{1E12}{(s^2 + 2.5E6s + 1E12)} = 0\ V$$

- Which is, in fact our assumed initial condition

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- Next, apply the ***final value theorem*** to the Laplace transform step response, (7)

$$v_o(\infty) = \lim_{s\to 0} sV_o(s) = \lim_{s\to 0} \frac{1E12}{(s^2 + 2.5E6s + 1E12)}$$

$$v_o(\infty) = \frac{1E12}{1E12} = 1\ V$$

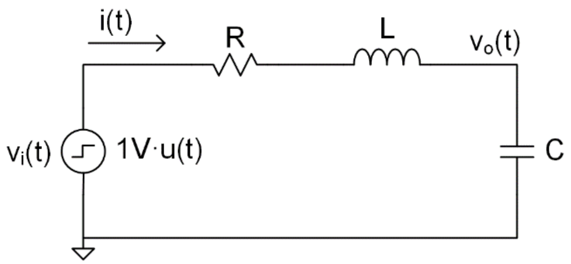
- This final value agrees with both intuition and our numerical analysis

# Inverse Laplace Transforms – Example 2



- ***Reduce the resistance*** and re-calculate the step response

$$R = 20\ \Omega$$
$$L = 10\ \mu H$$
$$C = 100\ nF$$




- Laplace transform of the step response becomes

$$V_o(s) = \frac{1E12}{s(s^2+2E6s+1E12)} \quad (13)$$

- Factoring the denominator

$$V_o(s) = \frac{1E12}{s(s+1E6)^2} \quad (14)$$

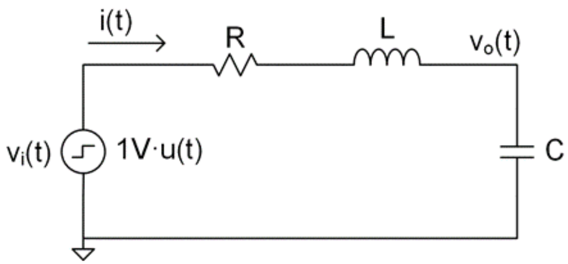
- In this case, the denominator polynomial has three ***real roots***, two of which are ***identical***

$$s_1 = 0, \qquad s_2 = -1E6, \qquad s_3 = -1E6$$

# Inverse Laplace Transforms – Example 2



- Partial fraction expansion of (14) has the form




$$V_o(s) = \frac{1E12}{s(s+1E6)^2} = \frac{r_1}{s} + \frac{r_2}{s+1E6} + \frac{r_3}{(s+1E6)^2} \qquad (15)$$

- Again, find residues by multiplying both sides of (15) by the left-hand side denominator

$$1E12 = r_1(s+1E6)^2 + r_2 s(s+1E6) + r_3 s$$

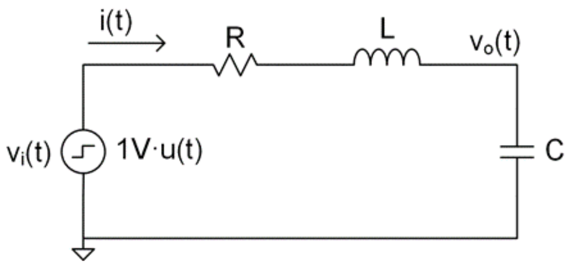
$$1E12 = r_1 s^2 + 2E6 r_1 s + 1E12 r_1 + r_2 s^2 + 1E6 r_2 s + r_3 s$$

- Collecting terms, we have

$$1E12 = s^2(r_1 + r_2) + s(2E6 r_1 + 1E6 r_2 + r_3) + 1E12 r_1 \qquad (16)$$

# Inverse Laplace Transforms – Example 2






- Equating coefficients of powers of $s$ on both sides of (16) gives a system of three equations in three unknowns

$$s^2: \quad 0 = r_1 + r_2$$
$$s^1: \quad 0 = 2E6r_1 + 1E6r_2 + r_3$$
$$s^0: \quad 1E12 = 1E12r_1$$

- Solving for the residues gives

$$r_1 = 1$$
$$r_2 = -1$$
$$r_3 = -1E6$$

- The Laplace transform of the step response is

$$V_o(s) = \frac{1}{s} - \frac{1}{s+1E6} - \frac{1E6}{(s+1E6)^2} \qquad (17)$$

- Equation (17) can now be transformed back to the time domain using the Laplace transform table

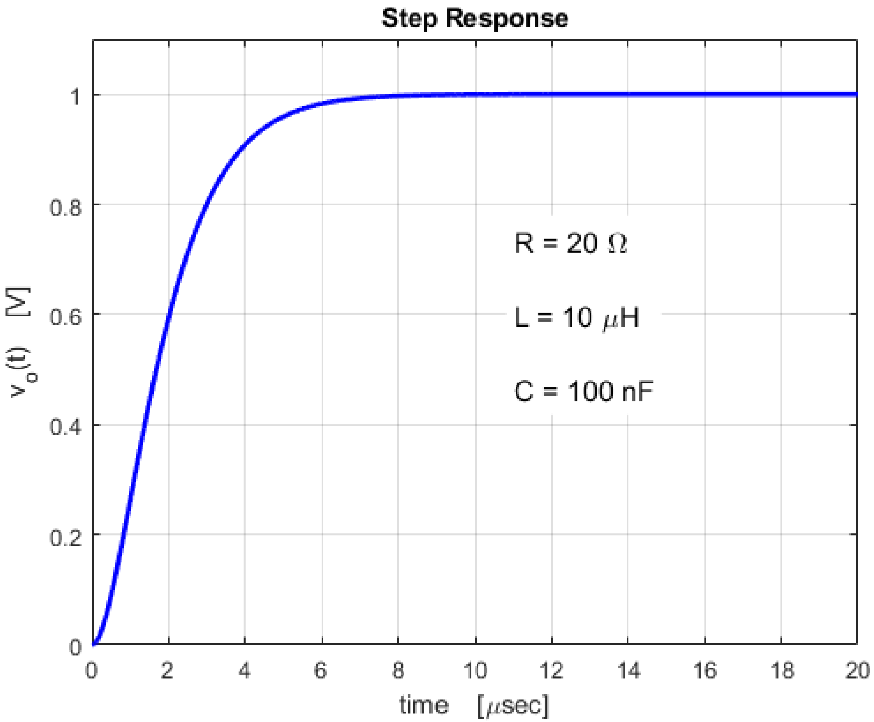
# Inverse Laplace Transforms – Example 2



- The time-domain step response of the system is the ***sum of a constant, a decaying exponential, and a decaying exponential scaled by time***:

$$v_o(t) = 1\,V - 1\,Ve^{-1E6t} - 1E6\frac{V}{s}te^{-1E6t} \tag{18}$$

- Step response plotted in MATLAB
- Again, characteristic of a signal having ***only real poles***
  - Similar to the last case
  - A bit faster – slower pole at $s = -500E3$ was eliminated



# Inverse Laplace Transforms – Example 3

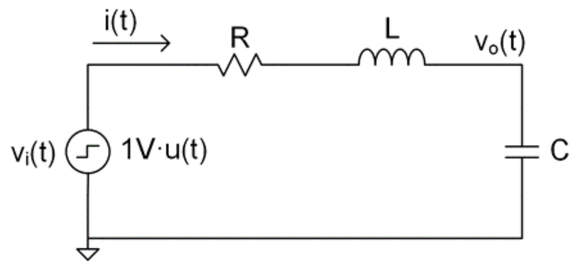


- ***Reduce the resistance even further*** and go through the process once again

$$R = 10\ \Omega$$

$$L = 10\ \mu H$$

$$C = 100\ nF$$




- Laplace transform of the step response becomes

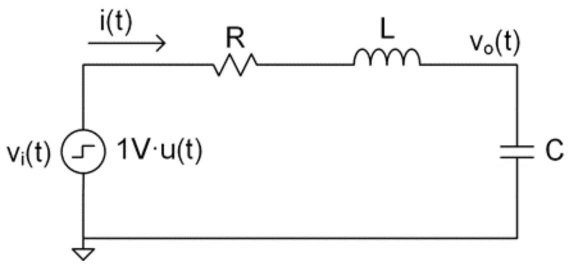
$$V_o(s) = \frac{1E12}{s(s^2+1E6s+1E12)} \tag{19}$$

- The second-order term in the denominator now has ***complex roots***, so we won't factor any further
- The denominator polynomial still has a root at zero and now has two roots which are a ***complex-conjugate pair***

$$s_1 = 0, \qquad s_2 = -500E3 + j866E3, \qquad s_3 = -500E3 - j866E3$$

# Inverse Laplace Transforms – Example 3






- Want to cast the partial fraction terms into forms that appear in the Laplace transform table
- Second-order terms should be of the form

$$\frac{r_i(s+\sigma)+r_{i+1}\omega}{(s+\sigma)^2+\omega^2} \qquad (20)$$

- This will transform into the sum of ***damped sine*** and ***cosine*** terms

$$\mathcal{L}^{-1}\left\{r_i\frac{(s+\sigma)}{(s+\sigma)^2+\omega^2}+r_{i+1}\frac{\omega}{(s+\sigma)^2+\omega^2}\right\}=r_ie^{-\sigma t}\cos(\omega t)+r_{i+1}e^{-\sigma t}\sin(\omega t)$$

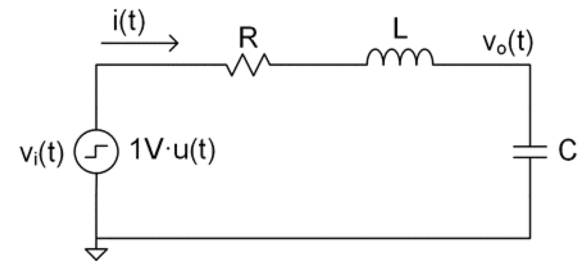
- To get the second-order term in the denominator of (19) into the form of (20), ***complete the square***, to give the following partial fraction expansion

$$V_o(s)=\frac{1E12}{s(s^2+1E6s+1E12)}=\frac{r_1}{s}+\frac{r_2(s+500E3)+r_3(866E3)}{(s+500E3)^2+(866E3)^2} \qquad (21)$$

# Inverse Laplace Transforms – Example 3



- Note that the $\sigma$ and $\omega$ terms in (20) and (21) are the ***real*** and ***imaginary parts*** of the complex-conjugate denominator roots




$$s_{2,3} = -\sigma \pm j\omega = -500E3 \pm j866E3$$

- Multiplying both sides of (21) by the left-hand-side denominator, equate coefficients and solve for residues as before:

$$r_1 = 1$$
$$r_2 = -1$$
$$r_3 = -0.577$$

- Laplace transform of the step response is

$$V_o(s) = \frac{1}{s} - \frac{(s+500E3)}{(s+500E3)^2+(866E3)^2} - \frac{0.577(866E3)}{(s+500E3)^2+(866E3)^2} \quad (22)$$

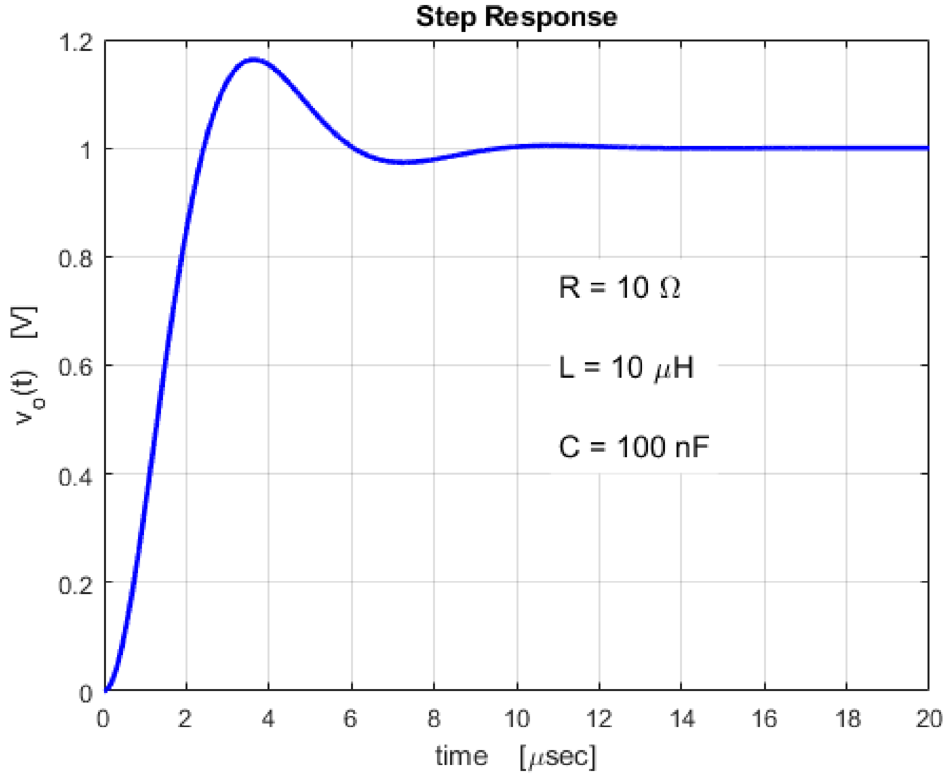
# Inverse Laplace Transforms – Example 3



- The time-domain step response of the system is the ***sum of a constant and two decaying sinusoids***:

$$y(t) = 1\,V - 1\,Ve^{-500E3t}\cos(866E3t) - 0.577\,Ve^{-500E3t}\sin(866E3t) \quad (23)$$

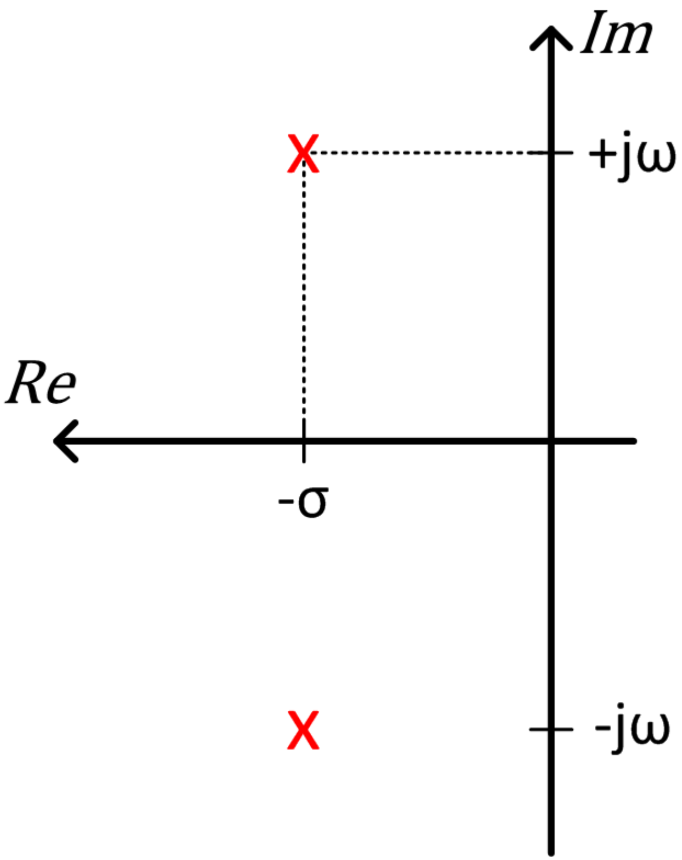
- Step response and individual components plotted in MATLAB
- Characteristic of a signal having ***complex poles***
  - Sinusoidal terms result in overshoot and (possibly) ringing



# Laplace-Domain Signals with Complex Poles



- The Laplace transform of the step response in the last example had ***complex poles***
    - A ***complex-conjugate pair***: $s = -\sigma \pm j\omega$




- Results in sine and cosine terms in the time domain

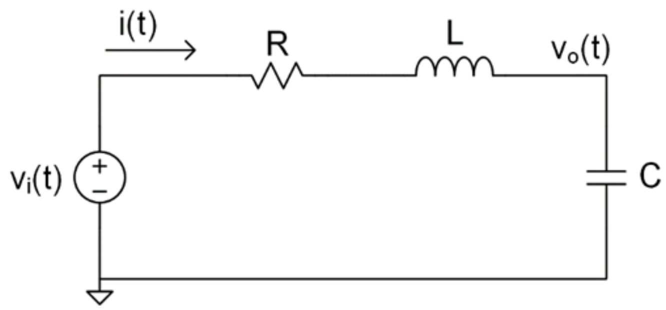
$$Ae^{-\sigma t}\cos(\omega t) + Be^{-\sigma t}\sin(\omega t)$$

- ***Imaginary part*** of the roots, $\omega$
    - ***Frequency of oscillation*** of sinusoidal components of the signal
- ***Real part*** of the roots, $\sigma$,
    - ***Rate of decay*** of the sinusoidal components
- Much more on this later

# Natural & Driven Responses

# Natural and Forced Responses





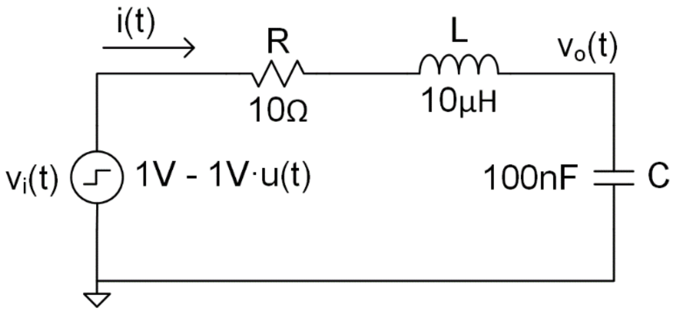

- In the previous section we used Laplace transforms to determine the response of a circuit to a step input, given zero initial conditions
  - The ***driven response***
- Now, consider the response of the same system to non-zero initial conditions only
  - The ***natural response***

# Natural Response



- Same ***under-damped*** RLC circuit
- Now the input steps from $1\ V$ to $0\ V$ at $t = 0$

$$v_i(t) = 1V - 1V\ u(t)$$




- Since $v_i(t \geq 0) = 0$, the governing equation becomes

$$\ddot{v}_o + \frac{R}{L}\dot{v}_o + \frac{1}{LC}v_o = 0 \qquad (24)$$

- Use the derivative property to Laplace transform (24)
  - Allow for non-zero initial-conditions

$$s^2V_o(s) - sv_o(0) - \dot{v}_o(0) + \frac{R}{L}sV_o(s) - \frac{R}{L}v_o(0) + \frac{1}{LC}V_o(s) = 0 \qquad (25)$$

# Natural Response



- Solving (25) for $V_o(s)$ gives the Laplace transform of the output due solely to ***initial conditions***
  - Laplace transform of the ***natural response***

$$V_o(s) = \frac{s v_o(0) + \dot{v}_o(0) + \frac{R}{L} v_o(0)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{26}$$

- For the given input, for $t < 0$:
  - $v_i(t < 0) = 1\ V$
  - $i(t < 0) = 0\ A$
  - $v_o(t < 0) = 1\ V$
- At $t = 0$, neither $i(t)$ nor $v_o(t)$ can change instantaneously, so the initial conditions are:

$$v_o(0) = 1\ V \quad \text{and} \quad \dot{v}_o(0) = 0\ V/s$$

# Natural Response



- Substituting component parameters and initial conditions into (26)

$$V_o(s) = \frac{s+1E6}{(s^2+1E6s+1E12)} \tag{27}$$

- Remember, it's the ***roots of the denominator polynomial*** that dictate the form of the response
  - **Real roots** – decaying exponentials
  - **Complex roots** – decaying sinusoids
- For the under-damped case, roots are complex
  - Complete the square
  - Partial fraction expansion has the form

$$V_o(s) = \frac{s+1E6}{(s^2+1E6s+1E12)} = \frac{r_1(s+500E3)+r_2(866E3)}{(s+500E3)^2+(866E3)^2} \tag{28}$$

# Natural Response



$$V_o(s) = \frac{s + 1E6}{(s^2 + 1E6s + 1E12)} = \frac{r_1(s + 500E3) + r_2(866E3)}{(s + 500E3)^2 + (866E3)^2} \quad (28)$$

- Multiply both sides of (28) by the denominator of the left-hand side

$$s + 1E6 = r_1 s + 500E3 r_1 + 866E3 r_2$$

- Equating coefficients and solving for $r_1$ and $r_2$ gives

$$r_1 = 1, \quad r_2 = 0.577$$

- The Laplace transform of the natural response:

$$V_o(s) = \frac{(s + 500E3)}{(s + 500E3)^2 + (866E3)^2} + \frac{0.577(866E3)}{(s + 500E3)^2 + (866E3)^2} \quad (29)$$

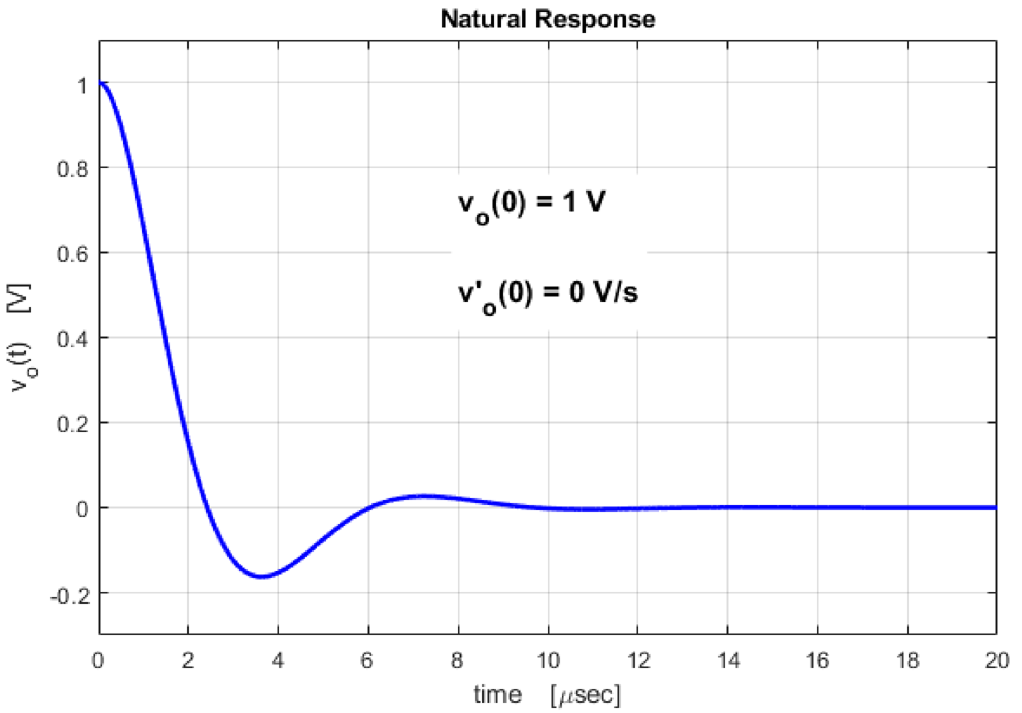
# Natural Response



- Inverse Laplace transform is the ***natural response***

$$y(t) = 1\,Ve^{-500E3t}\cos(866E3 \cdot t) + 0.577\,Ve^{-500E3t}\sin(866E3 \cdot t) \qquad (30)$$

- Under-damped response is the sum of ***decaying sine and cosine*** terms

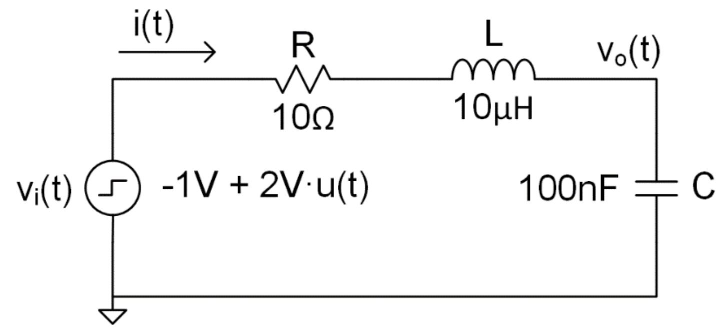


# Driven Response with Non-Zero I.C.s



- Now, change the source to provide both ***non-zero input*** (for $t \geq 0$) and ***non-zero initial conditions***:

$$v_i(t) = -1\,V + 2\,V \cdot u(t)$$




- The Laplace transform of the output including both input and initial conditions:

$$s^2 V_o(s) - s v_o(0) - \dot{v}_o(0) + \frac{R}{L} s V_o(s) - \frac{R}{L} v_o(0) + \frac{1}{LC} V_o(s) = \frac{1}{LC} V_i(s)$$

- Solving for $V_o(s)$ gives

$$V_o(s) = \frac{s v_o(0) + \dot{v}_o(0) + \frac{R}{L} v_o(0) + \frac{1}{LC} V_i(s)}{s^2 + \frac{R}{L} s + \frac{1}{LC}} \qquad (31)$$

# Driven Response with Non-Zero I.C.'s



- Laplace transform of the response has two components

$$V_o(s) = \frac{s\,v_o(0) + \dot{v}_o(0) + \frac{R}{L}v_o(0)}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} + \frac{\frac{1}{LC}F_{in}(s)}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} \qquad (32)$$

***Natural response*** - initial conditions

***Driven response*** - input

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- Total response is a superposition of the initial condition response and the driven response
- Both have the same denominator polynomial
  - Same roots, same type of response
    - Over-, under-, critically-damped

# Driven Response with Non-Zero I.C.s



- The input now is

$$v_i(t) = -1\,V + 2\,V \cdot u(t) = \begin{cases} -1\,V & t < 0 \\ +1\,V & t \geq 0 \end{cases}$$

- For $t \geq 0$, the input is $1\,V$
  - The same as a unit step, so it's Laplace transform is simply

$$V_i(s) = \frac{1}{s}$$

  - The fact that $v_i(t < 0) = -1\,V$ is accounted for by the initial conditions:

$$v_o(0) = -1\,V \quad \text{and} \quad \dot{v}_o(0) = 0\,V/s$$

# Driven Response with Non-Zero I.C.'s



- Substituting in component and input values gives the Laplace transform of the ***total*** response

$$V_o(s) = \frac{-s - 1E6 + \frac{1E12}{s}}{(s^2 + 1E6s + 1E12)} = \frac{-s^2 - 1E6s + 1E12}{s(s^2 + 1E6s + 1E12)}$$

- Transform back to time domain via partial fraction expansion

$$V_o(s) = \frac{r_1}{s} + \frac{r_2(s + 500E3)}{(s + 500E3)^2 + (866E3)^2} + \frac{r_3(866E3)}{(s + 500E3)^2 + (866E3)^2}$$

- Solving for the residues gives

$$r_1 = 1, \;\; r_2 = -2, \;\; r_3 = -1.15$$

# Driven Response with Non-Zero I.C.'s



- Total response:

$$v_o(t) = 1 - 2e^{-500E3t}\cos(866E3 \cdot t) - 1.15e^{-500E3t}\sin(866E3 \cdot t)$$

- Superposition of two components
  - ***Natural response*** due to initial conditions
  - ***Driven response*** due to the input