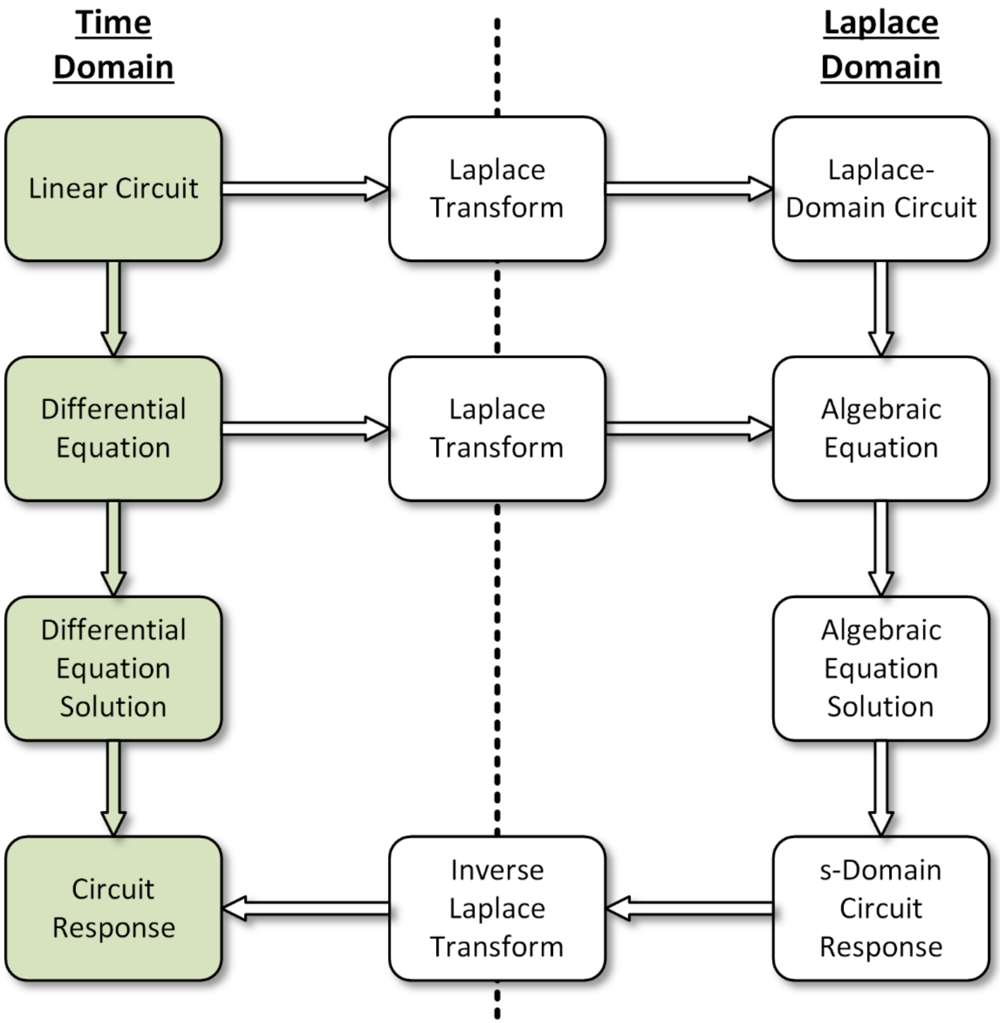
SECTION 4: LAPLACE-DOMAIN CIRCUIT ANALYSIS

ENGR 203 – Electrical Fundamentals III

Laplace-Domain Circuit Analysis

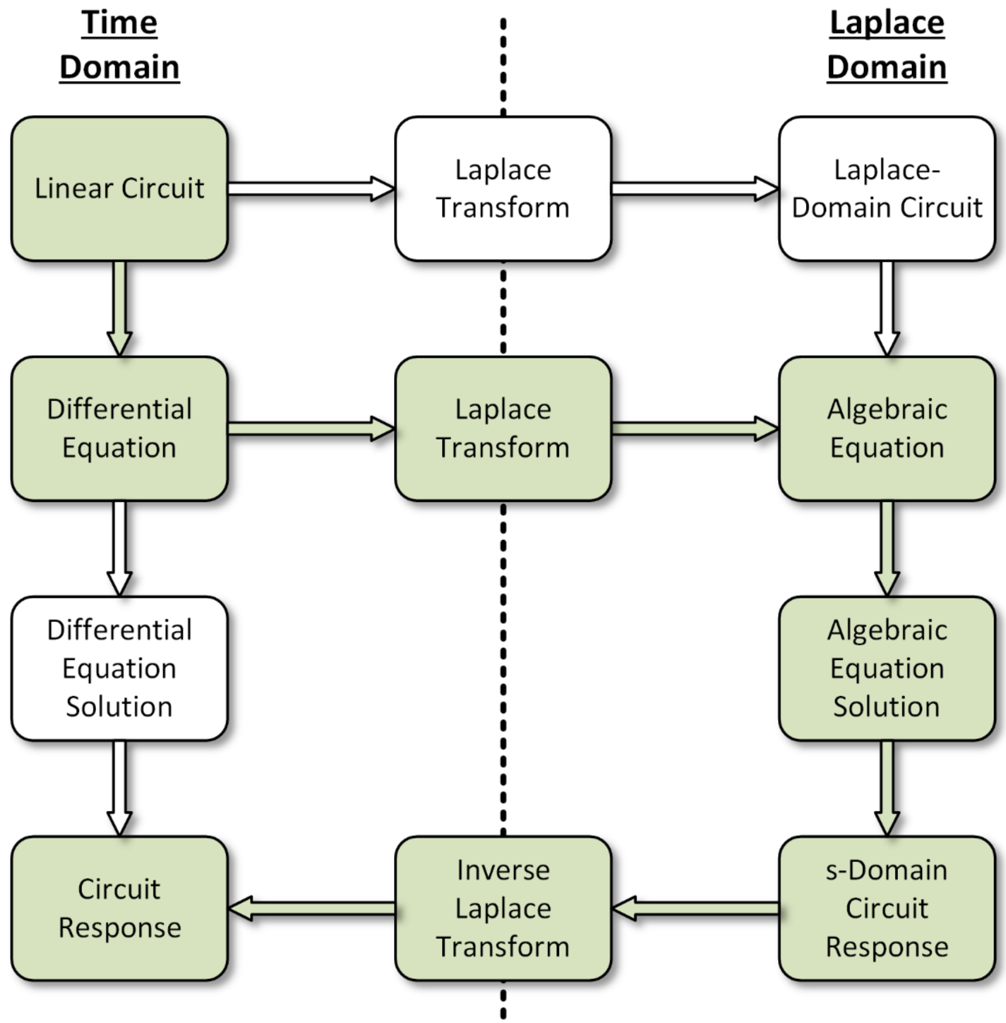
Transient Analysis – Time-

- ENGR 201/202 transient analysis procedure:



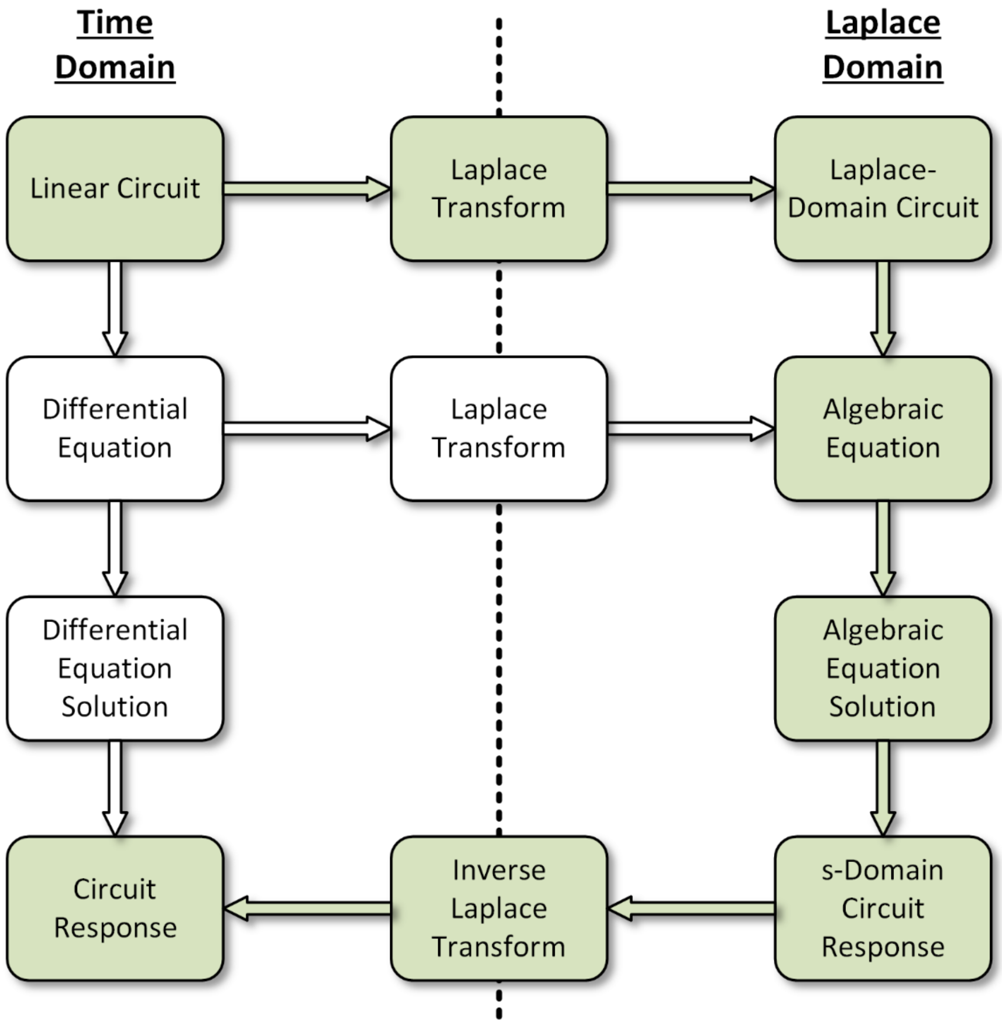
Laplace Transform Solution to ODE

- In the previous sections, we used Laplace transforms to solve a circuit's governing ODE:



Laplace-Domain Circuit Analysis

- Now, we will learn to analyze our circuits in the Laplace domain:



Laplace-Domain Circuit Analysis

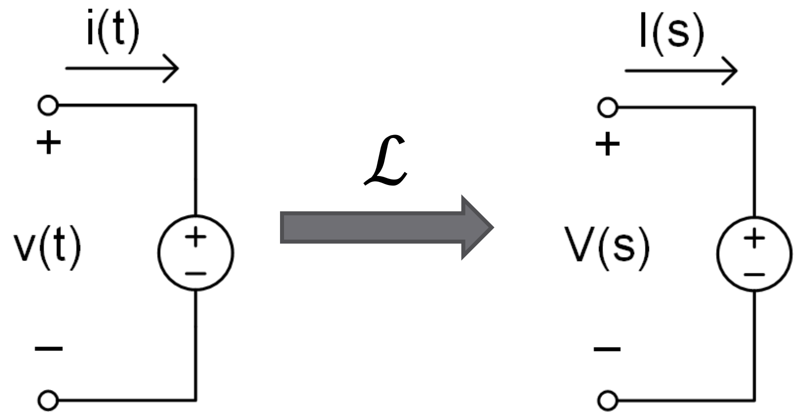
- Circuit analysis in the Laplace Domain:
  - ***Transform the circuit*** from the time domain to the Laplace domain
  - ***Analyze*** using the usual circuit analysis tools
    - Nodal analysis, voltage division, etc.
  - Solve ***algebraic*** circuit equations
    - Laplace transform of circuit response
  - ***Inverse transform*** back to the time domain

- First, we will learn to transform our circuits to the Laplace domain
  - Need Laplace-domain models for individual components

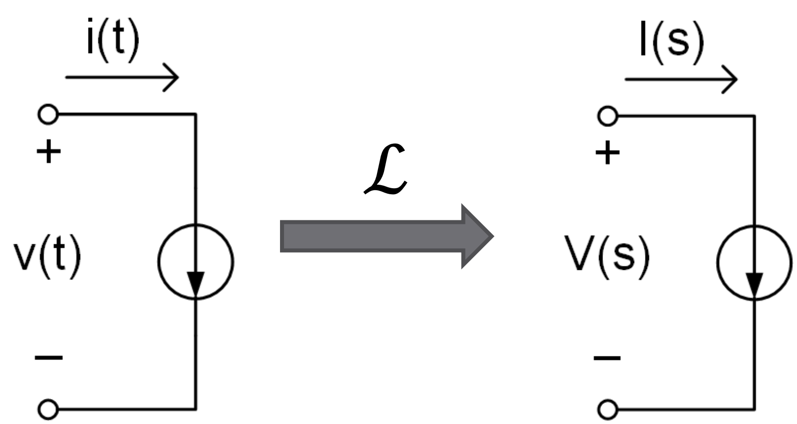
Laplace-Domain Component Models

Sources in the Laplace Domain

- Voltage source:



- Current source:



Resistors in the Laplace Domain

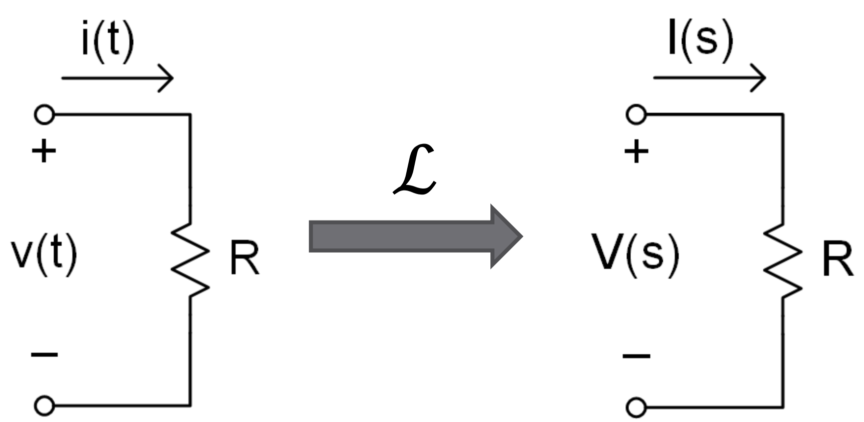
- Ohm's law governs the behavior of resistors

$$v(t) = R \cdot i(t) \quad \text{or} \quad i(t) = \frac{1}{R} v(t)$$

- Laplace transforming these expressions gives us Ohm's law in the Laplace domain:

$$V(s) = R \cdot I(s) \quad \text{or} \quad I(s) = \frac{1}{R} V(s)$$

- Resistors are the same in both domains:



Capacitors in the Laplace Domain

- Current-voltage relationship for a capacitor:

$$i(t) = C\frac{dv}{dt}$$

- Transform using the derivative property of the Laplace transform

$$I(s) = C \cdot sV(s) - C \cdot v(0)$$

- Two components to the Laplace-domain capacitor current:
  - One proportional to the capacitor voltage: $C \cdot sV(s)$
  - One proportional to the ***initial*** capacitor voltage: $-C \cdot v(0)$

- Assuming zero initial voltage, $v(0) = 0$, we have

$$I(s) = Cs \cdot V(s) = \frac{V(s)}{\frac{1}{Cs}} = \frac{V(s)}{Z_c(s)}$$

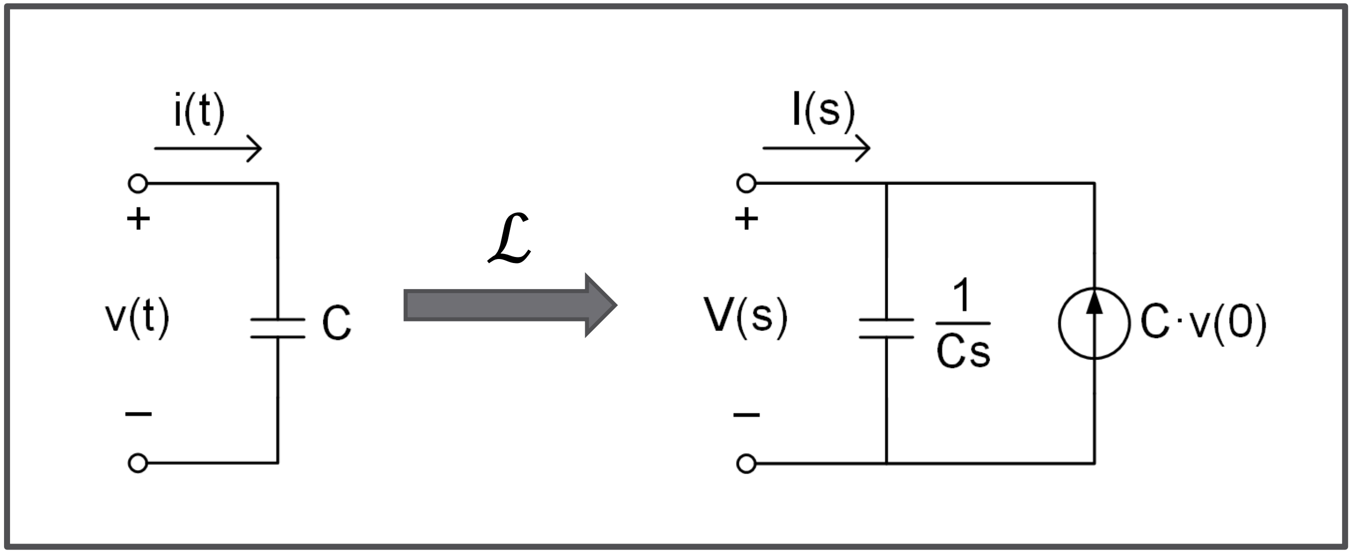
  - $Z_c(s)$ is the ***Laplace-domain capacitor impedance***

$$\boxed{Z_c = \frac{1}{Cs}}$$

Capacitors in the Laplace Domain

$$I(s) = Cs \cdot V(s) - C \cdot v(0)$$

- Capacitor model in the Laplace domain:
  - Two parallel branches, two currents
    - One given by Ohm's law
    - One set by the initial capacitor voltage



Capacitors in the Laplace Domain

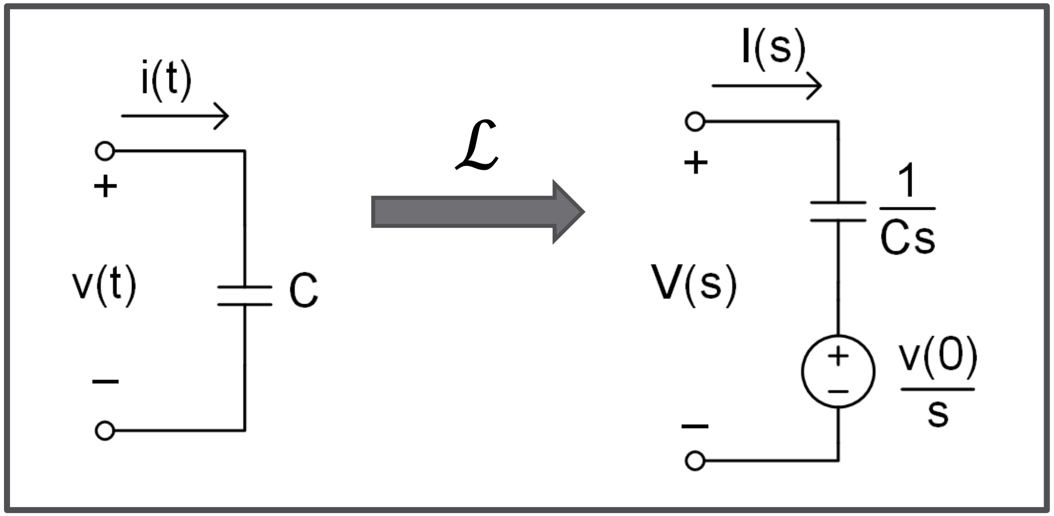
- Alternatively, the current-voltage relationship is:

$$v(t) = \frac{1}{C}\int i(t)dt + v(0)$$

- Transform using the integral property of the Laplace transform

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

- Two components to the Laplace-domain capacitor voltage:
  - One given by Ohm's law: $1/Cs \cdot I(s)$
  - One proportional to the initial capacitor voltage: $v(0)/s$



Inductors in the Laplace Domain

- Current-voltage relationship for an inductor:

$$v(t) = L\frac{di}{dt}$$

- Transform using the derivative property of the Laplace transform

$$V(s) = Ls \cdot I(s) - L \cdot i(0)$$

- Two components to the Laplace-domain inductor voltage:
  - One proportional to the inductor current: $Ls \cdot I(s)$
  - One proportional to the ***initial*** inductor current: $-L \cdot i(0)$

- Assuming zero initial current, $i(0) = 0$, we have

$$V(s) = Ls \cdot I(s) = Z_L(s) \cdot I(s)$$

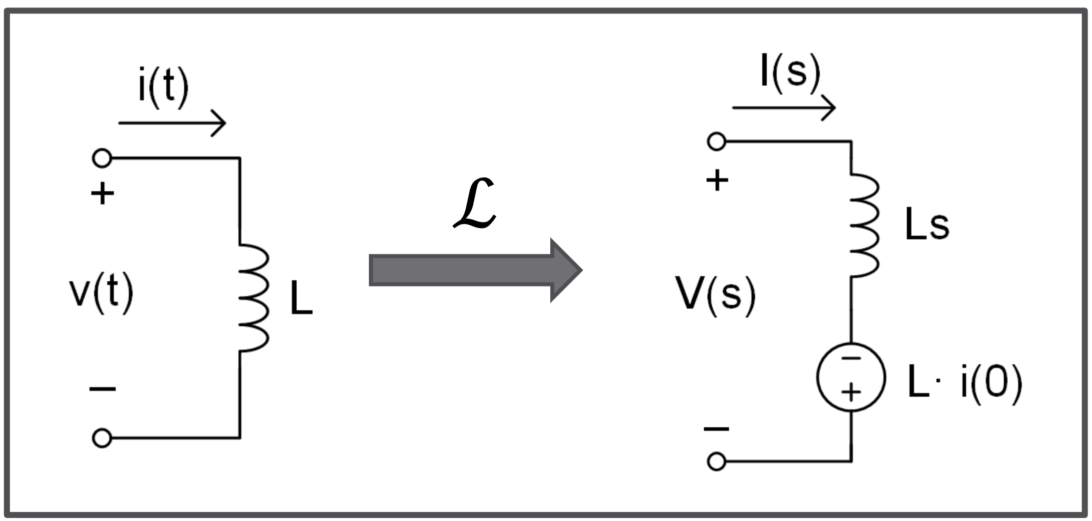
  - $Z_L(s)$ is the ***Laplace-domain inductor impedance***

$$\boxed{Z_L = Ls}$$

Inductors in the Laplace Domain

$$V(s) = Ls \cdot I(s) - L \cdot i(0)$$

- Inductor model in the Laplace domain:
  - Two series components, two voltages
    - One given by Ohm's law
    - One set by the initial inductor current



Inductors in the Laplace Domain

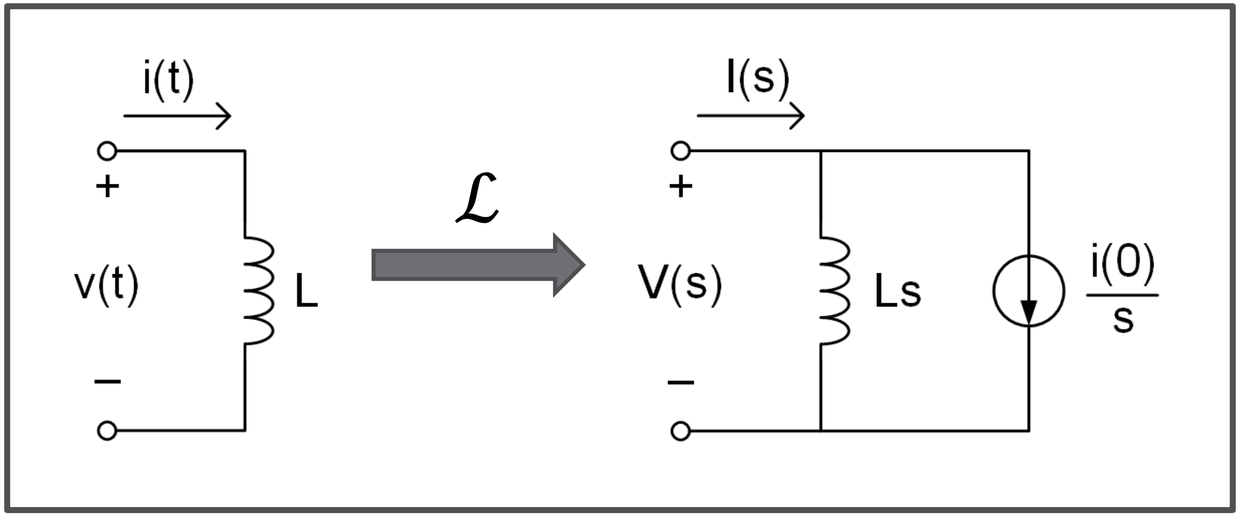
- Alternatively, using the integral form of the current-voltage relationship:

$$i(t) = \frac{1}{L}\int v(t)dt + i(0)$$

- Transform using the integral property of the Laplace transform

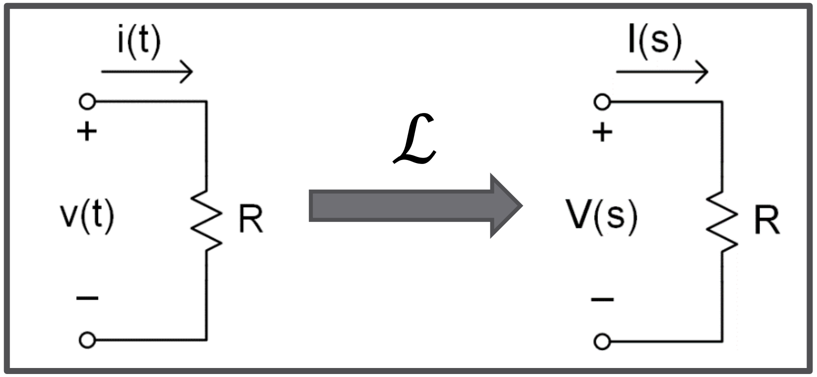
$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

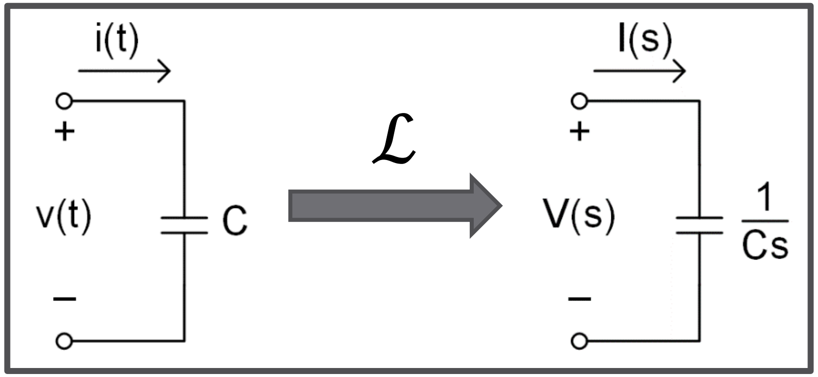
- Two components to the Laplace-domain inductor current:
  - One given by Ohm's law: $1/Ls \cdot V(s)$
  - One proportional to the initial inductor current: $i(0)/s$

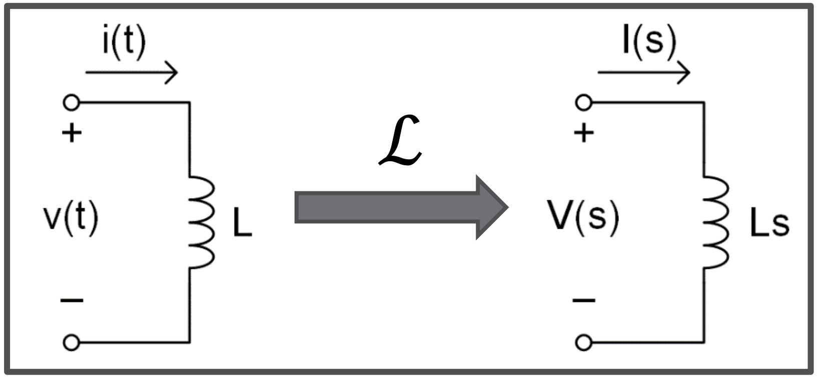


Zero Initial conditions

- It is very common to have ***zero initial conditions***
  - Simply replace components with their ***Laplace-domain impedances***







Laplace-Domain Circuit Analysis

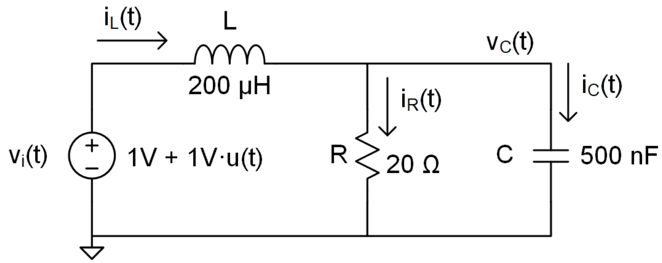
Laplace-Domain Circuit Analysis

- All the familiar circuit analysis tools still apply in the Laplace domain, e.g.:
  - Ohm's law
  - Kirchhoff's laws, KVL and KCL
  - Nodal analysis
  - Mesh analysis
  - Voltage and current division
  - Thevenin and Norton equivalent circuits

- General procedure:
  1) Determine initial conditions
  2) Transform the circuit to the Laplace domain
  3) Analyze the circuit to determine the Laplace transform of the quantity of interest (e.g., $V_o(s)$)
  4) Inverse transform back to the time domain via partial fraction expansion

Laplace-Domain Circuit Analysis – Example 1

- Determine $v_C(t)$ for the following circuit



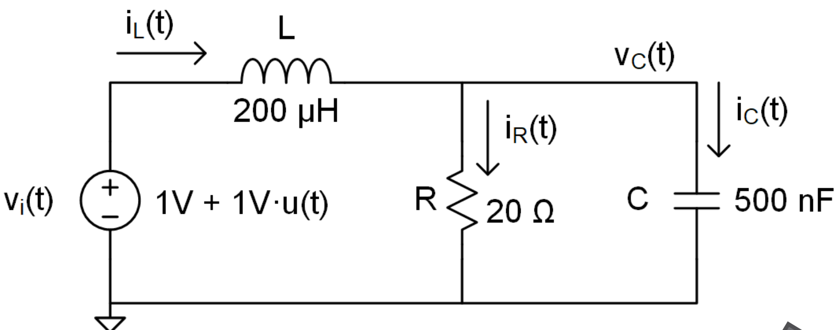
$$v_i(t) = 1V + 1V \cdot u(t) = \begin{cases} 1\ V & t < 0 \\ 2\ V & t \geq 0 \end{cases}$$

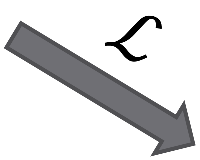
- First, determine the initial conditions, $v_C(0)$ and $i_L(0)$
  - For $t < 0$:
    - $v_i(t < 0) = 1\ V$
    - $i_L(t < 0) = 1\ V/20\ \Omega = 50\text{ mA}$
    - $v_C(t < 0) = 1\ V$
  - At $t = 0$
    - $v_i(0) = 2\ V$
    - $v_c(0) = 1\text{ V}$
    - $i_L(0) = 50\ mA$

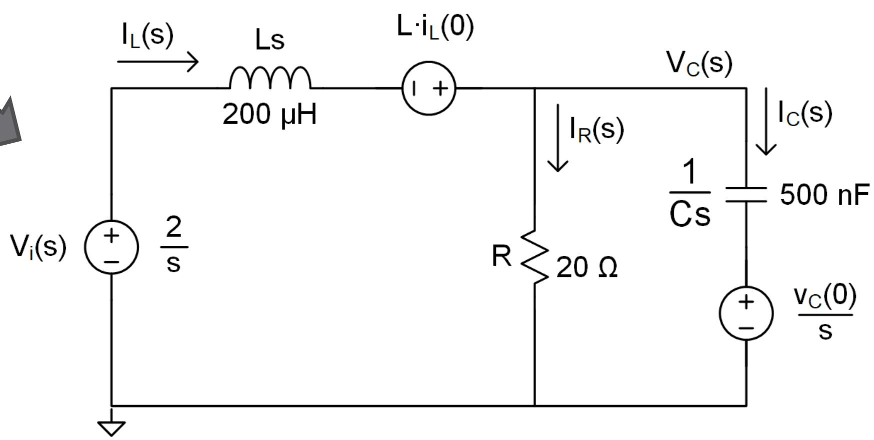
Laplace-Domain Circuit Analysis – Example 1

- Next, transform the circuit to the Laplace domain
  - Replace components with Laplace-domain equivalents
  - The input is $2\ V$ for $t \geq 0$, so

$$V_i(s) = \frac{2}{s}$$

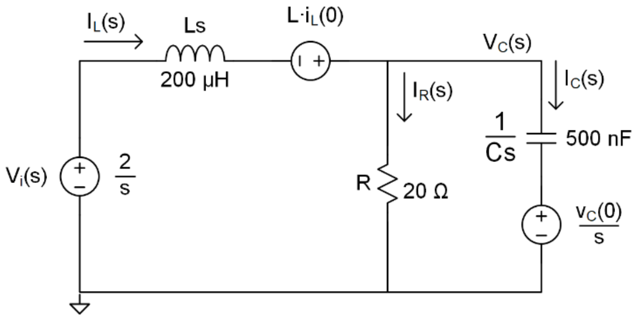






Laplace-Domain Circuit Analysis – Example 1

- Want $V_C(s)$, so do a nodal analysis
- KCL at node V_C:



$$I_L(s) - I_R(s) - I_C(s) = 0$$

$$\frac{V_i(s) - \left(V_C(s) - L \cdot i_L(0)\right)}{Ls} - \frac{V_C(s)}{R} - \frac{\left(V_C(s) - \frac{v_C(0)}{s}\right)}{{}^{1}/_{Cs}} = 0$$

$$\frac{V_i(s)}{Ls} + \left(-\frac{1}{Ls} - \frac{1}{R} - Cs\right)V_C(s) + \frac{i_L(0)}{s} + Cv_C(0) = 0$$

$$V_C(s)\left(\frac{1}{Ls} + \frac{1}{RC}s + Cs\right) = \frac{V_i(s)}{Ls} + \frac{i_L(0)}{s} + Cv_C(s)$$

$$V_C(s)\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right) = \frac{1}{LC}V_i(s) + \frac{1}{C}i_L(0) + v_C(0)s$$

Laplace-Domain Circuit Analysis – Example 1

- Solving for $V_C(s)$ gives the output in the Laplace domain:

$$V_C(s) = \frac{\frac{1}{LC}V_i(s) + \frac{1}{C}i_L(0) + v_C(0)s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

- Substituting in $V_i(s)$, initial conditions, and component values:

$$V_C(s) = \frac{s^2 + 100E3s + 20E9}{s(s^2 + 100E3s + 10E9)}$$

- One real pole and a complex-conjugate pair of poles:

$$s_1 = 0 \quad \text{and} \quad s_{2,3} = -50E3 \pm j86.6E3$$

Laplace-Domain Circuit Analysis – Example 1

- Inverse transform via partial fraction expansion

$$V_C(s) = \frac{s^2 + 100E3s + 20E9}{s(s^2 + 100E3s + 10E9)} = \frac{r_1}{s} + \frac{r_2(s + 50E3) + r_3(86.6E3)}{(s + 50E3)^2 + (86.6E3)^2}$$

- Multiply through by the LHS denominator

$$s^2 + 100E3s + 20E9 = r_1 s^2 + 100E3r_1 s + 10E9r_1 + r_2 s^2 + 50E3r_2 s + 86.6E3r_3 s$$

$$s^2 + 100E3s + 20E9 = (r_1 + r_2)s^2 + (100E3r_1 + 50E3r_2 + 86.6E3r_3)s + 10E9r_1$$

- Equating coefficients gives three equations for the residues:

$$s^2\!: \;\; 1 = r_1 + r_2$$

$$s^1\!: \;\; 100E3 = 100E3r_1 + 50E3r_2 + 86.6E3r_3$$

$$s^0\!: \;\; 20E9 = 10E9r_1$$

- Solving for the residues:

$$r_1 = 2 \;, \quad r_2 = -1 \;, \quad r_3 = -0.577$$

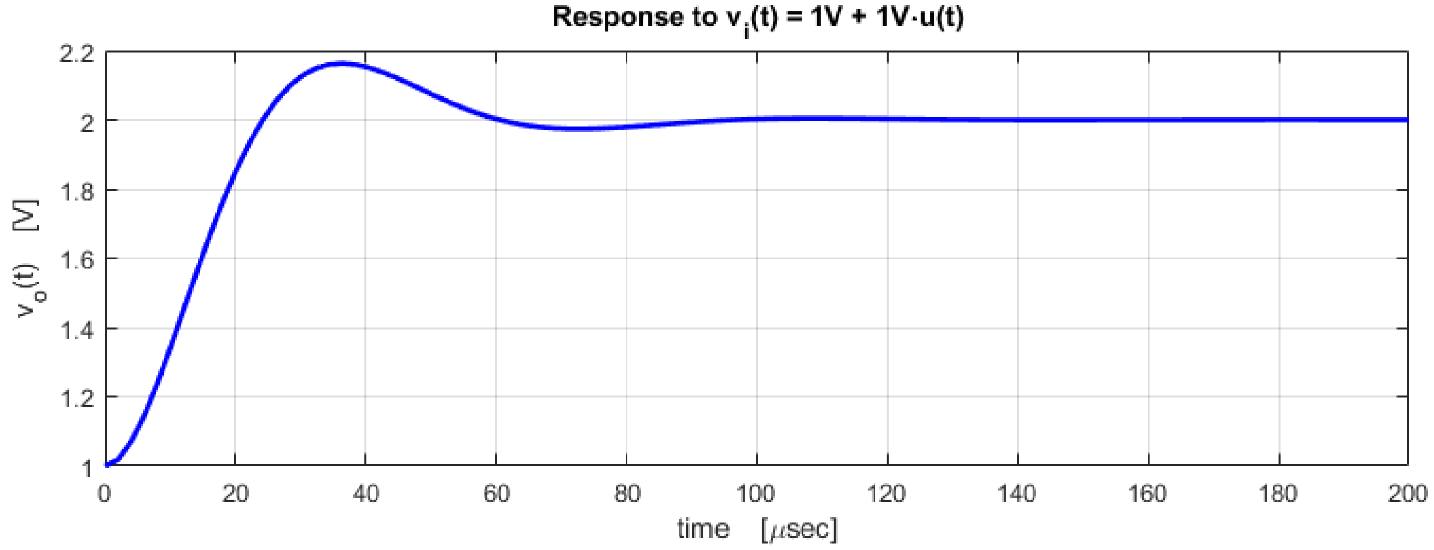
Laplace-Domain Circuit Analysis – Example 1

- The Laplace transform of the output voltage:

$$V_C(s) = \frac{2}{s} - \frac{(s + 50E3)}{(s + 50E3)^2 + (86.6E3)^2} - 0.577\frac{(86.6E3)}{(s + 50E3)^2 + (86.6E3)^2}$$

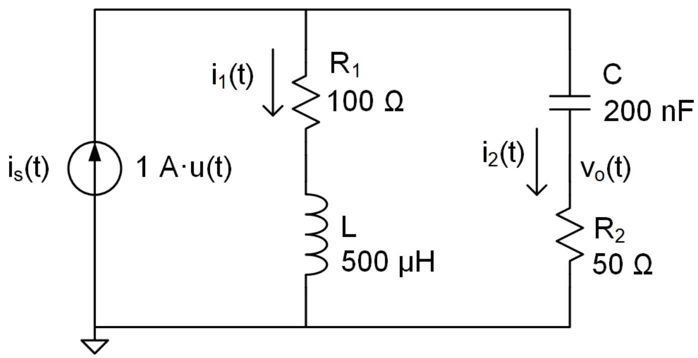
- Inverse transforming to the time domain:

$$v_C(t) = 2\,V - 1\,Ve^{-50E3t}\cos(86.6E3t) - 0.577e^{-50E3t}\sin(86.6E3t)$$

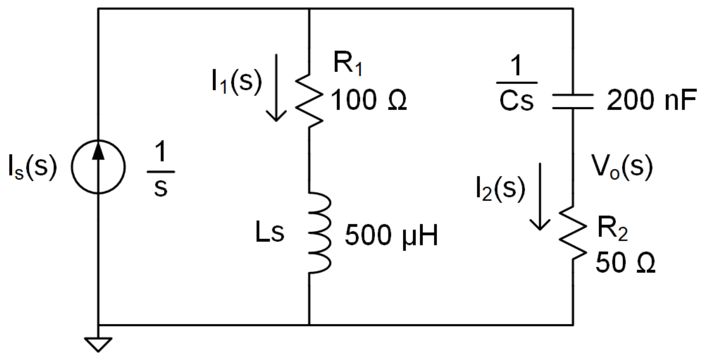


Laplace-Domain Circuit Analysis – Example 2

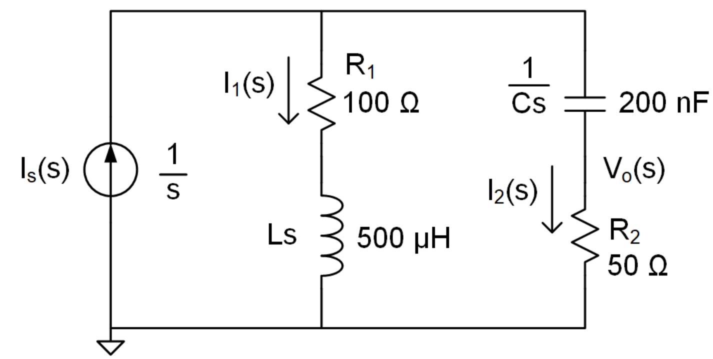
- Determine $v_o(t)$ for the following circuit



- Now, the initial conditions are zero
  - $v_c(0) = 0\text{ V}$
  - $i_L(0) = 0\ A$
  - Laplace-domain component models are simplified
    - No sources



Laplace-Domain Circuit Analysis – Example 2



- The output voltage is given by

$$V_o(s) = I_2(s) \cdot R_2$$

- Apply current division to find $I_2(s)$

$$I_2(s) = I_s(s) \cdot \frac{Y_2(s)}{Y_1(s) + Y_2(s)}$$

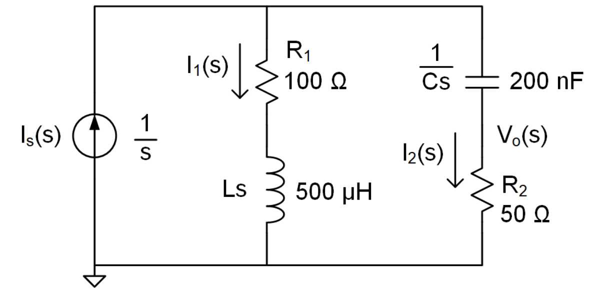
- The admittances of the parallel branches:

$$Y_1(s) = Z_1(s)^{-1} = \frac{1}{R_1 + Ls}$$

$$Y_2(s) = Z_1(s)^{-1} = \left(R_2 + \frac{1}{Cs}\right)^{-1} = \frac{Cs}{R_2 Cs + 1}$$

Laplace-Domain Circuit Analysis – Example 2

- Current division



$$I_2(s) = I_s(s) \cdot \frac{Y_2(s)}{Y_1(s) + Y_2(s)}$$

$$I_2(s) = \frac{1}{s} \cdot \frac{\frac{Cs}{R_2Cs + 1}}{\frac{1}{R_1 + Ls} + \frac{Cs}{R_2Cs + 1}} = \frac{1}{s} \cdot \frac{\frac{Cs}{R_2Cs + 1}}{\frac{R_2Cs + 1 + R_1Cs + LCs^2}{(R_1 + Ls)(R_2Cs + 1)}}$$

$$I_2(s) = \frac{1}{s} \cdot \frac{(R_1 + Ls)Cs}{R_2Cs + 1 + R_1Cs + LCs^2} = \frac{LCs^2 + R_1Cs}{s(LCs^2 + (R_1 + R_2)Cs + 1)}$$

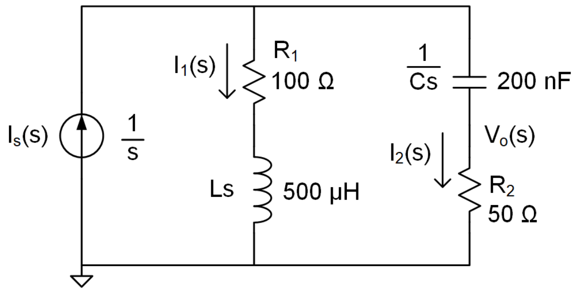
$$I_2(s) = \frac{LCs + R_1C}{LCs^2 + (R_1 + R_2)Cs + 1} = \frac{s + \frac{R_1}{L}}{s^2 + \frac{(R_1 + R_2)}{L}s + \frac{1}{LC}}$$

Laplace-Domain Circuit Analysis – Example 2

- Ohm's law give the output voltage

$$V_o(s) = R_2 I_2(s)$$

$$V_o(s) = \frac{R_2\left(s + \frac{R_1}{L}\right)}{s^2 + \frac{(R_1 + R_2)}{L}s + \frac{1}{LC}}$$



- Substituting in component values:

$$V_o(s) = \frac{50(s + 200E3)}{s^2 + 300E3s + 10E9}$$

- The denominator has two real roots
  - The voltage response is over-damped

$$V_o(s) = \frac{50(s + 200E3)}{(s + 38.2E3)(s + 261.8E3)}$$

Laplace-Domain Circuit Analysis – Example 2

- Inverse transform via partial fraction expansion

$$V_o(s) = \frac{50(s + 200E3)}{(s + 38.2E3)(s + 261.8E3)} = \frac{r_1}{(s + 38.2E3)} + \frac{r_2}{(s + 261.8E3)}$$

- Multiply through by the LHS denominator

$$50s + 10E6 = r_1 s + 261.8E3r_1 + r_2 s + 38.2E3r_2$$

$$50s + 10E6 = (r_1 + r_2)s + 261.8E3r_1 + 38.2E3r_2$$

- Equating coefficients gives three equations for the residues:

$$s^1:\ \ 50 = r_1 + r_2$$

$$s^0:\ \ 10E6 = 261.8E3r_1 + 38.2E3r_2$$

- Solving for the residues:

$$r_1 = 36.2\ \ ,\quad r_2 = 13.8$$

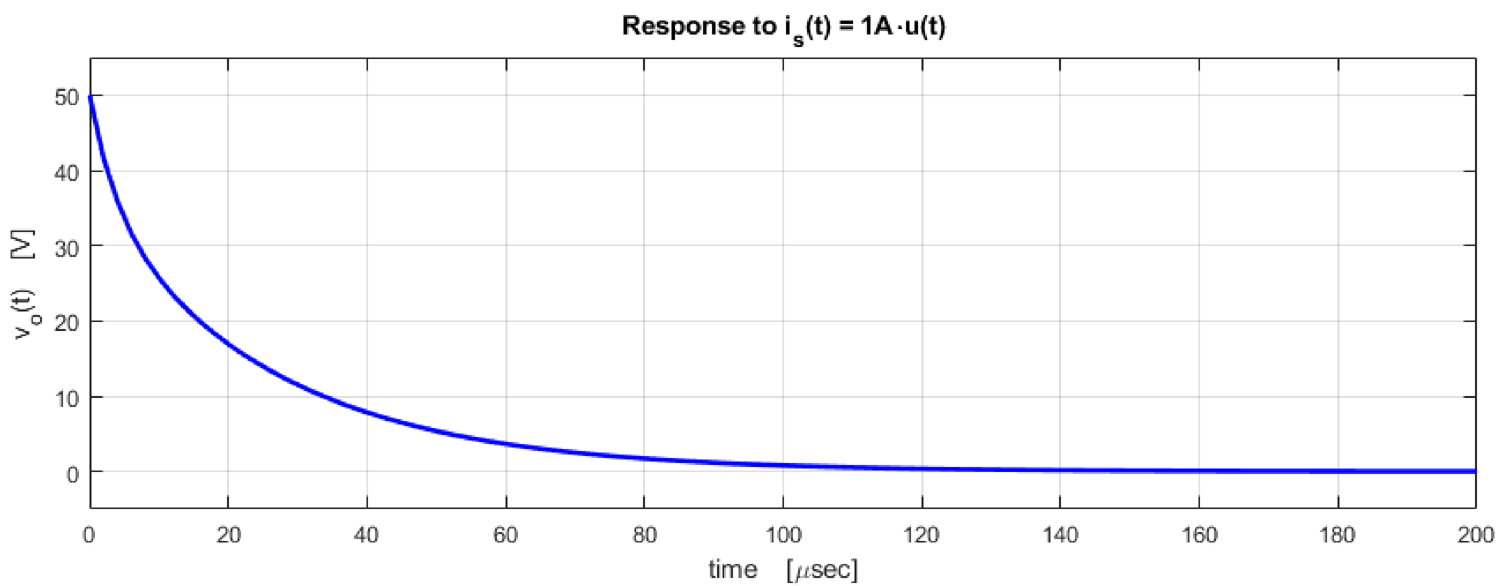
Laplace-Domain Circuit Analysis – Example 2

- The Laplace transform of the output voltage:

$$V_o(s) = \frac{36.2}{(s + 38.2E3)} + \frac{13.8}{(s + 261.8E3)}$$

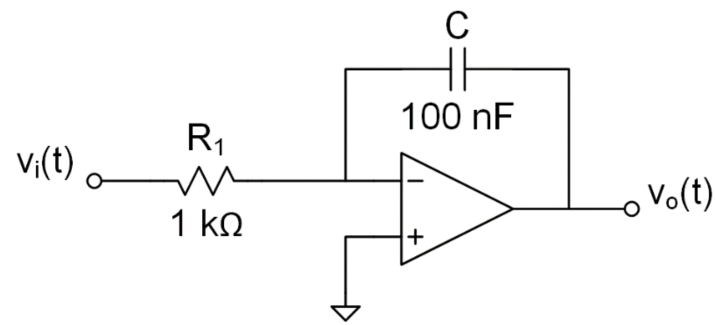
- Inverse transforming to the time domain:

$$v_o(t) = 36.2\,Ve^{-38.2E3t} + 13.8\,Ve^{-261.8E3t}$$

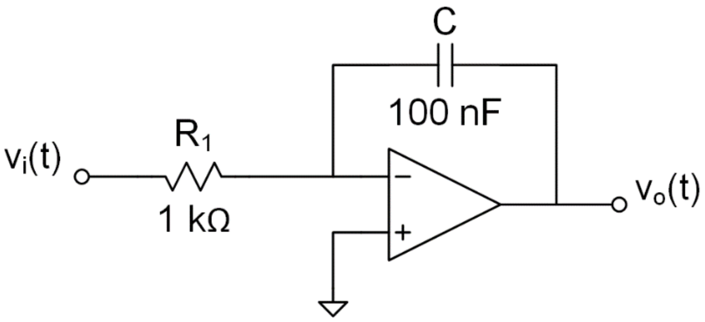


Example Problems

- Determine the output of the following opamp circuit for $v_i(t) = 1\,V\cos(100E3 \cdot t)$
  - What is the circuit's function?



- Determine the Laplace-domain step response.
- Apply the final value theorem to determine the steady-state output.
- Find the time-domain step response.



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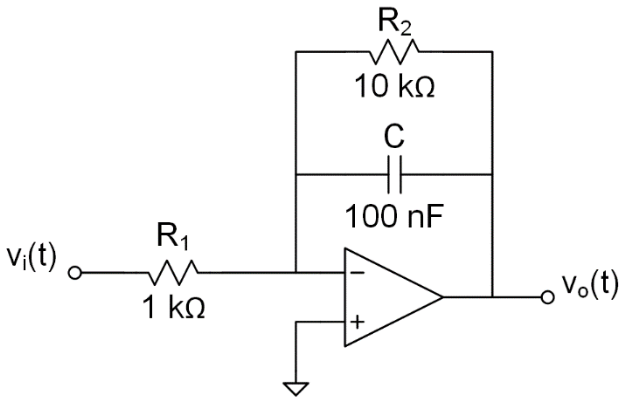

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