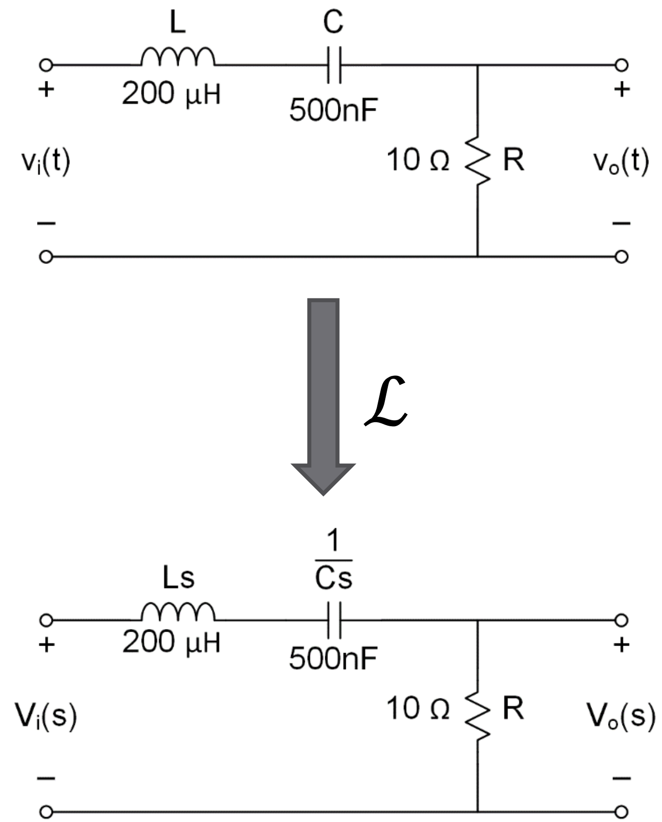
SECTION 5: TRANSFER FUNCTIONS

ENGR 203 – Electrical Fundamentals III

Transfer Function

Transfer Function

- Determine the circuit output, $v_o(t)$
  - Assume zero initial conditions
- First, transform circuit to the Laplace domain
- Apply voltage division

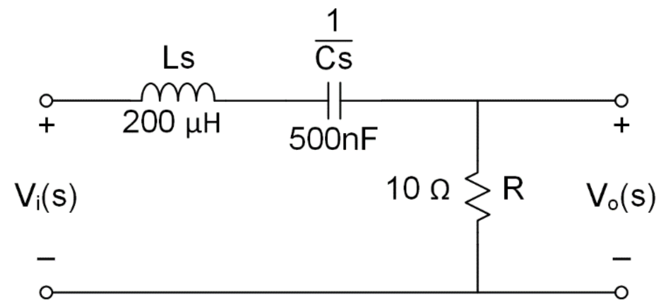


$$V_o(s) = V_i(s)\frac{R}{R + Ls + \frac{1}{Cs}}$$

$$V_o(s) = V_i(s)\frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Transfer Function

$$V_o(s) = V_i(s)\left(\frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right)$$



- The Laplace-domain output is the Laplace-domain input multiplied by a scaling factor
- Dividing by the input, $V_i(s)$, gives the ***transfer function***

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

- ***Transfer function:*** *the ratio of the Laplace-domain output to the Laplace-domain input, assuming zero initial conditions*

Transfer function

- The transfer function is a ***mathematical model*** for a circuit
  - Describes the input-to-output relationship
  - Laplace-domain expression – algebraic
  - An alternative to the differential-equation model

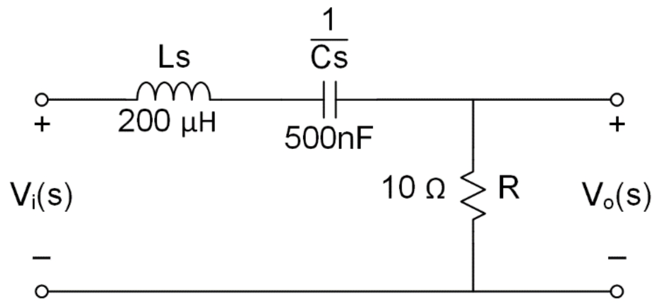
- Can use the transfer function to determine a circuit's output in response to a particular input

$$V_o(s) = V_i(s) \cdot G(s) = V_i(s) \cdot \frac{V_o(s)}{V_i(s)} = V_o(s)$$

- Can think of the transfer function as the ***circuit's Laplace-domain gain***

Example 1 – Transfer Function

- Determine the step response for our example circuit
- Transfer function:



$$G(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

- Substituting in component values:

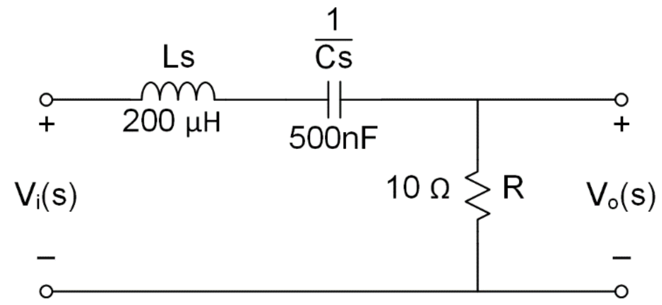
$$G(s) = \frac{50E3s}{s^2 + 50E3s + 10E9}$$

- Laplace-domain step response:

$$V_o(s) = V_i(s) \cdot G(s) = \frac{1}{s} \cdot \frac{50E3s}{s^2 + 50E3s + 10E9} = \frac{50E3}{s^2 + 50E3s + 10E9}$$

Example 1 – Transfer Function

$$V_o(s) = \frac{50E3}{s^2 + 50E3s + 10E9}$$



- Roots of the denominator polynomial are

$$s_{1,2} = -25E3 \pm j96.8E3$$

- Under-damped, so response is a sum of decaying sinusoids
- Inverse transform via partial fraction expansion:

$$V_o(s) = \frac{50E3}{s^2 + 50E3s + 10E9} = \frac{r_1(s + 25E3)}{(s + 25E3)^2 + (96.8E3)^2} + \frac{r_2 96.8E3}{(s + 25E3)^2 + (96.8E3)^2}$$

- Solving for residues, as usual, gives

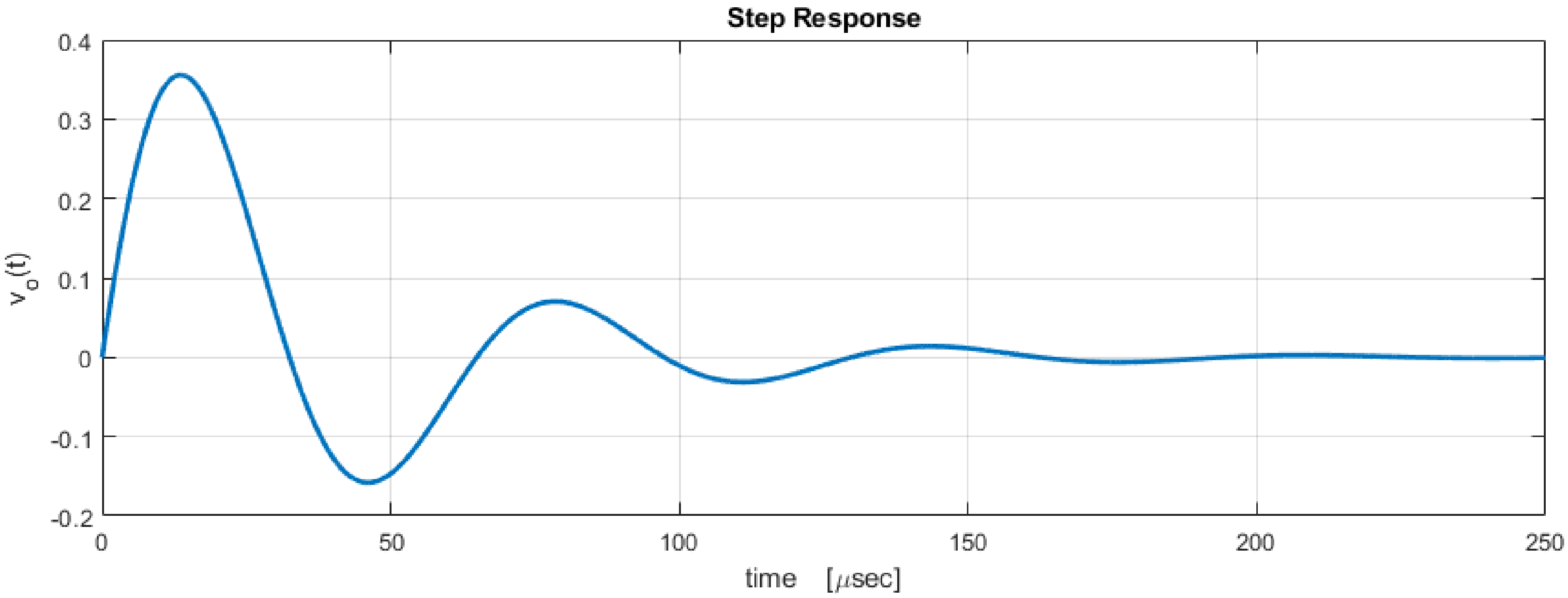
$$r_1 = 0 \quad \text{and} \quad r_2 = 0.519$$

Transfer Function - Example

$$V_o(s) = 0.519\,\frac{96.8E3}{(s+25E3)^2 + (96.8E3)^2}$$

- $\cos(\omega t)$ term goes to zero, only a $\sin(\omega t)$ term left

$$v_o(t) = 0.519\,V\,\mathrm{e}^{-25E3t}\,\sin(96.8E3t)$$



Characteristic Polynomial

- Roots of the transfer function's denominator polynomial determine:
  - Nature of circuit response
  - Damping
  - Terms in the time-domain response
- These roots are called the circuit ***poles***
- Denominator of the transfer function is the ***characteristic polynomial***

$$G(s) = \frac{B(s)}{\Delta(s)}$$

where

$B(s)$: numerator polynomial
$\Delta(s)$: characteristic polynomial

Characteristic Polynomial

- Roots of the characteristic polynomial, $\Delta(s)$, are the solutions to the ***characteristic equation***:

$$\Delta(s) = 0$$

  - These are the circuit ***poles***
  - Describe circuit behavior

- Next, we'll see how pole location in the complex plane is related to time-domain circuit response
  - We'll focus on second-order circuits

Second-Order Circuit Poles

Second-Order Systems

- Second-order transfer function

$$G(s) = \frac{Num(s)}{s^2+a_1 s+a_0} = \frac{Num(s)}{(s+\sigma)^2+\omega_d^2} \quad (1)$$

where ω_d is the ***damped natural frequency***

- Can also express the 2nd-order transfer function as

$$G(s) = \frac{Num(s)}{s^2+2\zeta\omega_0 s+\omega_0^2} = \frac{Num(s)}{s^2+2\sigma s+\omega_0^2} \quad (2)$$

where ω_o is the ***un-damped natural frequency***, and ζ is the ***damping ratio***

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$\zeta = \frac{\sigma}{\omega_0}$$

- Two poles at

$$s_{1,2} = -\sigma \pm \sqrt{\sigma^2-\omega_0^2} = -\zeta\omega_o \pm \omega_0\sqrt{\zeta^2-1}$$

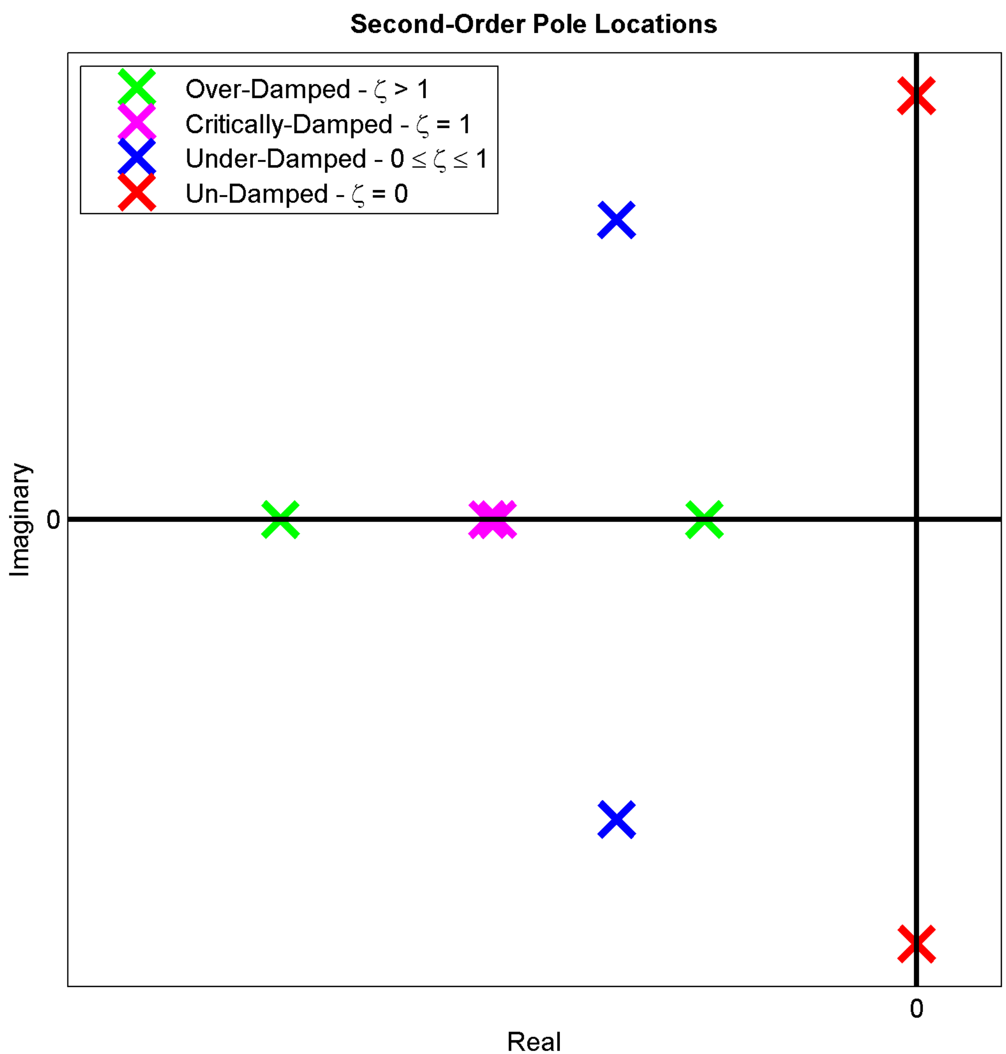
Categories of Second-Order Systems

- The 2nd-order system poles are

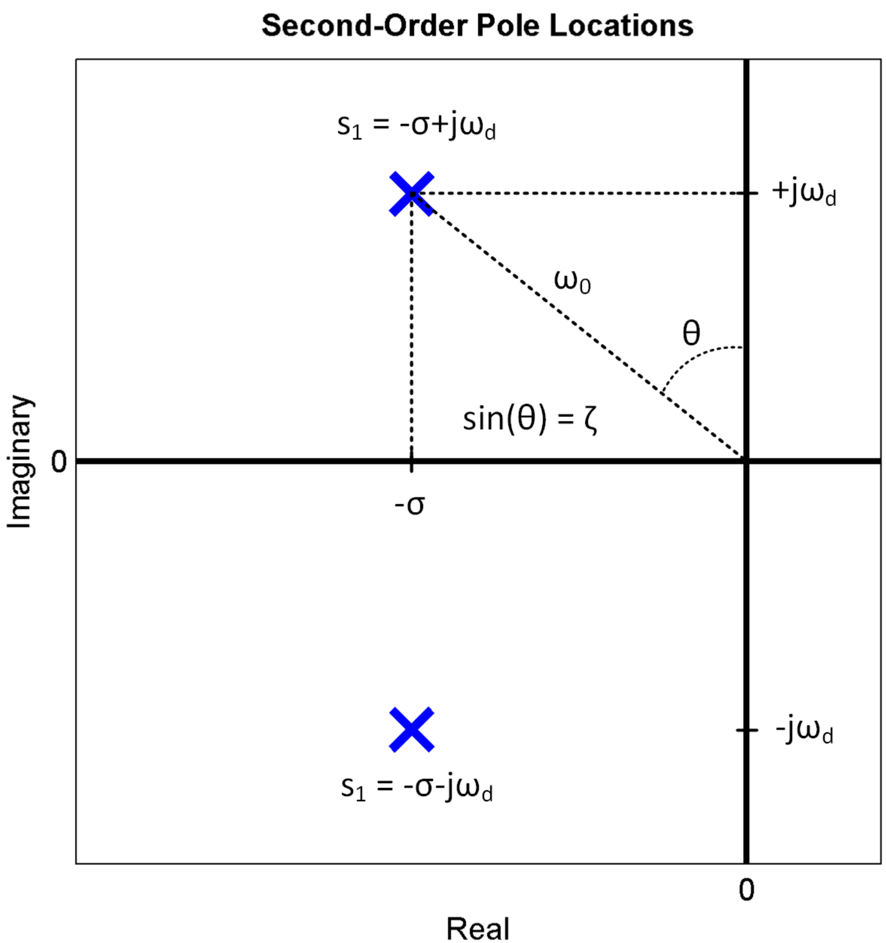
$$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

- Value of ζ determines the nature of the poles and, therefore, the response
- $\boldsymbol{\zeta > 1}$: ***<u>Over-damped</u>***
  - Two distinct, real poles – sum of decaying exponentials – treat as two first-order terms
  - $s_1 = -\sigma_1,\ s_2 = -\sigma_2$
- $\boldsymbol{\zeta = 1}$: ***<u>Critically-damped</u>***
  - Two identical, real poles – time-scaled decaying exponentials
  - $s_{1,2} = -\sigma = -\zeta\omega_0 = -\omega_0$
- $\boldsymbol{0 < \zeta < 1}$: ***<u>Under-damped</u>***
  - Complex-conjugate pair of poles – sum of decaying sinusoids
  - $s_{1,2} = -\sigma \pm j\omega_d = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}$
- $\boldsymbol{\zeta = 0}$: ***<u>Un-damped</u>***
  - Purely-imaginary, conjugate pair of poles – sum of non-decaying sinusoids
  - $s_{1,2} = \pm j\omega_0$

2^{nd}-Order Pole Locations and Damping



Second-Order Poles - $0 \le \zeta \le 1$



- Can relate σ, ω_d, ω_0, and ζ to pole location geometry
- ω_0 is the magnitude of the poles
- ζ is a measure of system damping

$$\zeta = \frac{\sigma}{\omega_0} = \sin(\theta)$$

- $\zeta = 0$
  - Two purely imaginary poles
- $\zeta = 1$
  - Two identical real poles

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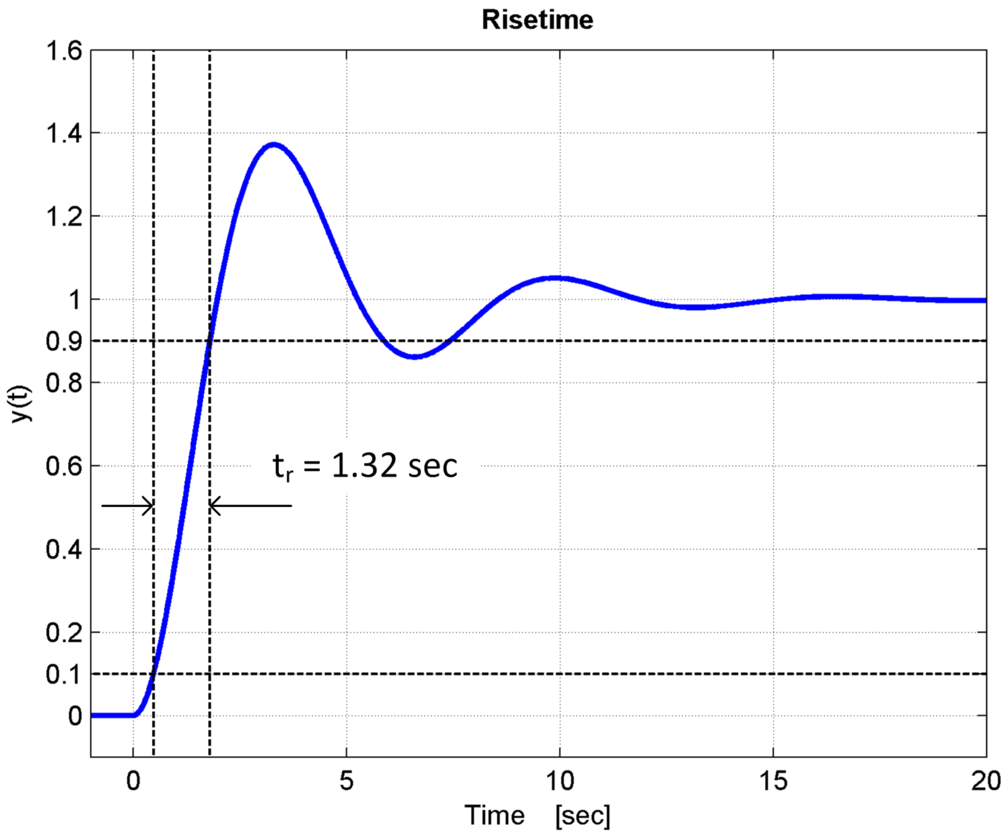
Step Response Characteristics

Step Response – Risetime

- ***Risetime*** is the time it takes a signal to transition between two set levels
  - Typically, 10% to 90% of full transition
  - Sometimes 20% to 80%
- A measure of the speed of a response
- Very rough approximation:

$$t_r \approx \frac{1.8}{\omega_0}$$

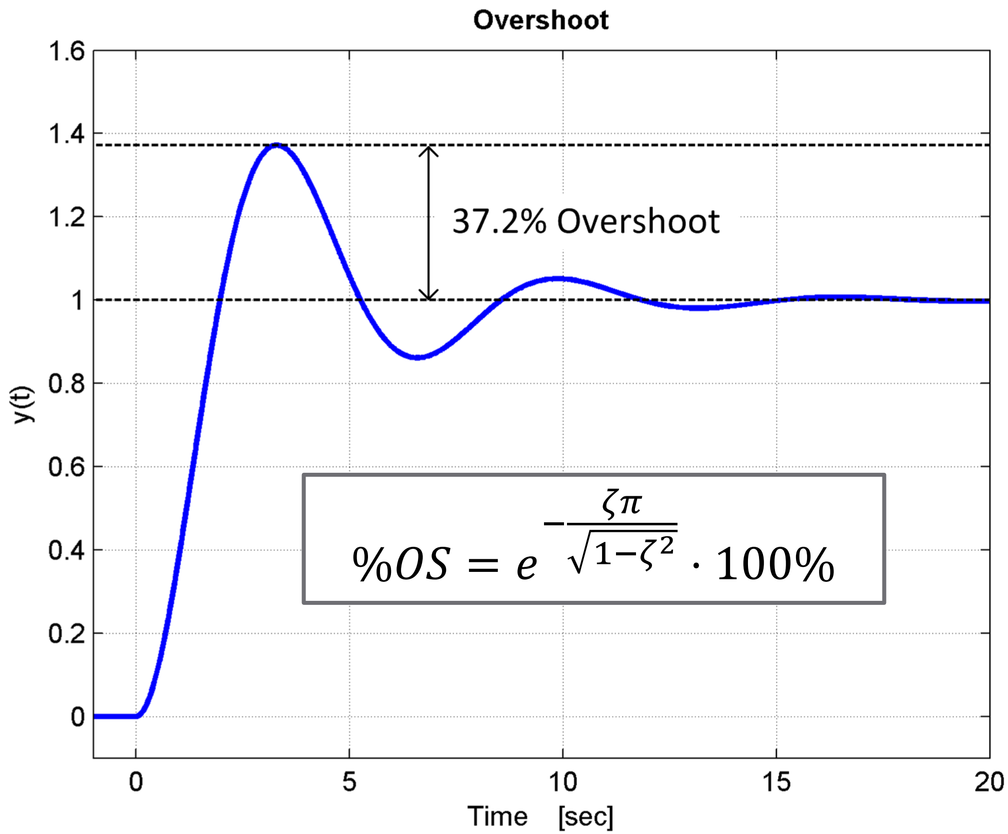


Step Response – Overshoot

- ***Overshoot*** is the magnitude of a signal's excursion beyond its final value
  - Expressed as a percentage of full-scale swing

- Overshoot increases as ζ decreases

ζ	%OS
0.45	20
0.5	16
0.6	10
0.7	5



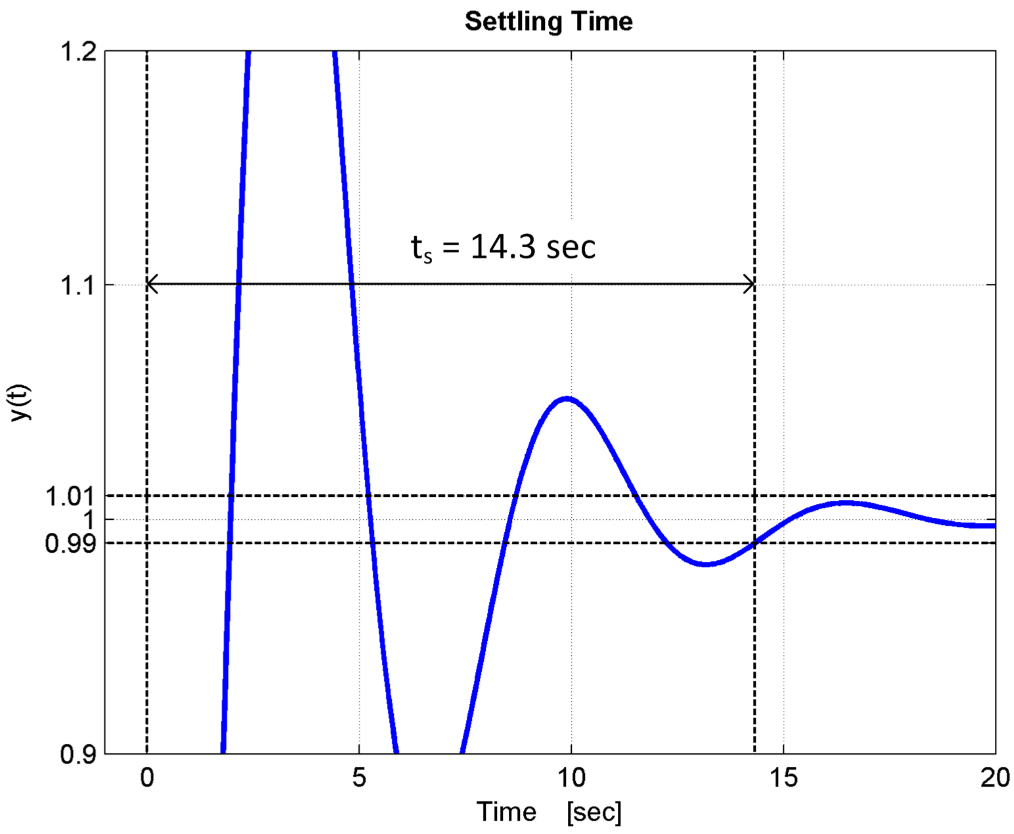
$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \cdot 100\%$$

$$\zeta = \frac{-\ln\left(\%OS/100\right)}{\sqrt{\pi^2 + \ln^2\left(\%OS/100\right)}}$$

Step Response –Settling Time

- ***Settling time*** is the time it takes a signal to settle, finally, to within some percentage of its final value
  - Typically, $\pm 1\%$ or $\pm 5\%$
- Inversely proportional to the real part of the poles, σ
- For $\pm 1\%$ settling:

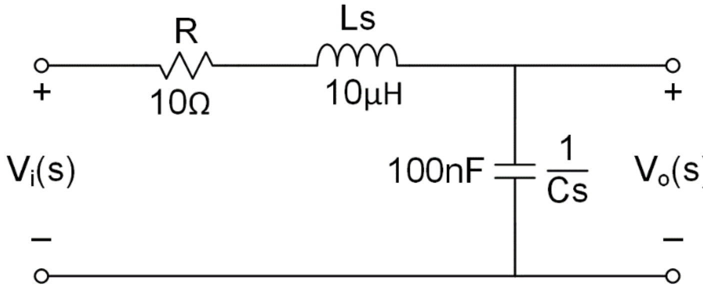
$$t_s \approx \frac{4.6}{\sigma} = \frac{4.6}{\zeta\omega_0}$$



Transfer Function & Circuit Response

Example 2 – Transfer Function

- Revisit the RLC low-pass circuit from Section 3



- Determine the transfer function
  - Apply voltage division

$$G(s) = \frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

  - Substitute component values

$$G(s) = \frac{1E12}{s^2 + 1E6s + 1E12}$$

- The characteristic polynomial:

$$s^2 + 1E6 + 1E12$$

Example 2 – Transfer Function

- Circuit poles are the roots of the characteristic polynomial

$$s_{1,2} = -500E3 \pm j866E3 = -\sigma \pm j\omega_d$$

  - Under-damped circuit

- Can write the characteristic polynomial as

$$s^2 + 1E6s + 1E12 = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

or

$$(s + 500E3)^2 + (866E3)^2 = (s + \sigma)^2 + \omega_d^2$$

so

$$\sigma = 500E3\, rad/sec$$
$$\omega_0 = 1E6\, rad/sec$$
$$\omega_d = 866E3\, rad/sec$$
$$\zeta = 0.5$$

Example 2 – Poles

- Pole locations tell us about circuit response

- Overshoot:

$$\%OS = e^{-\frac{0.5\pi}{\sqrt{1-0.5^2}}} \cdot 100\%$$

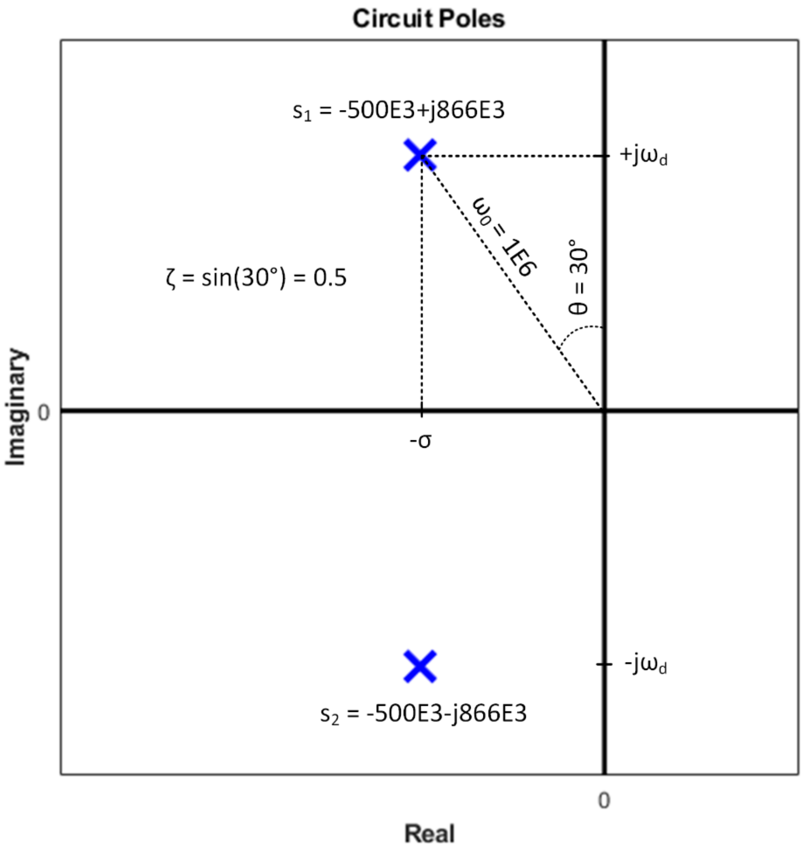
$$\%OS = 16.3\%$$

- Risetime:

$$t_r \approx \frac{1.8}{\omega_0} = \frac{1.8}{1E6} = 1.8\ \mu sec$$

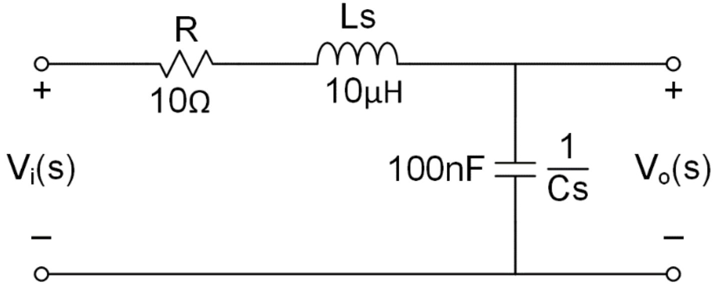
- Settling time:

$$t_s \approx \frac{4.6}{\sigma} = \frac{4.6}{500E3} = 9.2\ \mu sec$$



Example 2 – Step Response

$$G(s) = \frac{1E12}{s^2 + 1E6s + 1E12}$$



- Use the transfer function to determine the step response

$$V_o(s) = V_i(s) \cdot G(s) = \frac{1}{s} \cdot \frac{1E12}{s^2 + 1E6s + 1E12}$$

- Inverse transform via partial fraction expansion

$$V_o(s) = \frac{r_1}{s} + \frac{r_2(s + 500E3)}{(s + 500E3)^2 + (866E3)^2} + \frac{r_3 866E3}{(s + 500E3)^2 + (866E3)^2}$$

- Residues:

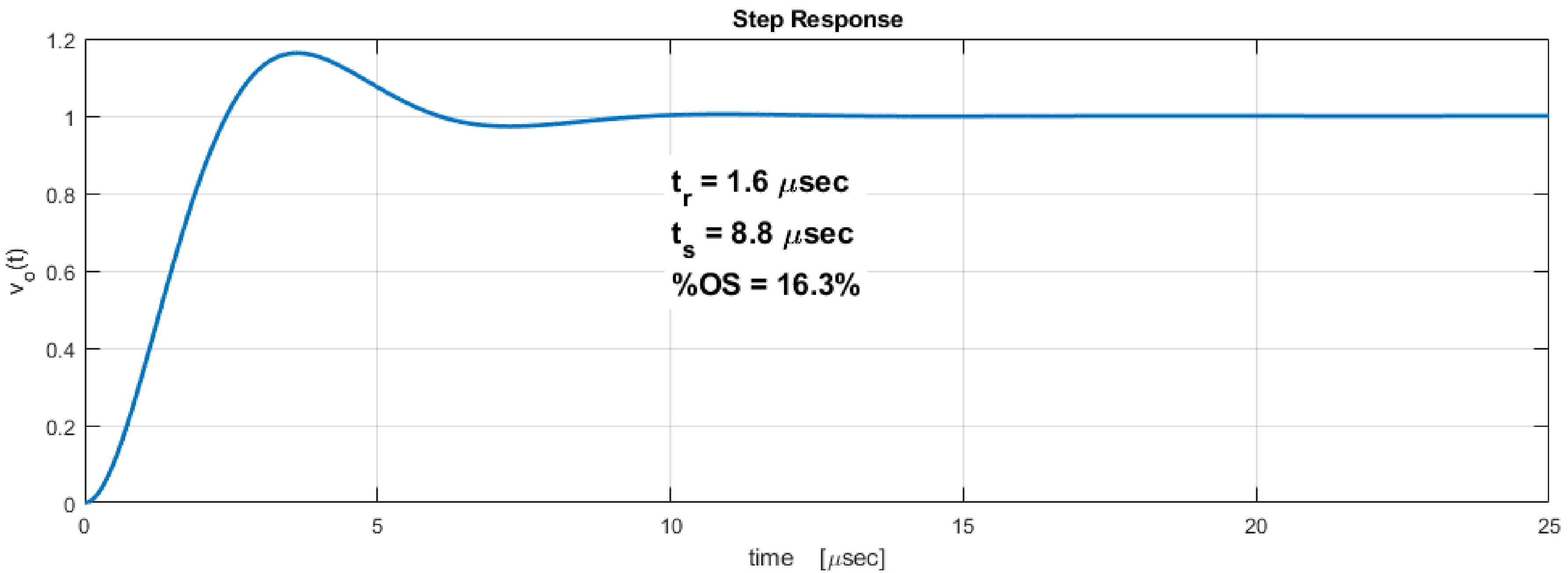
$$r_1 = 1\,,\;\; r_2 = -1\,,\;\; r_3 = -0.577$$

Example 2 – Step Response

$$V_o(s) = \frac{1}{s} - \frac{(s + 500E3)}{(s + 500E3)^2 + (866E3)^2} - \frac{0.577(866E3)}{(s + 500E3)^2 + (866E3)^2}$$

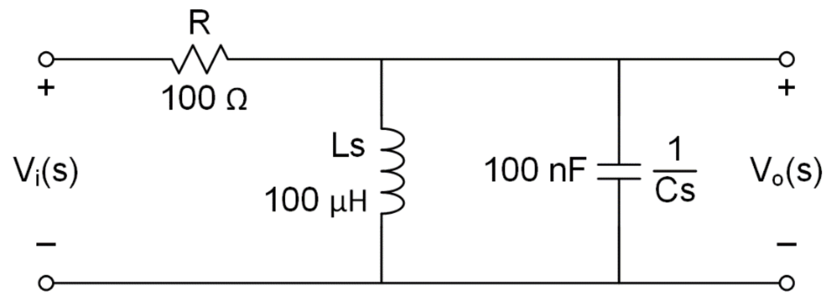
$$v_o(t) = 1\,V - 1\,V \cdot e^{-500E3t}\cos(866E3 \cdot t) - 0.577\,V \cdot e^{-500E3t}\sin(866E3 \cdot t)$$

- Step response metrics agree well with approximations



Example 3 – Impulse Response

- Determine the ***impulse response***
- Transfer function:



  - Apply voltage division

$$G(s) = \frac{\frac{Ls}{LCs^2+1}}{R+\frac{Ls}{LCs^2+1}} = \frac{\frac{1}{RC}s}{s^2+\frac{1}{RC}s+\frac{1}{LC}}$$

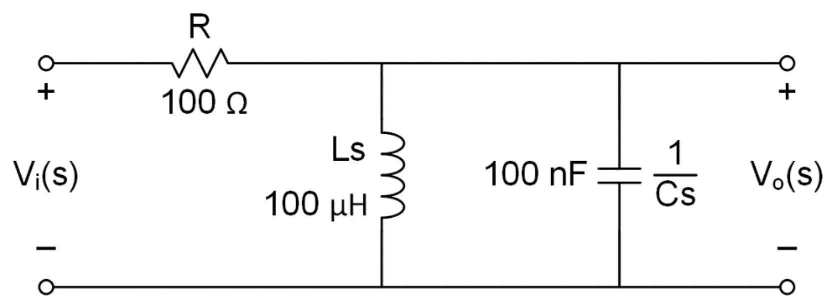
  - Substitute component values

$$G(s) = \frac{100E3s}{s^2+100E3s+100E9}$$

Example 3 – Impulse Response

- The input is an impulse function
  - Laplace transform:

$$\mathcal{L}\{\delta(t)\} = 1$$



- The output is Laplace-domain impulse response

$$V_o(s) = V_i(s) \cdot G(s) = 1 \cdot G(s) = G(s)$$

- The time-domain impulse response

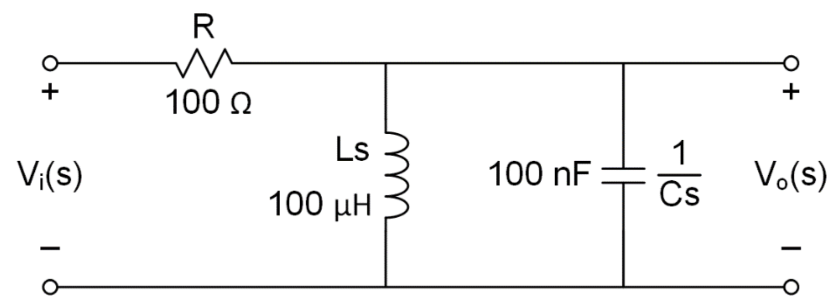
$$g(t) = v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\{G(s)\}$$

- ***The transfer function is the Laplace transform of the impulse response***

Example 3 – Impulse Response

- To determine the impulse response, inverse Laplace transform the transfer function

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$



- Partial fraction expansion:

$$G(s) = \frac{100E3s}{s^2 + 100E3s + 100E9}$$

$$G(s) = \frac{r_1(s + 50E3)}{(s + 50E3)^2 + (312\text{E}3)^2} + \frac{r_2(312E3)}{(s + 50E3)^2 + (312\text{E}3)^2}$$

- Residues:

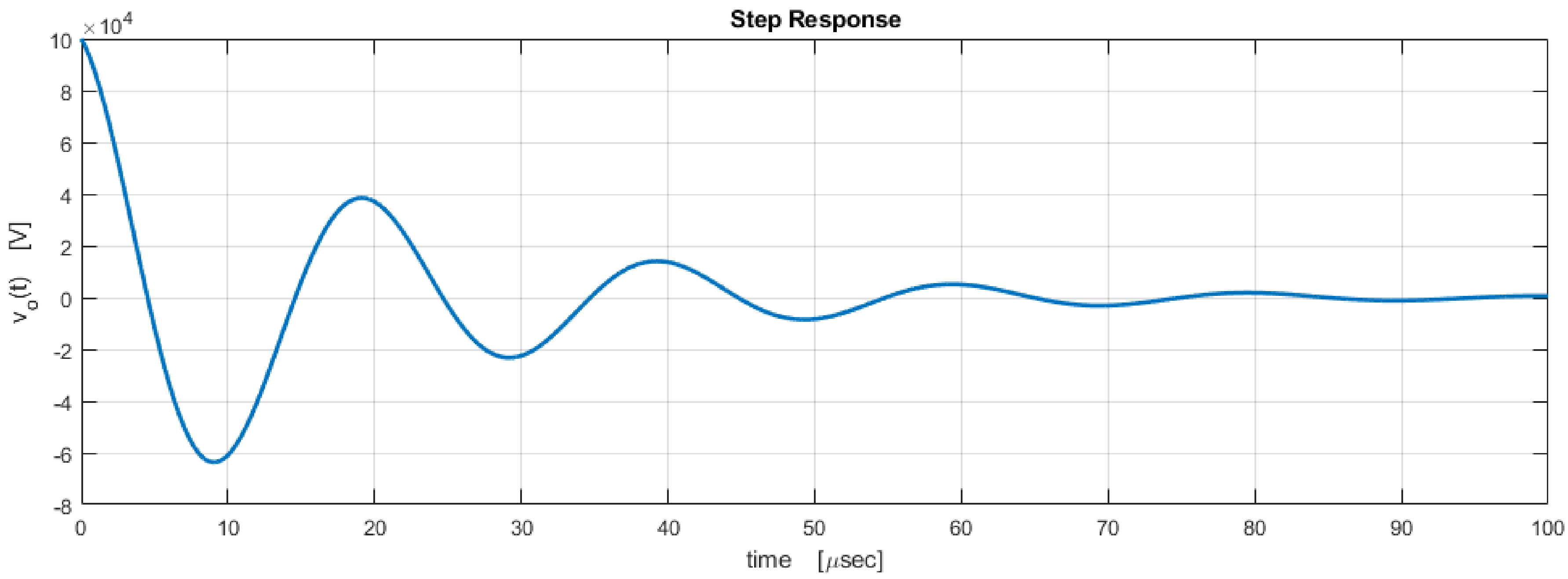
$$r_1 = 100E3 \quad \text{and} \quad r_2 = -16.03E3$$

Example 3 – Impulse Response

$$G(s) = \frac{100E3(s + 50E3)}{(s + 50E3)^2 + (312\text{E}3)^2} - \frac{16.03E3(312E3)}{(s + 50E3)^2 + (312\text{E}3)^2}$$

$$g(t) = 100E3\,V \cdot e^{-50E3t}\cos(312\text{E}3 \cdot t) - 16.03E3 \cdot e^{-50E3t}\sin(312\text{E}3 \cdot t)$$

- ***Impulse response is the inverse transform of the transfer function***



The Convolution Integral

Convolution Integral

- Laplace transform of a circuit output is given by the product of the transform of the input signal and the transfer function

$$Y(s) = G(s) \cdot U(s)$$

- Recall that ***multiplication in the Laplace domain corresponds to convolution in the time domain***

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = g(t) * u(t)$$

- ***Time-domain output given by the convolution of the input signal and the impulse response***

$$y(t) = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

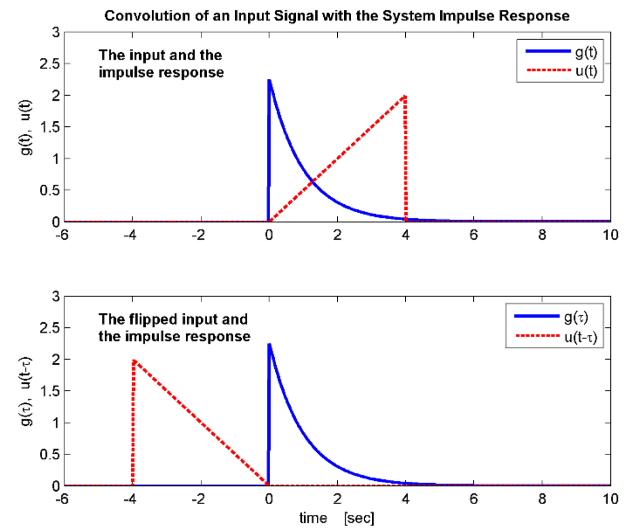
Convolution

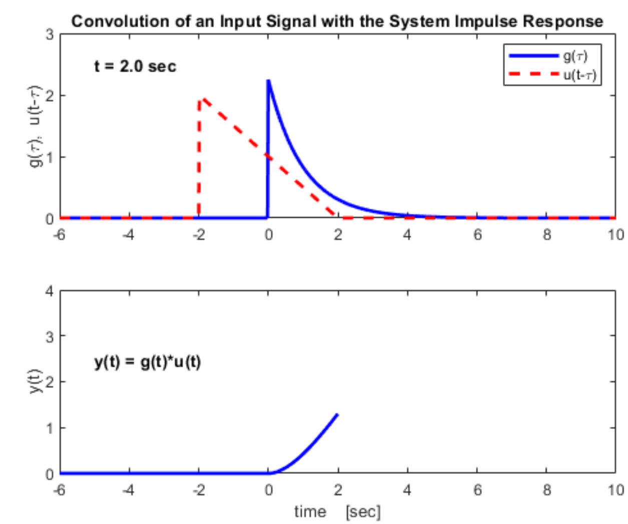
- Time-domain output is the input ***convolved*** with the impulse response

$$y(t) = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

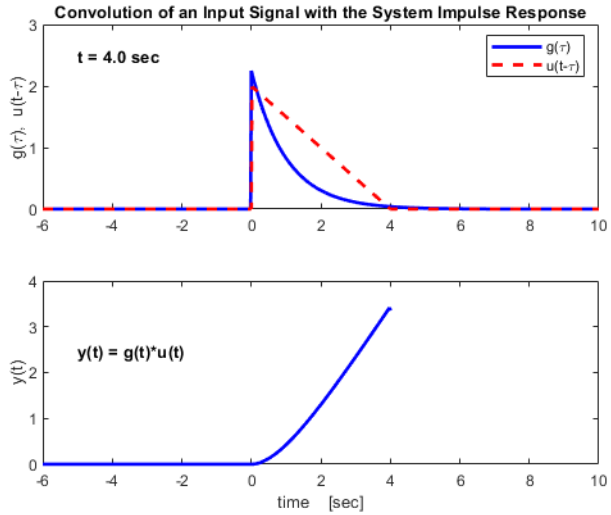
  - Input is flipped in time and shifted by t
  - Multiply impulse response and flipped/shifted input
  - Integrate over $\tau = 0 \ldots t$

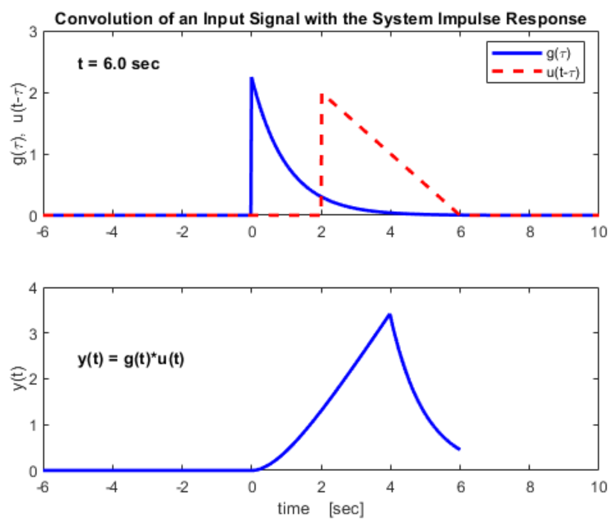
- Each time point of $y(t)$ given by result of integral with $u(-\tau)$ shifted by t

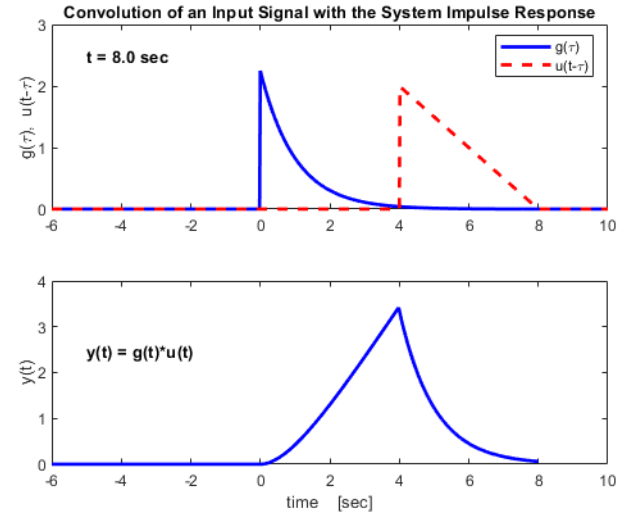


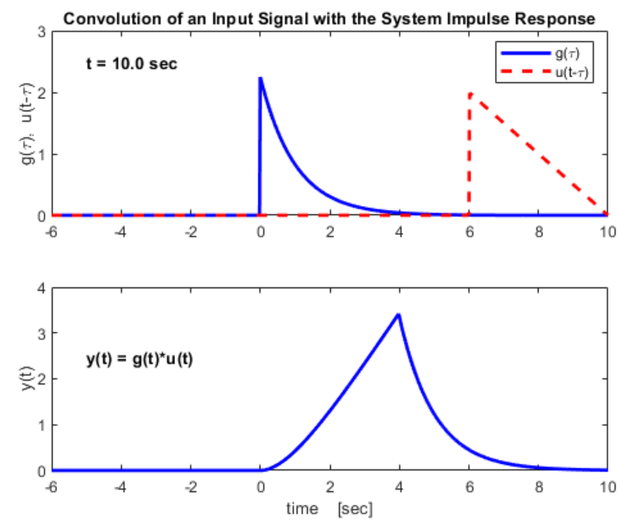


Convolution







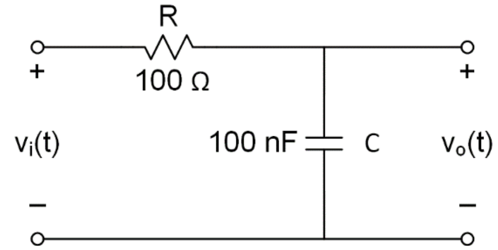


Convolution

Convolution – Example

- Determine the response of the circuit to an exponential step input

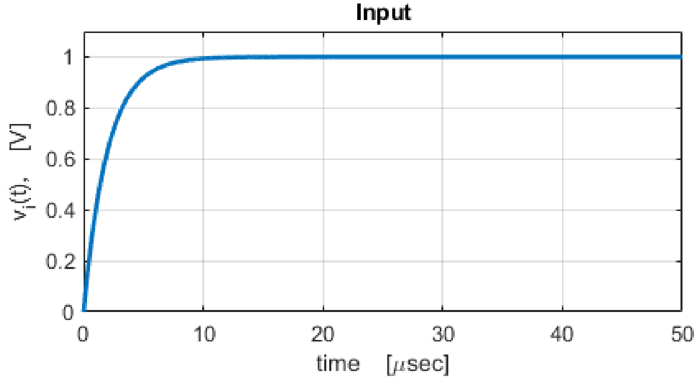
$$v_i(t) = 1\,V - 1\,Ve^{-500E3t}$$



- Circuit transfer function:

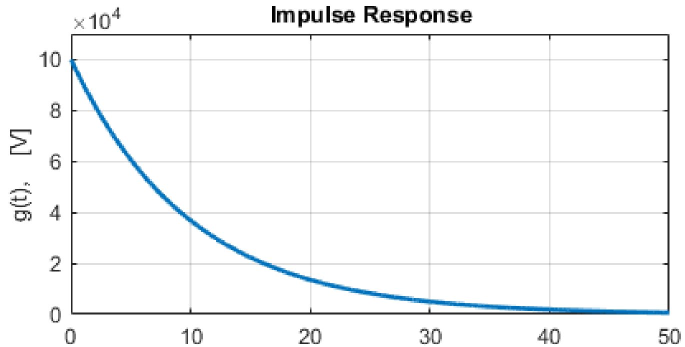
$$G(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$G(s) = \frac{100E3}{s + 100E3}$$



- Impulse response

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = 100E3\; e^{-100E3t}$$



Convolution – Example

- Output given by the convolution of the input and the impulse response

$$v_0(t) = g(t) * v_i(t)$$

$$v_o(t) = \int_0^t g(\tau) v_i(t-\tau) d\tau$$

$$v_o(t) = \int_0^t 100E3\, e^{-100E3\tau} \left(1 - e^{-500E3(t-\tau)}\right) d\tau$$

$$v_o(t) = 100E3 \left[\int_0^t e^{-100E3\tau} d\tau - \int_o^t e^{-100E3\tau} e^{-500E3(t-\tau)} d\tau \right] \qquad (1)$$

- Solving the first integral in (1)

$$100E3 \int_0^t e^{-100E3\tau} d\tau = \frac{100E3}{-100E3} e^{-100E3\tau} \Big|_0^t = -e^{-100E3\tau} \Big|_0^t$$

$$100E3 \int_0^t e^{-100E3\tau} d\tau = -e^{-100E3t} + 1 \qquad (2)$$

Convolution – Example

- Solving the second integral in (1)

$$-100E3\int_o^t e^{-100E3\tau}e^{-500E3(t-\tau)}d\tau = -100E3\int_o^t e^{-500E3t}e^{400E3\tau}d\tau$$

$$= -100E3\ e^{-500E3t}\int_o^t e^{400E3\tau}d\tau$$

$$= -100E3\ e^{-500E3t}\cdot\frac{1}{400E3}e^{400E3\tau}\Big|_0^t$$

$$= -0.25e^{-500E3t}\cdot[e^{400E3t}-1]$$

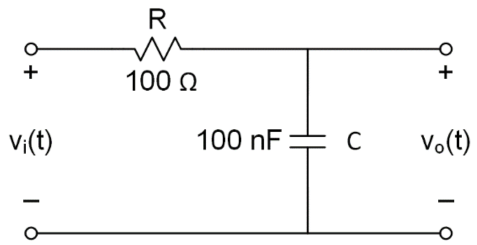
$$= -0.25e^{-100E3t}+0.25e^{-500E3t} \qquad (3)$$

- Recombining the two integral solutions, (2) and (3), we have

$$v_o(t) = 1 - e^{-100E3t} - 0.25e^{-100E3t} + 0.25e^{-500E3t}$$

$$v_o(t) = 1 - 1.25e^{-100E3t} + 0.25e^{-500E3t} \qquad (4)$$

Convolution – Example



$$v_0(t) = g(t) * v_i(t)$$

$$v_o(t) = 1 - 1.25e^{-100E3t} + 0.25e^{-500E3t}$$