SECTION 6: FREQUENCY RESPONSE ANALYSIS

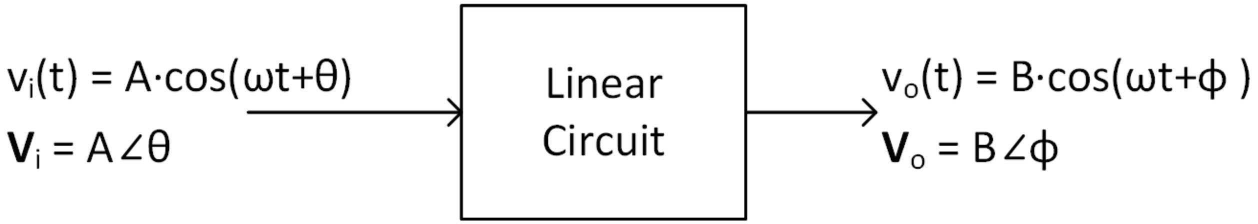
ENGR 203 – Electrical Fundamentals III

2

Frequency Response

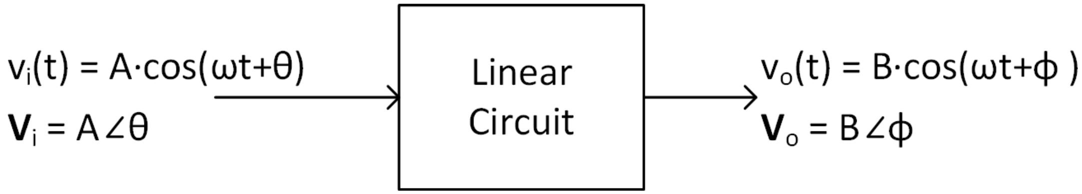
Frequency Response

- Apply a sinusoidal input to a linear circuit
  - Output is sinusoidal
  - Same frequency
  - In general, different amplitude and phase



- Input/output relationship given by circuit's ***gain*** and ***phase*** responses

Frequency Response



- **<u>Gain</u>**: ratio of output amplitude to input amplitude

$$\text{Gain} = \frac{|V_o|}{|V_i|} = \frac{B}{A}$$

- **<u>Phase</u>**: phase shift from input to output

$$\text{Phase} = \angle V_o - \angle V_i = \phi - \theta$$

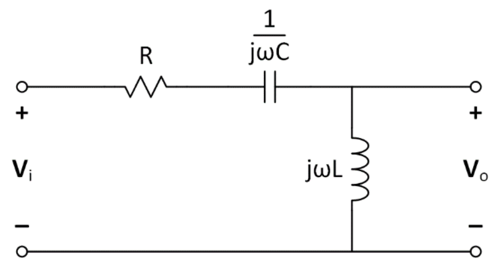
- Gain and phase relationships given by the circuit's ***frequency response function***, $G(j\omega)$

$$\text{Gain} = |G(j\omega)|$$

$$\text{Phase} = \angle G(j\omega)$$

Frequency Response

- In ENGR 202, we saw how to derive a circuit's frequency response function in the ***phasor domain***
- For example:



$$\frac{\mathbf{V}_o}{\mathbf{V}_i} = G(j\omega) = \frac{j\omega L}{j\omega L + R + \dfrac{1}{j\omega C}}$$

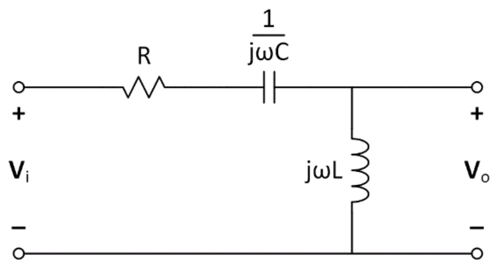
$$G(j\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC + RCj\omega + 1}$$

$$G(j\omega) = \frac{(j\omega)^2}{(j\omega)^2 + \dfrac{R}{L} j\omega + \dfrac{1}{LC}}$$

Frequency Response

- $G(j\omega)$ is the ***frequency response function***
  - Ratio of ***output phasor*** to ***input phasor***

$$G(j\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i}$$



$$G(j\omega) = \frac{(j\omega)^2}{(j\omega)^2 + \frac{R}{L} j\omega + \frac{1}{LC}}$$

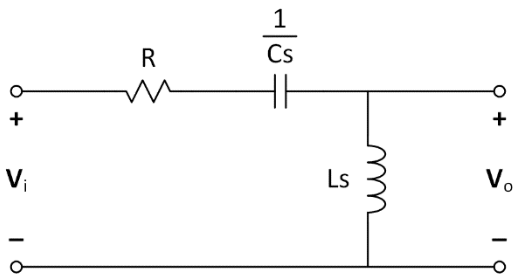
- A complex-valued function of frequency
  - $|G(j\omega)|$ at each ω is the ***gain*** at that frequency
    - Ratio of output amplitude to input amplitude

$$|G(j\omega)| = \frac{|\mathbf{V}_o|}{|\mathbf{V}_i|}$$

  - $\angle G(j\omega)$ at each ω is the ***phase*** at that frequency
    - Phase shift between input and output sinusoids

$$\angle G(j\omega) = \angle \mathbf{V}_o - \angle \mathbf{V}_i$$

Frequency Response



- Consider the same circuit transformed to the Laplace domain
- The transfer function is

$$G(s) = \frac{Ls}{Ls + Rs + \frac{1}{Cs}} = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

- Note the similarity to the frequency response function:

$$G(s) = \frac{s^2}{s^2+\frac{R}{L}s+\frac{1}{LC}} \qquad G(j\omega) = \frac{(j\omega)^2}{(j\omega)^2+\frac{R}{L}j\omega+\frac{1}{LC}}$$

- ***The frequency response function is the transfer function with s replaced by jω***

$$G(j\omega) = G(s)\Big|_{s\rightarrow j\omega}$$

Plotting the Frequency Response Function

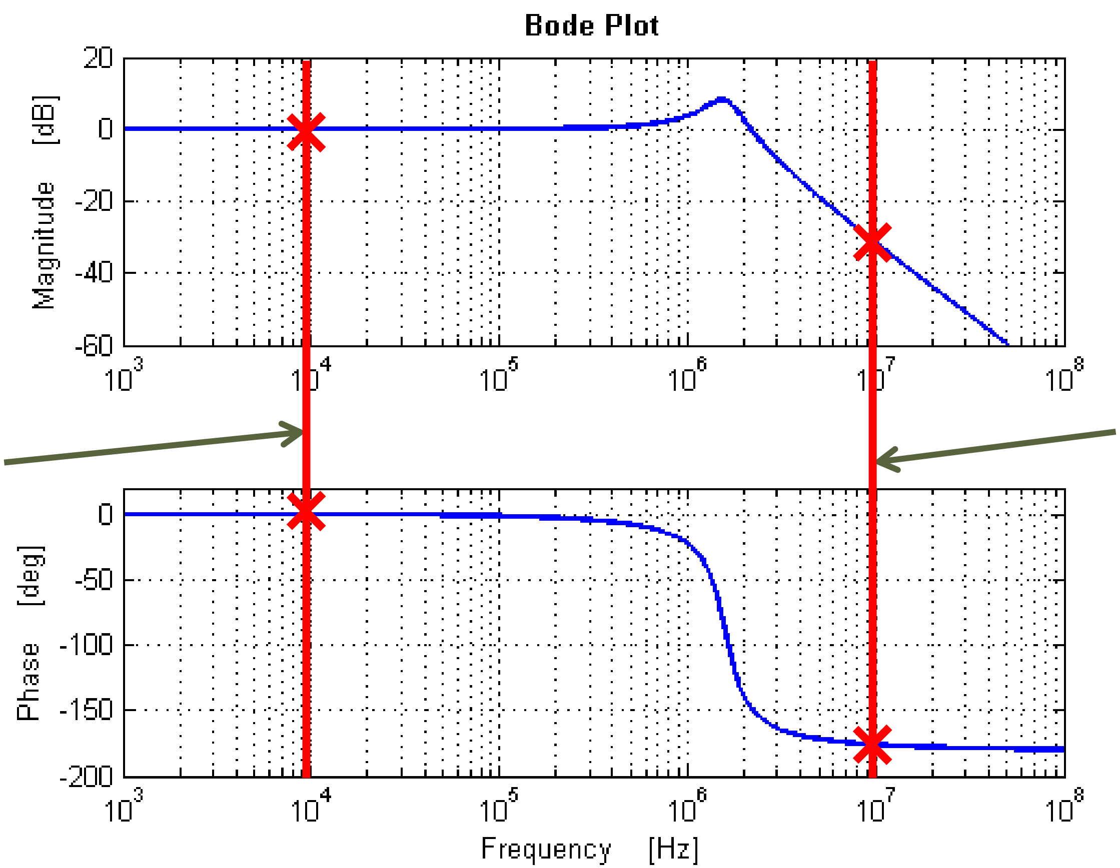
- $G(j\omega)$ is a complex-valued function of frequency
  - Has both magnitude and phase
  - Plot gain and phase separately
- Frequency response plots formatted as ***<u>Bode plots</u>***
  - Two sets of axes: gain on top, phase below
  - Identical, logarithmic frequency axes
  - Gain axis is logarithmic – either explicitly or as units of decibels (dB)
  - Phase axis is linear with units of degrees

Interpreting Bode Plots

Bode plots tell you the gain and phase shift at all frequencies:

choose a frequency, read gain and phase values from the plot

For a 10KHz sinusoidal input, the gain is 0dB (1) and the phase shift is 0°.



For a 10MHz sinusoidal input, the gain is -32dB (0.025), and the phase shift is -176°.

Response of 1st- and 2nd-Order Factors

Transfer Function Factors

- We've already seen that a transfer function denominator can be factored into first- and second-order terms

$$G(s)=\frac{Num(s)}{(s-p_1)(s-p_2)\cdots\left(s^2+2\zeta_1\omega_{0,1}s+\omega_{0,1}^2\right)\left(s^2+2\zeta_2\omega_{0,2}s+\omega_{0,2}^2\right)\cdots}$$

- The same is true of the numerator

$$G(s)=\frac{(s-z_1)(s-z_2)\cdots\left(s^2+2\zeta_a\omega_{0,a}s+\omega_{0,a}^2\right)\left(s^2+2\zeta_2\omega_{0,b}s+\omega_{0,b}^2\right)\cdots}{(s-p_1)(s-p_2)\cdots\left(s^2+2\zeta_1\omega_{0,1}s+\omega_{0,1}^2\right)\left(s^2+2\zeta_2\omega_{0,2}s+\omega_{0,2}^2\right)\cdots}$$

- Can think of the transfer function as a product of the individual factors
- For example, consider the following system

$$G(s)=\frac{(s-z_1)}{(s-p_1)\left(s^2+2\zeta_1\omega_{0,1}s+\omega_{0,1}^2\right)}$$

- Can rewrite as

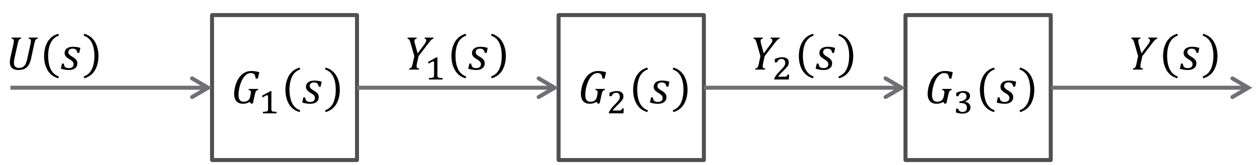
$$G(s)=(s-z_1)\cdot\frac{1}{(s-p_1)}\cdot\frac{1}{\left(s^2+2\zeta_1\omega_{0,1}s+\omega_{0,1}^2\right)}$$

Transfer Function Factors

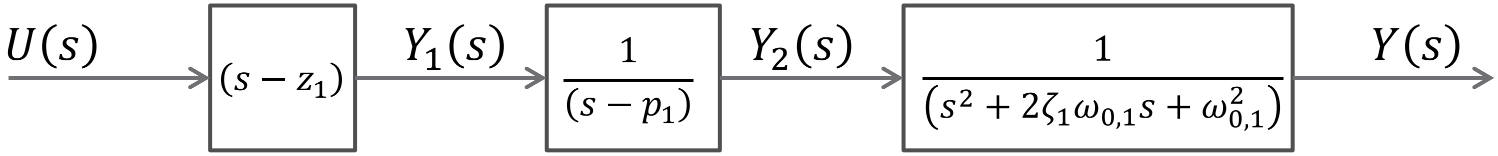
$$G(s) = (s - z_1) \cdot \frac{1}{(s - p_1)} \cdot \frac{1}{\left(s^2 + 2\zeta_1\omega_{0,1}s + \omega_{0,1}^2\right)}$$

- Think of this as three cascaded transfer functions

$$G_1(s) = (s - z_1), \quad G_2(s) = \frac{1}{(s-p_1)}, \quad G_3(s) = \frac{1}{\left(s^2+2\zeta_1\omega_{0,1}s+\omega_{0,1}^2\right)}$$

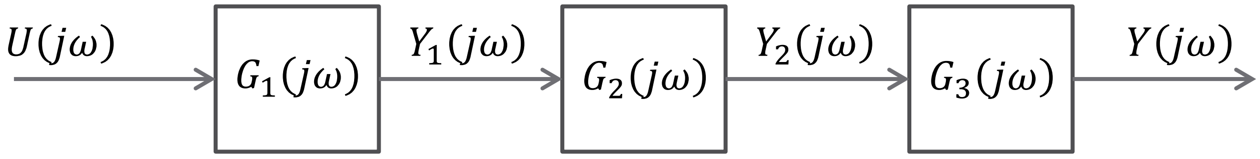


or



Transfer Function Factors

- Overall transfer function – and therefore, frequency response – is the product of individual first- and second-order factors
- Instructive, therefore, to understand the responses of the individual factors
  - ***First- and second-order poles and zeros***



First-Order Factors

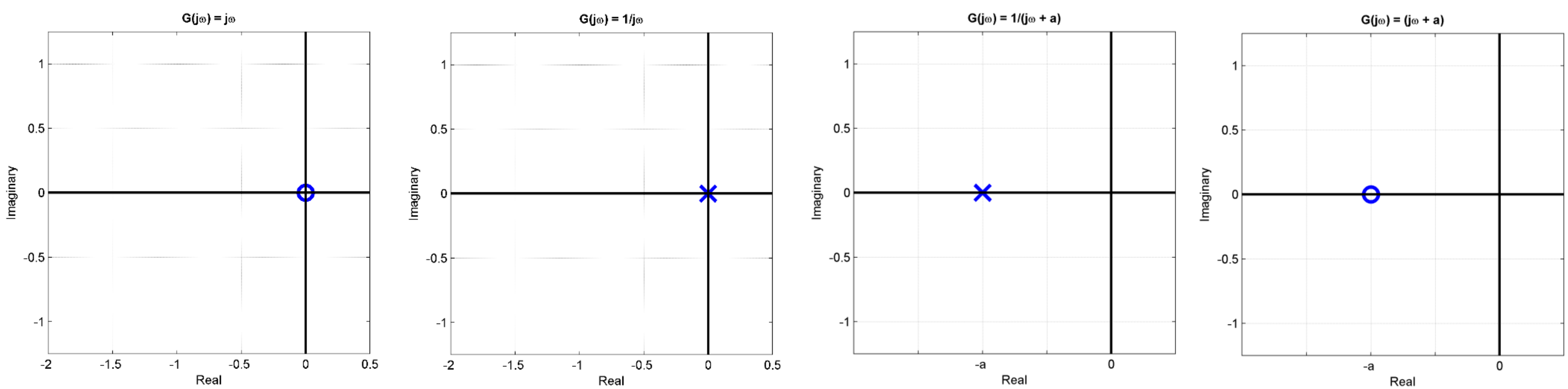
- First-order factors
  - Single, real poles or zeros
- In the Laplace domain:

$$G(s) = s, \quad G(s) = \frac{1}{s}, \quad G(s) = s + a, \quad G(s) = \frac{1}{s+a}$$

- In the frequency domain

$$G(j\omega) = j\omega, \quad G(j\omega) = \frac{1}{j\omega}, \quad G(j\omega) = j\omega + a, \quad G(j\omega) = \frac{1}{j\omega+a}$$

- Pole/zero plots:



First-Order Factors – Zero at the Origin

- A ***differentiator***

$$G(s) = s$$

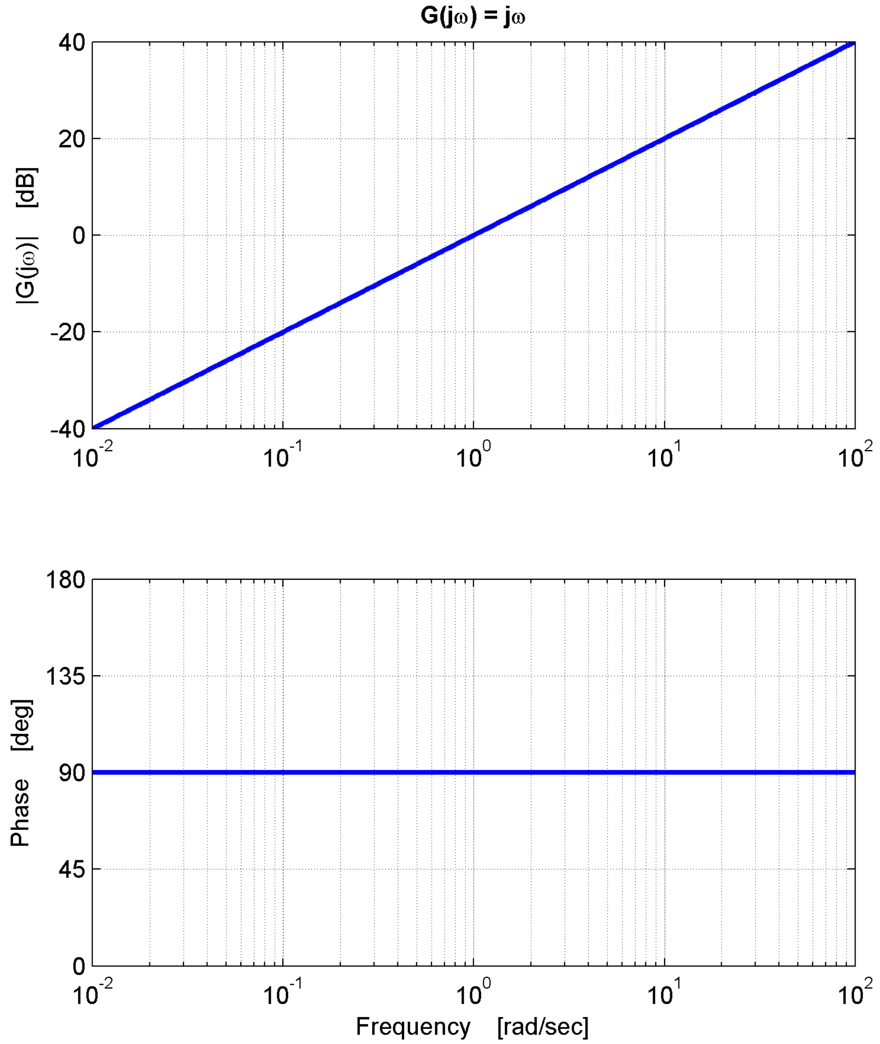
$$G(j\omega) = j\omega$$

- **Gain**:

$$|G(j\omega)| = |j\omega| = \omega$$

- **Phase**:

$$\angle G(j\omega) = +90°, \quad \forall\omega$$



First-Order Factors – Pole at the Origin

- An ***integrator***

$$G(s) = \frac{1}{s}$$

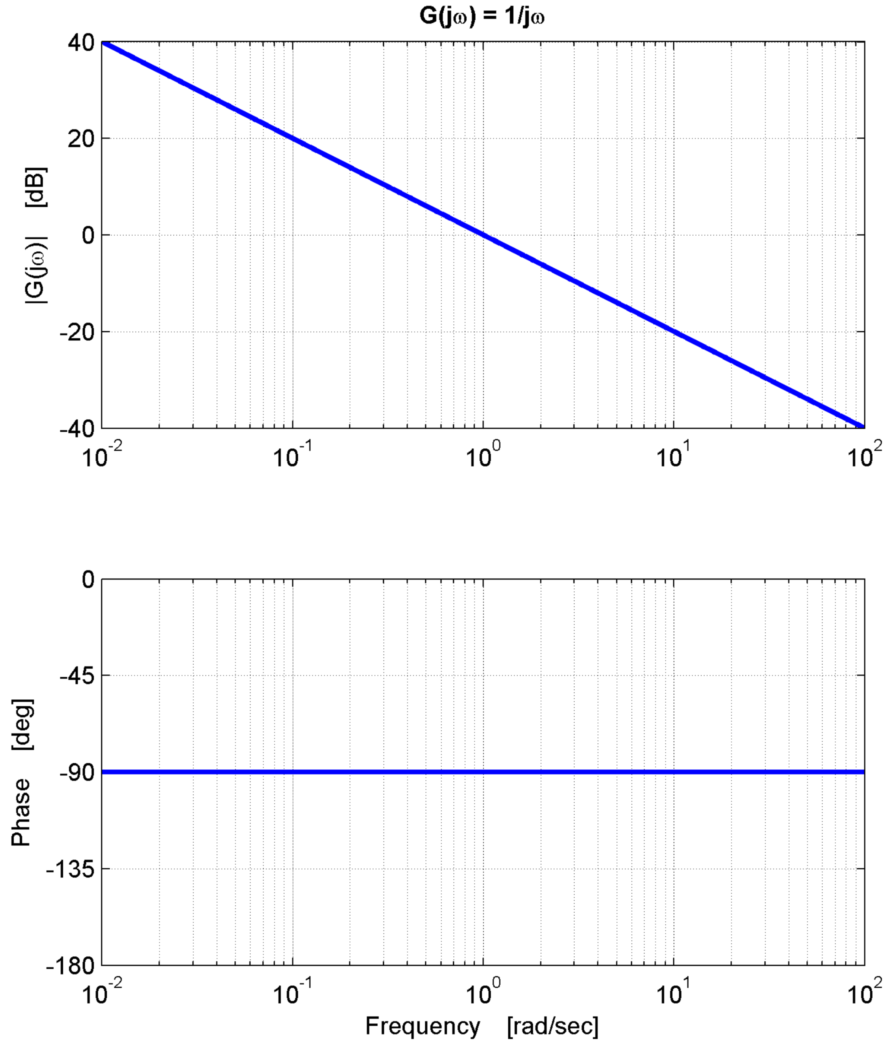
$$G(j\omega) = \frac{1}{j\omega}$$

- **Gain**:

$$|G(j\omega)| = \left|\frac{1}{j\omega}\right| = \frac{1}{\omega}$$

- **Phase**:

$$\angle G(j\omega) = \angle - j\frac{1}{\omega} = -90°, \quad \forall \omega$$



First-Order Factors – Single, Real Zero

- Single, real zero at $s = -a$

$$G(j\omega) = j\omega + a$$

- **Gain**:

$$|G(j\omega)| = \sqrt{\omega^2 + a^2}$$

for $\omega \ll a$

$$|G(j\omega)| \approx a$$

for $\omega \gg a$

$$|G(j\omega)| \approx \omega$$

- **Phase**:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$$

for $\omega \ll a$

$$\angle G(j\omega) \approx \angle a = 0°$$

for $\omega \gg a$

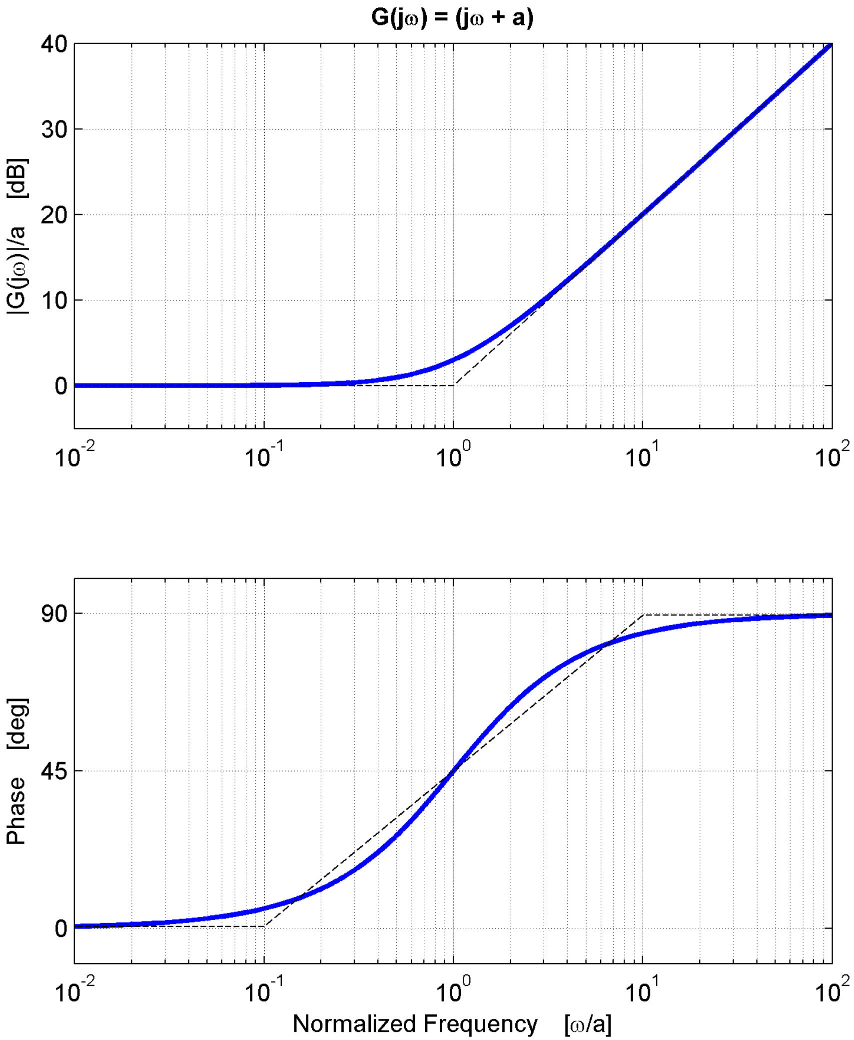
$$\angle G(j\omega) \approx \angle j\omega = 90°$$

First-Order Factors – Single, Real Zero

- Corner frequency:

  $$\omega_c = a$$

  - $|G(j\omega_c)| = a\sqrt{2} = 1.414 \cdot a$
  - $|G(j\omega_c)|_{dB} = (a)_{dB} + 3dB$
  - $\angle G(j\omega_c) = +45°$
- For $\omega \gg \omega_c$, gain increases at:
  - $20dB/dec$
  - $6dB/oct$
- From $\sim 0.1\omega_c$ to $\sim 10\omega_c$, phase increases at a rate of:
  - $\sim 45°/dec$
  - Rough approximation



First-Order Factors – Single, Real Pole

- Single, real pole at $s = -a$

$$G(j\omega) = \frac{1}{j\omega + a}$$

- **Gain**:

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2 + a^2}}$$

for $\omega \ll a$

$$|G(j\omega)| \approx \frac{1}{a}$$

for $\omega \gg a$

$$|G(j\omega)| \approx \frac{1}{\omega}$$

- **Phase**:

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

for $\omega \ll a$

$$\angle G(j\omega) \approx \angle \frac{1}{a} = 0°$$

for $\omega \gg a$

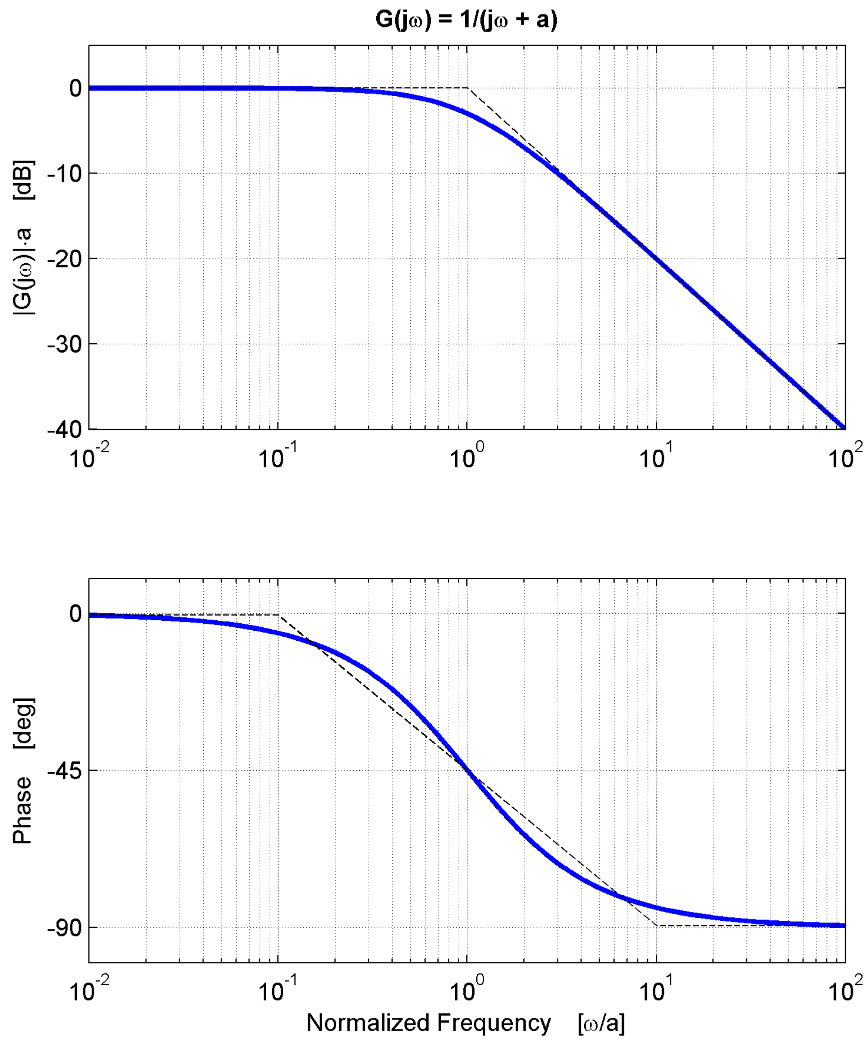
$$\angle G(j\omega) \approx \angle \frac{1}{j\omega} = -90°$$

First-Order Factors – Single, Real Pole

- Corner frequency:

$$\omega_c = a$$

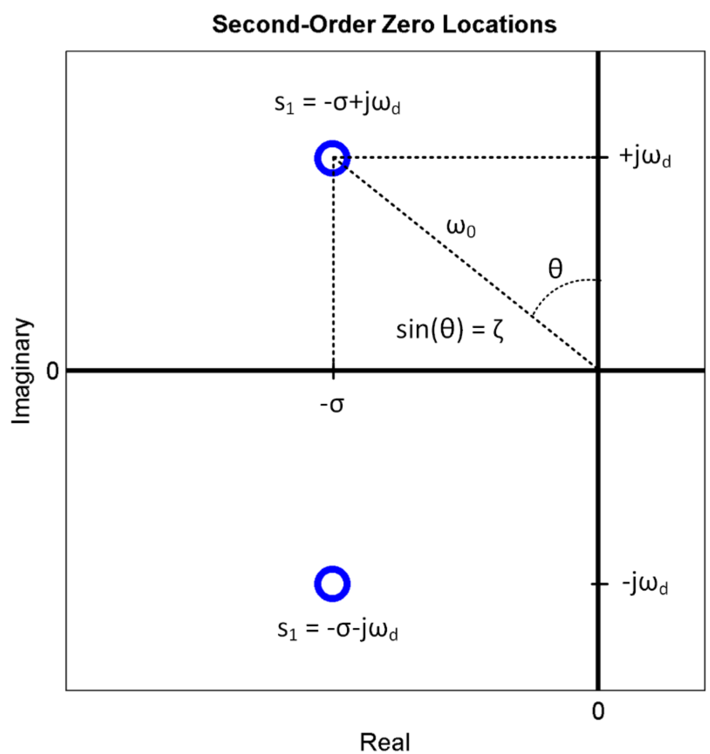
  - $|G(j\omega_c)| = \frac{1}{a\sqrt{2}} = 0.707 \cdot \frac{1}{a}$
  - $|G(j\omega_c)|_{dB} = \left(\frac{1}{a}\right)_{dB} - 3dB$
  - $\angle G(j\omega_c) = -45°$
- For $\omega \gg \omega_c$, gain decreases at:
  - $-20dB/dec$
  - $-6dB/oct$
- From $\sim 0.1\omega_c$ to $\sim 10\omega_c$, phase decreases at a rate of:
  - $\sim -45°/dec$
  - Rough approximation



Second-Order Factors

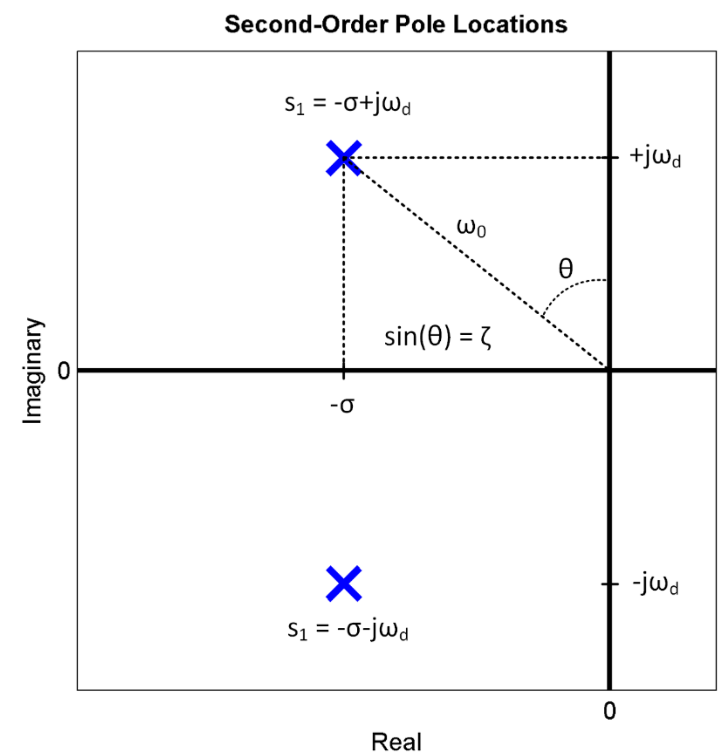
- Complex-conjugate zeros

$$G(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$



- Complex-conjugate poles

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



$$\sigma = \zeta\omega_0, \ \omega_d = \omega_0\sqrt{1-\zeta^2}$$

2^{nd}-Order Factors – Complex-Conjugate Zeros

- Complex-conjugate zeros at $s = -\sigma \pm j\omega_d$

$$G(j\omega) = (j\omega)^2 + 2\zeta\omega_0(j\omega) + \omega_0^2$$

- **Gain**:

for $\omega \ll \omega_0$

$$|G(j\omega)| \approx \omega_0^2$$

for $\omega = \omega_0$

$$|G(j\omega)| = 2\zeta\omega_0^2$$

for $\omega \gg \omega_0$

$$|G(j\omega)| \approx \omega^2$$

- **Phase**:

for $\omega \ll \omega_0$

$$\angle G(j\omega) \approx \angle \omega_0^2 = 0°$$

for $\omega = \omega_0$

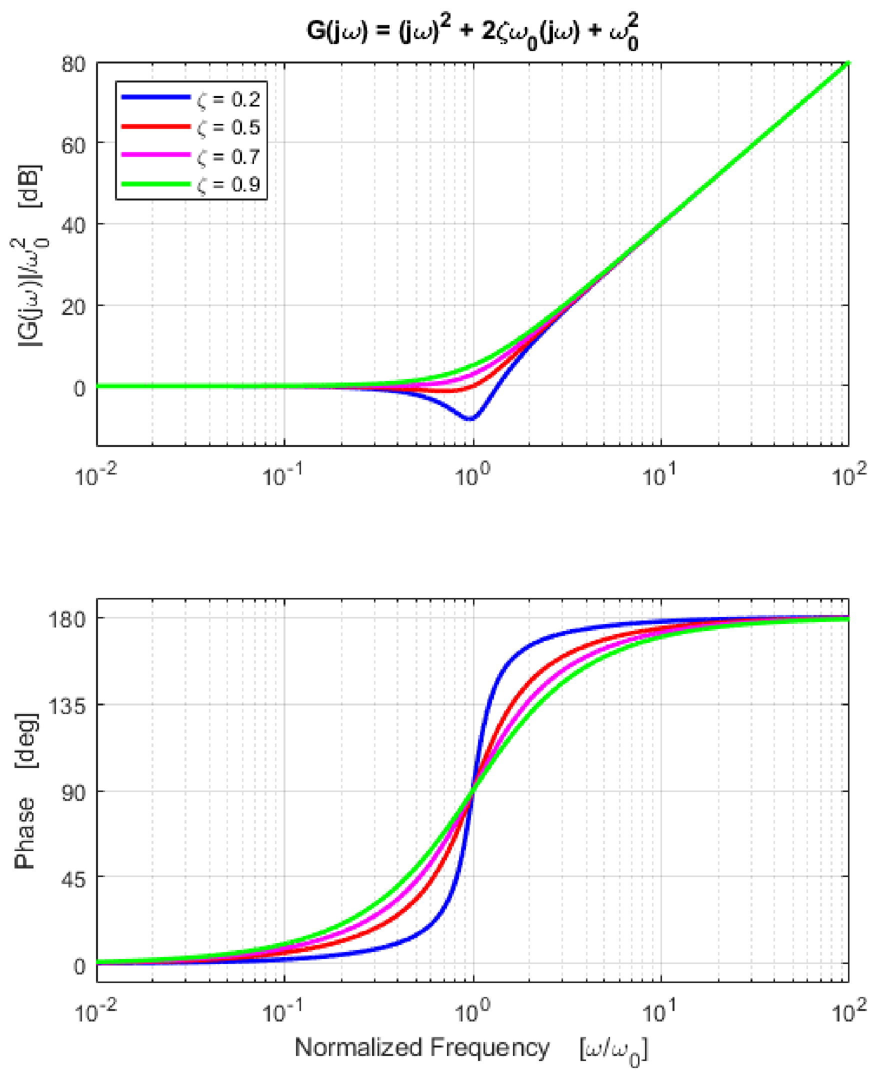
$$\angle G(j\omega) = \angle j2\zeta\omega_0^2 = +90°$$

for $\omega \gg \omega_0$

$$\angle G(j\omega) \approx \angle -\omega^2 = +180°$$

2^{nd}-Order Factors – Complex-Conjugate Zeros

- Response may dip below low-freq. value near ω_0
  - Peaking increases as ζ decreases
- Gain increases at $+40dB/dec$ or $+12dB/oct$ for $\omega \gg \omega_0$
- Corner frequency depends on damping ratio, ζ
  - ω_c increases as ζ decreases
- At $\omega = \omega_c, \angle G(j\omega) = 90°$
- Phase transition abruptness depends on ζ



2^{nd}-Order Factors – Complex-Conjugate Poles

- Complex-conjugate poles at $s = -\sigma \pm j\omega_d$

$$G(j\omega) = \frac{1}{(j\omega)^2 + 2\zeta\omega_0(j\omega) + \omega_0^2}$$

- **Gain**:

for $\omega \ll \omega_0$

$$|G(j\omega)| \approx \frac{1}{\omega_0^2}$$

for $\omega = \omega_0$

$$|G(j\omega)| = \frac{1}{2\zeta\omega_0^2}$$

for $\omega \gg \omega_0$

$$|G(j\omega)| \approx \frac{1}{\omega^2}$$

- **Phase**:

for $\omega \ll \omega_0$

$$\angle G(j\omega) \approx \angle \frac{1}{\omega_0^2} = 0°$$

for $\omega = \omega_0$

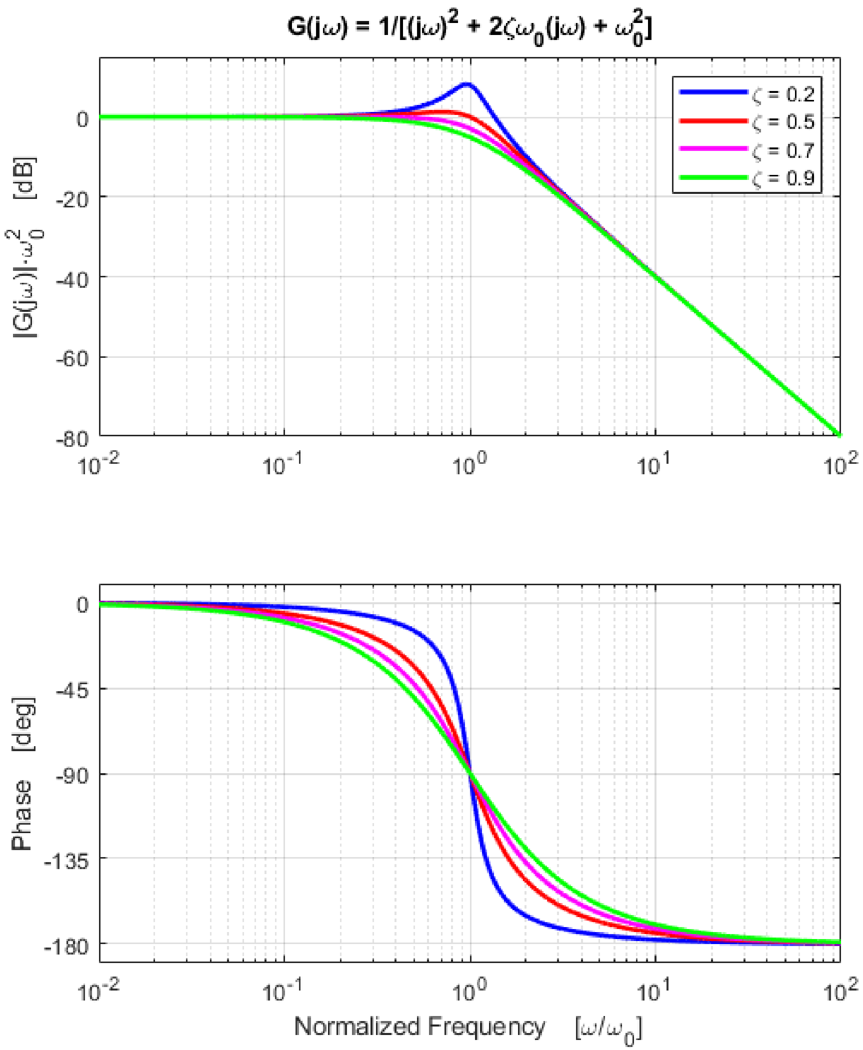
$$\angle G(j\omega) = \angle \frac{1}{j2\zeta\omega_0^2} = -90°$$

for $\omega \gg \omega_0$

$$\angle G(j\omega) \approx \angle -\frac{1}{\omega^2} = -180°$$

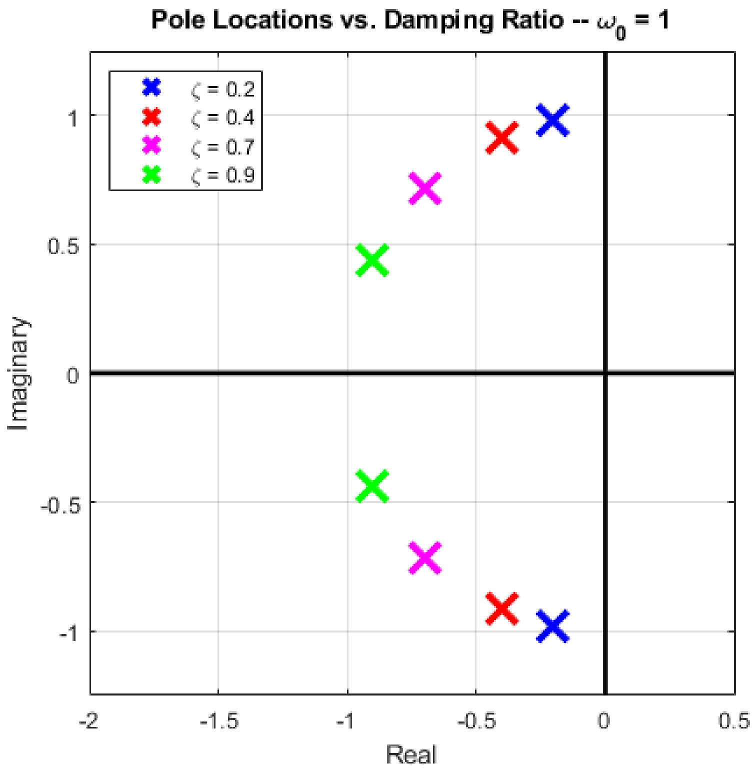
2^{nd}-Order Factors – Complex-Conjugate Poles

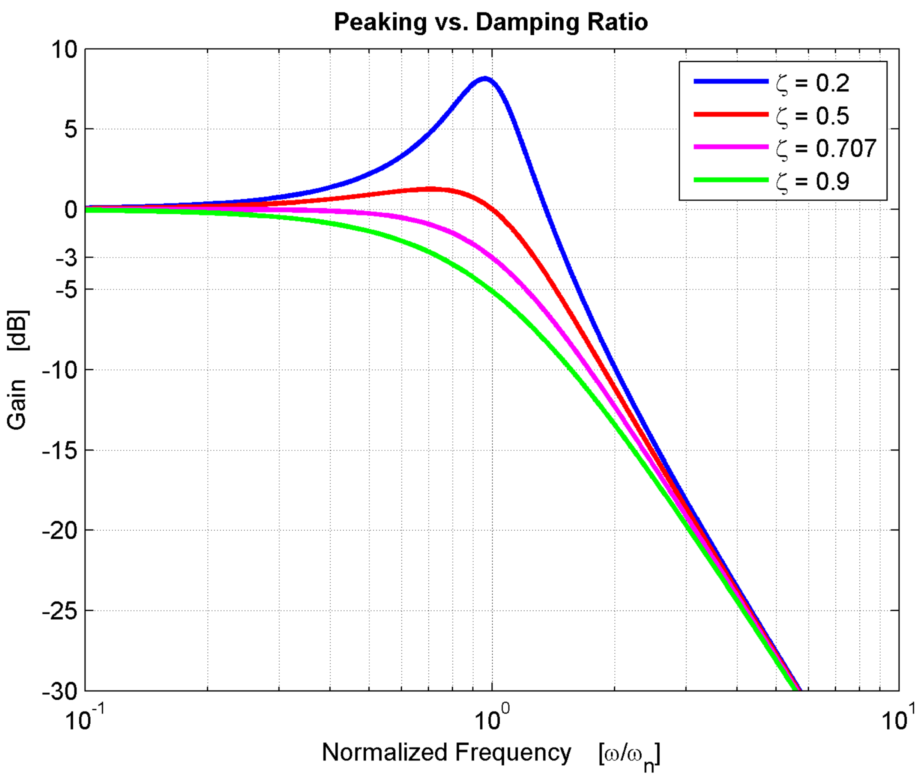
- Response may peak above low-freq. value near ω_0
  - Peaking increases as ζ decreases
- Gain decreases at $-40dB/dec$ or $-12dB/oct$ for $\omega \gg \omega_0$
- Corner frequency depends on damping ratio, ζ
  - ω_c increases as ζ decreases
- At $\omega = \omega_c, \angle G(j\omega) = -90°$
- Phase transition abruptness depends on ζ



Pole Location and Peaking

- Peaking is dependent on ζ – pole locations
  - No peaking at all for $\zeta \geq 1/\sqrt{2} = 0.707$
  - $\zeta = 0.707$ – ***maximally-flat*** or ***Butterworth*** response





Frequency Response Components - Example

- Consider the following system

$$G(s) = \frac{20(s+20)}{(s+1)(s+100)}$$

- The system's frequency response function is

$$G(j\omega) = \frac{20(j\omega+20)}{(j\omega+1)(j\omega+100)}$$

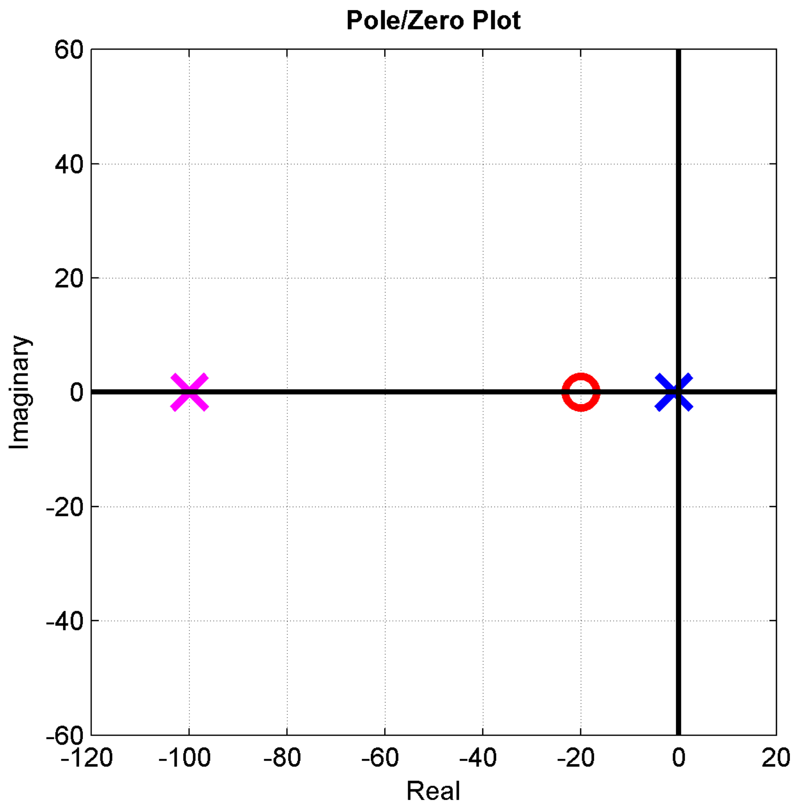
- As we've seen we can consider this a product of individual frequency response factors

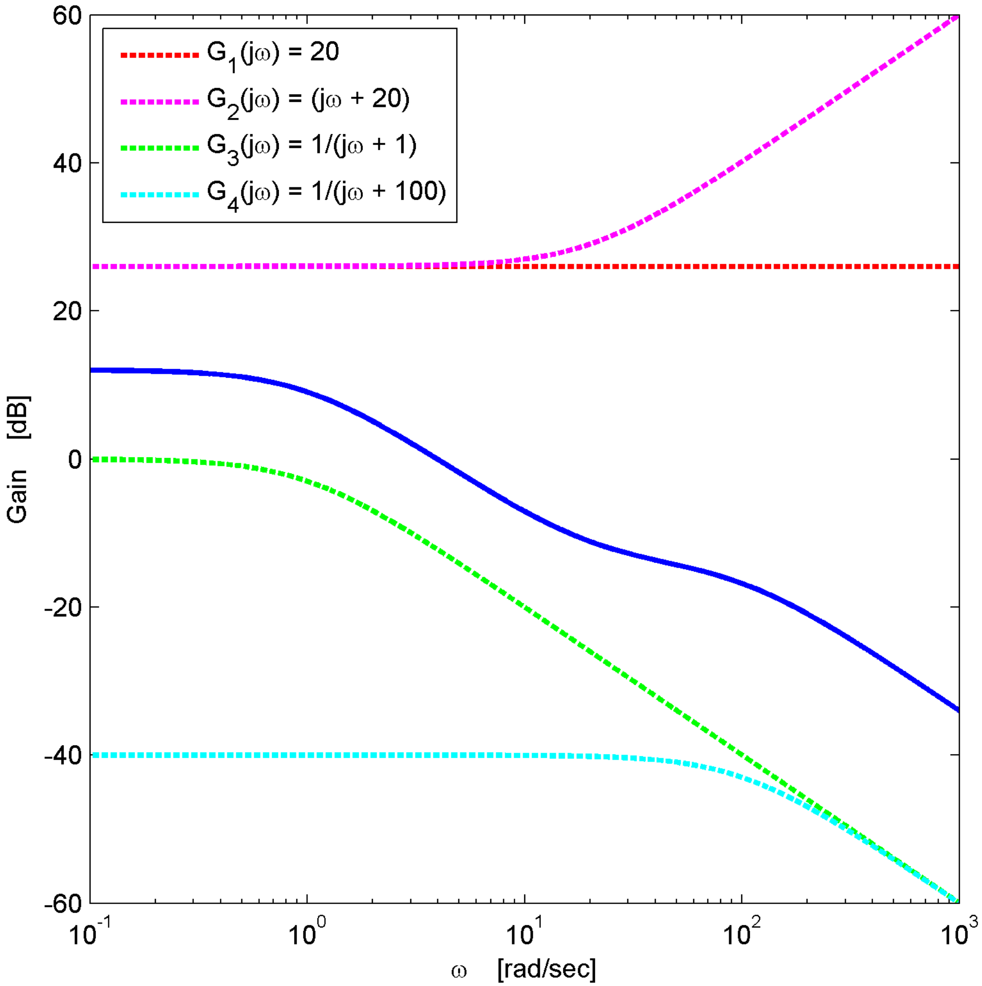
$$G(j\omega) = 20 \cdot (j\omega+20) \cdot \frac{1}{(j\omega+1)} \cdot \frac{1}{(j\omega+100)}$$

- Overall response is the composite of the individual responses
  - Product of individual gain responses – sum in dB
  - Sum of individual phase responses

Frequency Response Components - Example

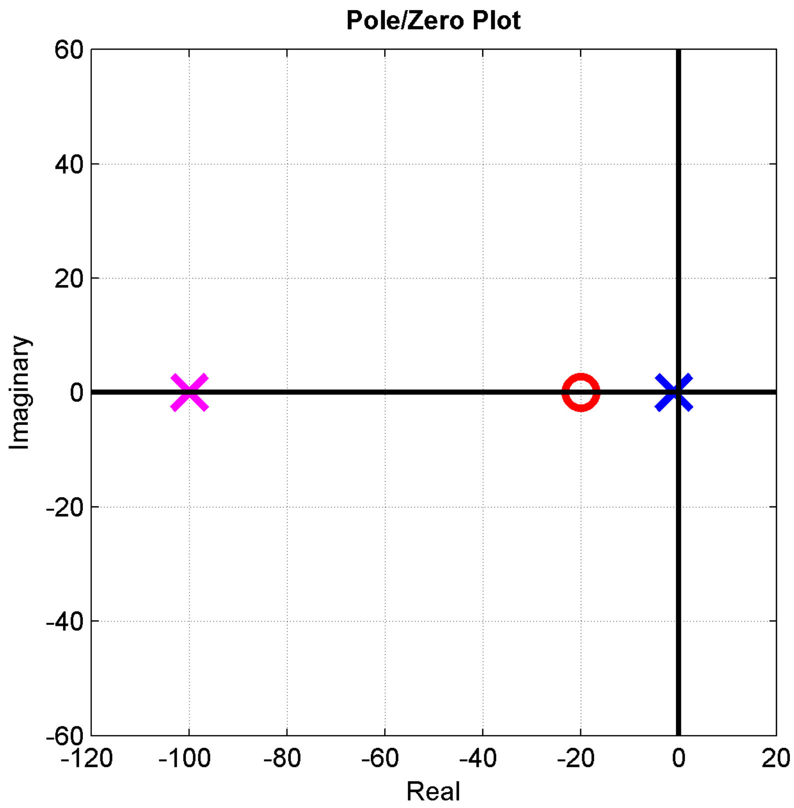
- Gain response

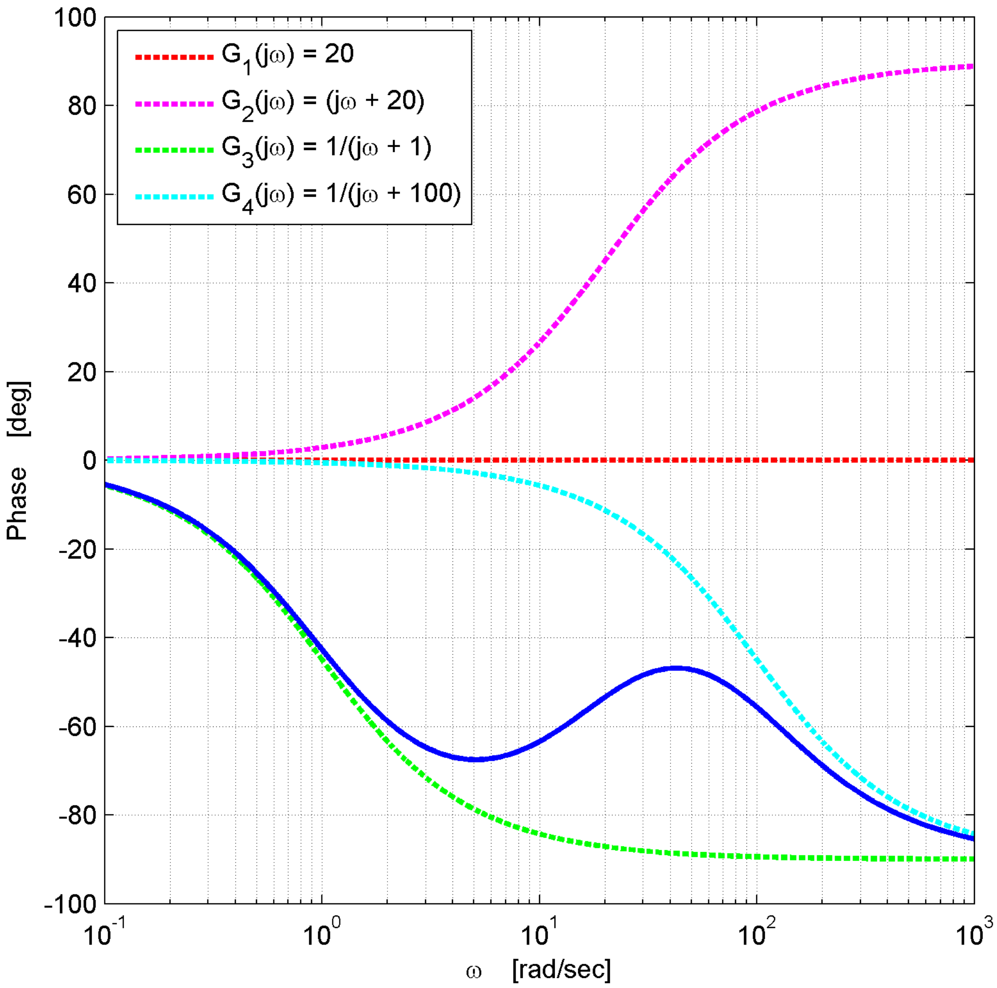




Frequency Response Components - Example

- Phase response





Bode Plot Construction

Bode Plot Construction

- We've just seen that a system's transfer function can be factored into first- and second-order terms
  - Each factor contributes a component to the overall gain and phase responses
- Now, we'll look at a technique for ***manually sketching a system's Bode plot***
  - In practice, you'll almost always plot with a computer
  - But, learning to do it by hand provides valuable insight
- We'll look at how to approximate Bode plots for each of the different factors

Bode Form of the Transfer function

- Consider the general transfer function form:

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots\left(s^2+2\zeta_a\omega_{0,a}s+\omega_{0,a}^2\right)\cdots}{(s-p_1)(s-p_2)\cdots\left(s^2+2\zeta_1\omega_{0,1}s+\omega_{0,1}^2\right)\cdots}$$

- We first want to put this into ***Bode form***:

$$G(s) = K_0\frac{\left(\frac{s}{\omega_{ca}}+1\right)\left(\frac{s}{\omega_{cb}}+1\right)\cdots\left(\frac{s^2}{\omega_{0,a}^2}+\frac{2\zeta_a}{\omega_{0,a}}s+1\right)\cdots}{\left(\frac{s}{\omega_{c1}}+1\right)\left(\frac{s}{\omega_{c2}}+1\right)\cdots\left(\frac{s^2}{\omega_{0,1}^2}+\frac{2\zeta_1}{\omega_{0,1}}s+1\right)\cdots}$$

- Putting $G(s)$ into Bode form requires putting each of the ***first- and second-order factors into Bode form***

First-Order Factors in Bode Form

- ***First-order transfer function factors*** include:

$$G(s) = s^n, \quad G(s) = s + \sigma, \quad G(s) = \frac{1}{s+\sigma}$$

- For the first factor, $G(s) = s^n$, n is a positive or negative integer
  - Already in Bode form
- For the second two, divide through by σ, giving

$$G(s) = \sigma\left(\frac{s}{\sigma} + 1\right) \quad \text{and} \quad G(s) = \frac{1}{\sigma\left(\frac{s}{\sigma}+1\right)}$$

- Here, $\sigma = \omega_c$, the ***corner frequency*** associated with that zero or pole, so

$$G(s) = \omega_c\left(\frac{s}{\omega_c} + 1\right) \quad \text{and} \quad G(s) = \frac{1}{\omega_c\left(\frac{s}{\omega_c}+1\right)}$$

Second-Order Factors in Bode Form

- ***Second-order transfer function factors*** include:

$$G(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2 \quad \text{and} \quad G(s) = \frac{1}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

- Again, normalize the s^0 coefficient, giving

$$G(s) = \omega_0^2\left[\frac{s^2}{\omega_0^2} + \frac{2\zeta}{\omega_0}s + 1\right] \quad \text{and} \quad G(s) = \frac{1/\omega_0^2}{\frac{s^2}{\omega_0^2}+\frac{2\zeta}{\omega_0}s+1}$$

- Putting each factor into its Bode form involves factoring out any DC gain component
- Lump all of ***DC gains*** together into a single gain constant, K_0

$$G(s) = K_0\frac{\left(\frac{s}{\omega_{ca}}+1\right)\left(\frac{s}{\omega_{cb}}+1\right)\cdots\left(\frac{s^2}{\omega_{0,a}^2}+\frac{2\zeta_a}{\omega_{0,a}}s+1\right)\cdots}{\left(\frac{s}{\omega_{c1}}+1\right)\left(\frac{s}{\omega_{c2}}+1\right)\cdots\left(\frac{s^2}{\omega_{0,1}^2}+\frac{2\zeta_1}{\omega_{0,1}}s+1\right)\cdots}$$

Bode Plot Construction

- Transfer function in Bode form

$$G(s) = K_0 \frac{\left(\frac{s}{\omega_{ca}}+1\right)\left(\frac{s}{\omega_{cb}}+1\right)\cdots\left(\frac{s^2}{\omega_{0,a}^2}+\frac{2\zeta_a}{\omega_{0,a}}s+1\right)\cdots}{\left(\frac{s}{\omega_{c1}}+1\right)\left(\frac{s}{\omega_{c2}}+1\right)\cdots\left(\frac{s^2}{\omega_{0,1}^2}+\frac{2\zeta_1}{\omega_{0,1}}s+1\right)\cdots}$$

- Product of a constant DC gain factor, K_0, and first- and second-order factors

- Plot the frequency response of each factor individually, then combine graphically
  - Overall response is the product of individual factors
    - Product of gain responses – sum on a dB scale
    - Sum of phase responses

Bode Plot Construction

- **Bode plot construction procedure**:
  1. Put the transfer function into ***Bode form***
  2. Draw a ***straight-line asymptotic approximation*** for the gain and phase response of each individual factor
  3. ***Graphically add*** all individual response components and sketch the result

- Note that we are really plotting the frequency response function, $G(j\omega)$
  - We use the transfer function, $G(s)$, to simplify notation

- Next, we'll look at the straight-line asymptotic approximations for the Bode plots for each of the transfer function factors

Bode Plot – Constant Gain Factor

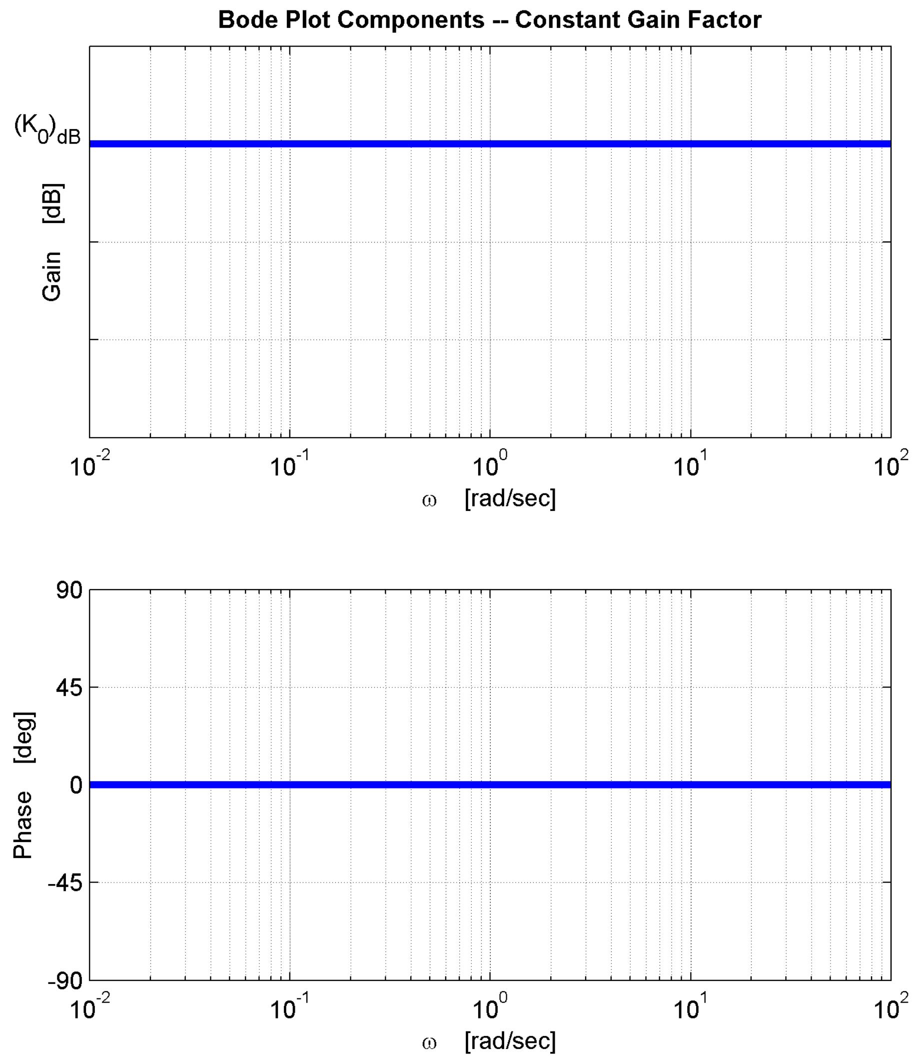
$$G(s) = K_0$$

- Constant gain

$$|G(s)| = K_0$$

- Constant Phase

$$\angle G(s) = 0°$$

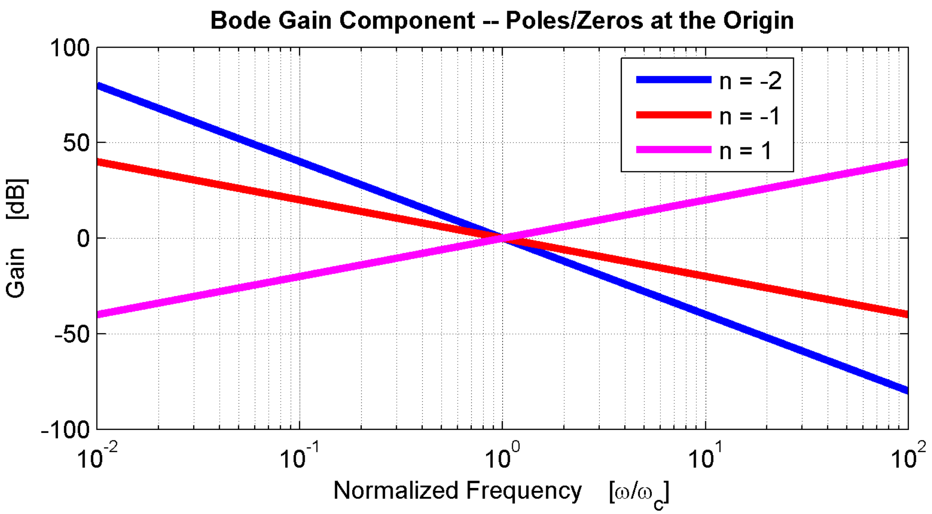


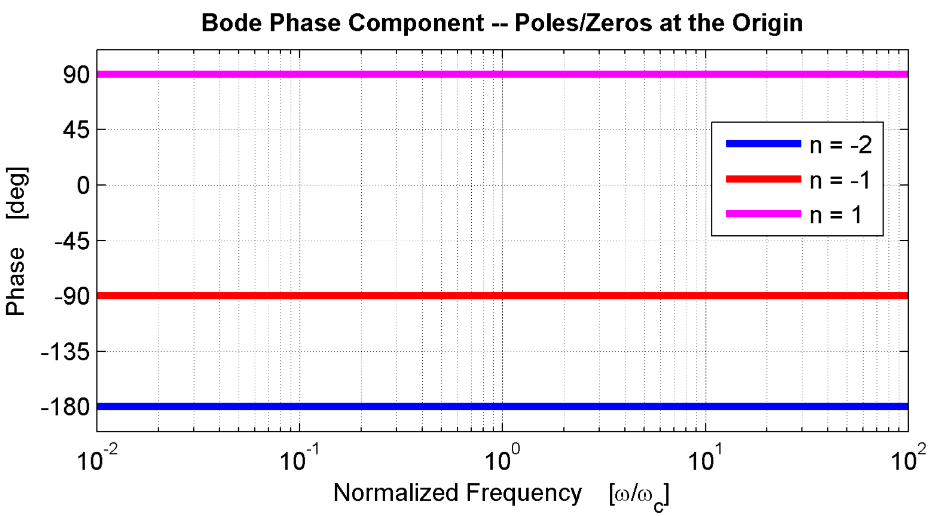
Bode Plot – Poles/Zeros at the Origin

$$G(s) = s^n$$

- $n > 0$:
  - n zeros at the origin
- $n < 0$:
  - n poles at the origin
- **Gain**:
  - Straight line
  - Slope $= n \cdot 20 \frac{dB}{dec} = n \cdot 6 \frac{dB}{oct}$
  - $0dB$ at $\omega = 1$
- **Phase**:

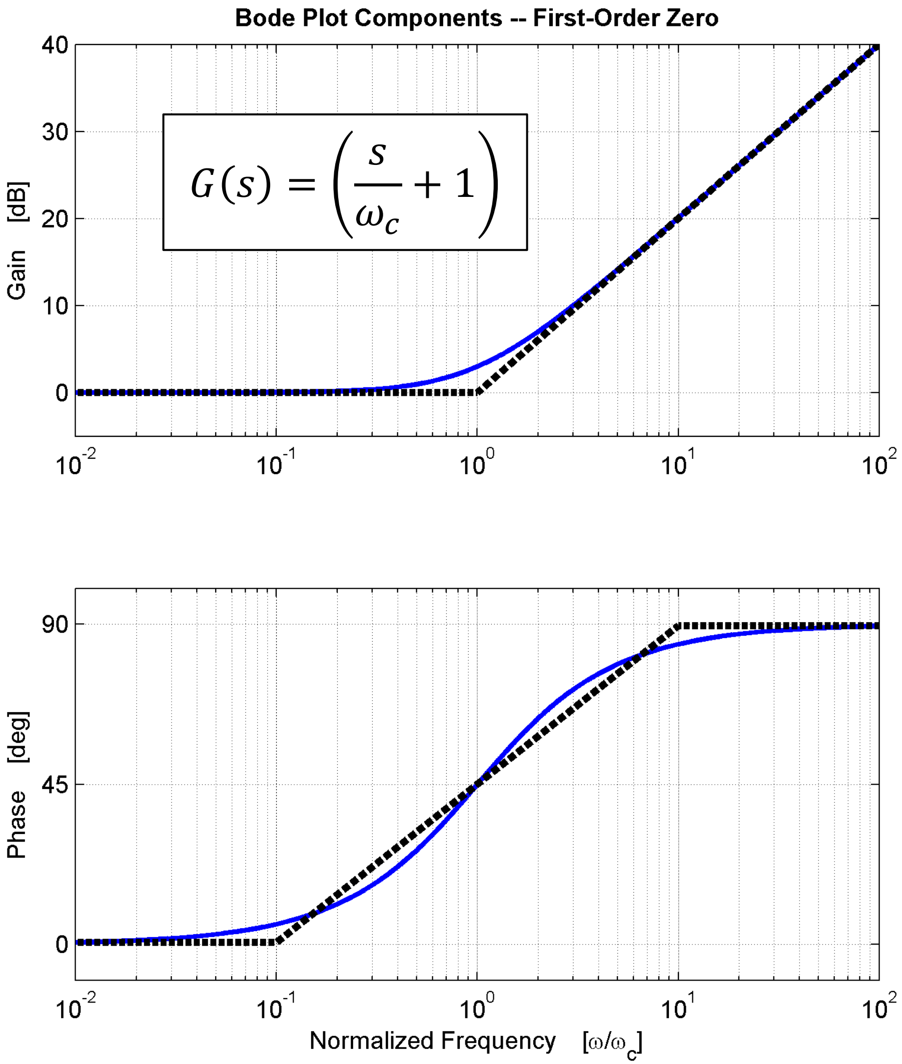
$$\angle G(s) = n \cdot 90°$$





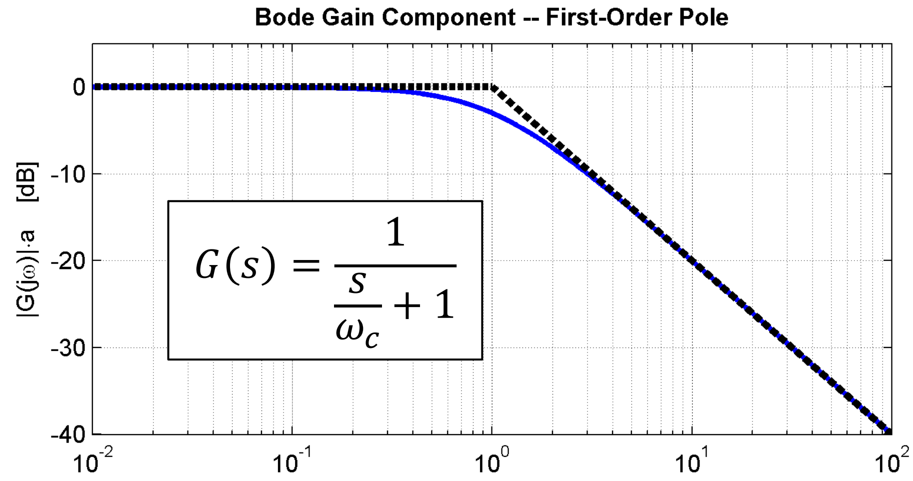
Bode Plot – First-Order Zero

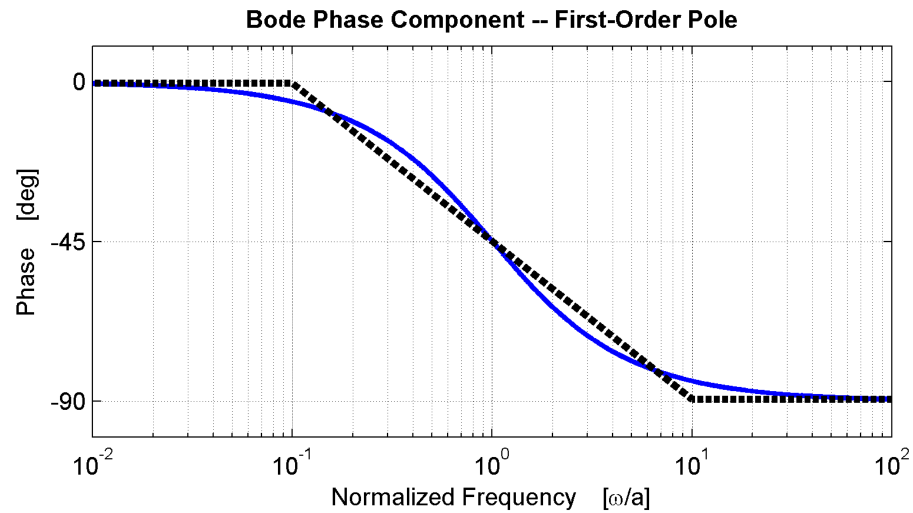
- Single real zero at $s = -\omega_c$
- **Gain**:
  - $0dB$ for $\omega < \omega_c$
  - $+20\frac{dB}{dec} = +6\frac{dB}{oct}$ for $\omega > \omega_c$
  - Straight-line asymptotes intersect at $(\omega_c, 0dB)$
- **Phase**:
  - $0°$ for $\omega \leq 0.1 \cdot \omega_c$
  - $45°$ for $\omega = \omega_c$
  - $90°$ for $\omega \geq 10 \cdot \omega_c$
  - $+45°/dec$ through ω_c



Bode Plot – First-Order Pole

- Single real pole at $s = -\omega_c$
- **Gain**:
  - $0dB$ for $\omega < \omega_c$
  - $-20\frac{dB}{dec} = -6\frac{dB}{oct}$ for $\omega > \omega_c$
  - Straight-line asymptotes intersect at $(\omega_c, 0dB)$
- **Phase**:
  - $0°$ for $\omega \le 0.1 \cdot \omega_c$
  - $-45°$ for $\omega = \omega_c$
  - $-90°$ for $\omega \ge 10 \cdot \omega_c$
  - $-45°/dec$ through ω_c



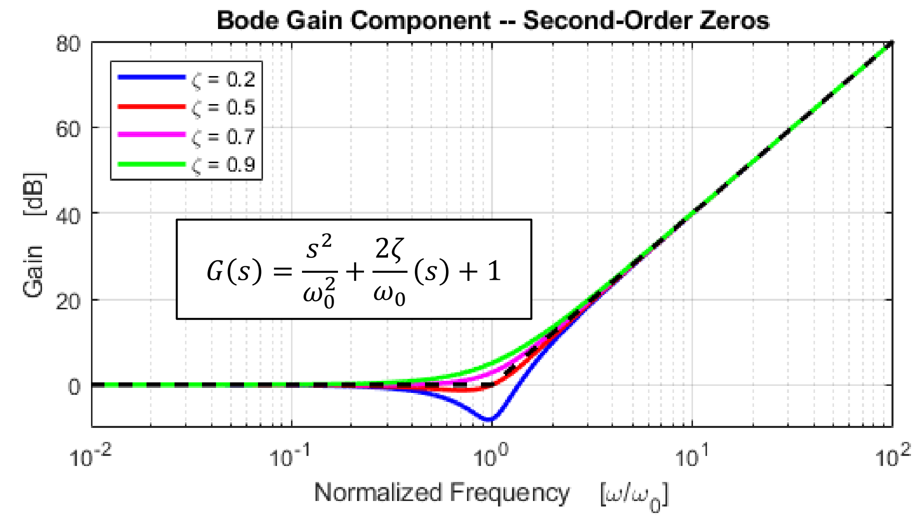


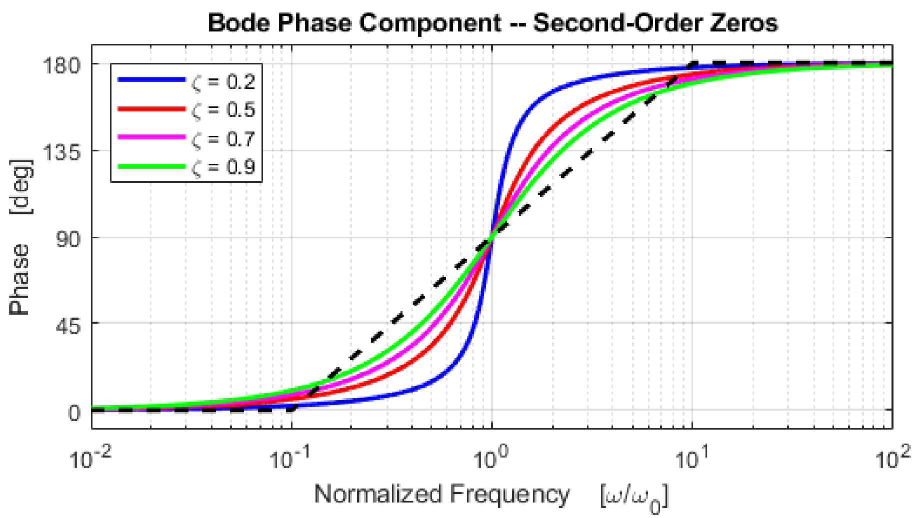
Bode Plot – Second-Order Zero

- Complex-conjugate zeros:

$$s_{1,2} = -\sigma \pm j\omega_d$$

- **Gain**:
  - $0dB$ for $\omega \leq \omega_0$
  - $+40\frac{dB}{dec} = +12\frac{dB}{oct}$ for $\omega > \omega_0$
  - Straight-line asymptotes intersect at $(\omega_0, 0dB)$
  - ζ-dependent peaking around ω_0
- **Phase**:
  - $0°$ for $\omega \leq 0.1 \cdot \omega_0$
  - $90°$ for $\omega = \omega_0$
  - $180°$ for $\omega \geq 10 \cdot \omega_0$
  - $+90°/dec$ through ω_0





Bode Plot – Second-Order Pole

- Complex-conjugate poles:

$$s_{1,2} = -\sigma \pm j\omega_d$$

- **Gain**:
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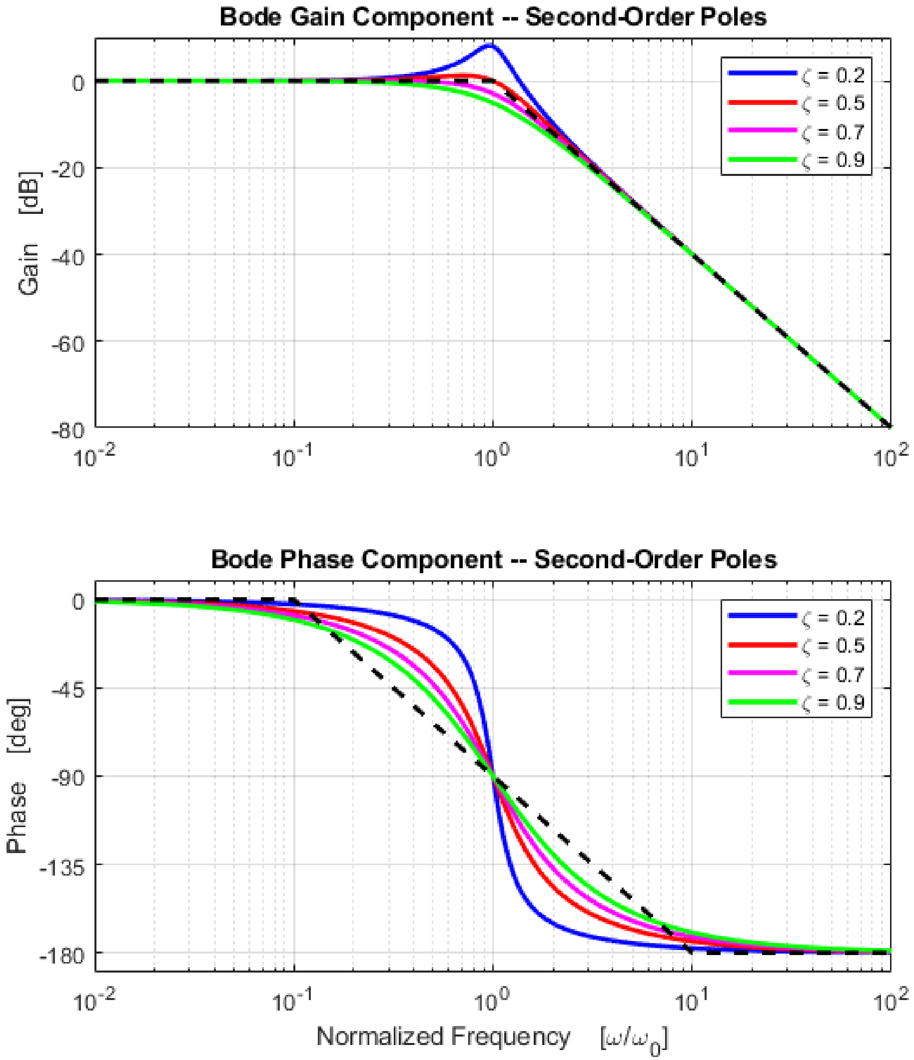
  - Straight-line asymptotes intersect at $(\omega_0, 0dB)$
  - ζ-dependent peaking around ω_0
- **Phase**:
  - $0°$ for $\omega \leq 0.1 \cdot \omega_0$
  - $-90°$ for $\omega = \omega_0$
  - $-180°$ for $\omega \geq 10 \cdot \omega_0$
  - $-90°/dec$ through ω_0



Bode Plot Construction – Example

- Consider a system with the following ***transfer function***

$$G(s) = \frac{10(s+20)}{s(s+400)}$$

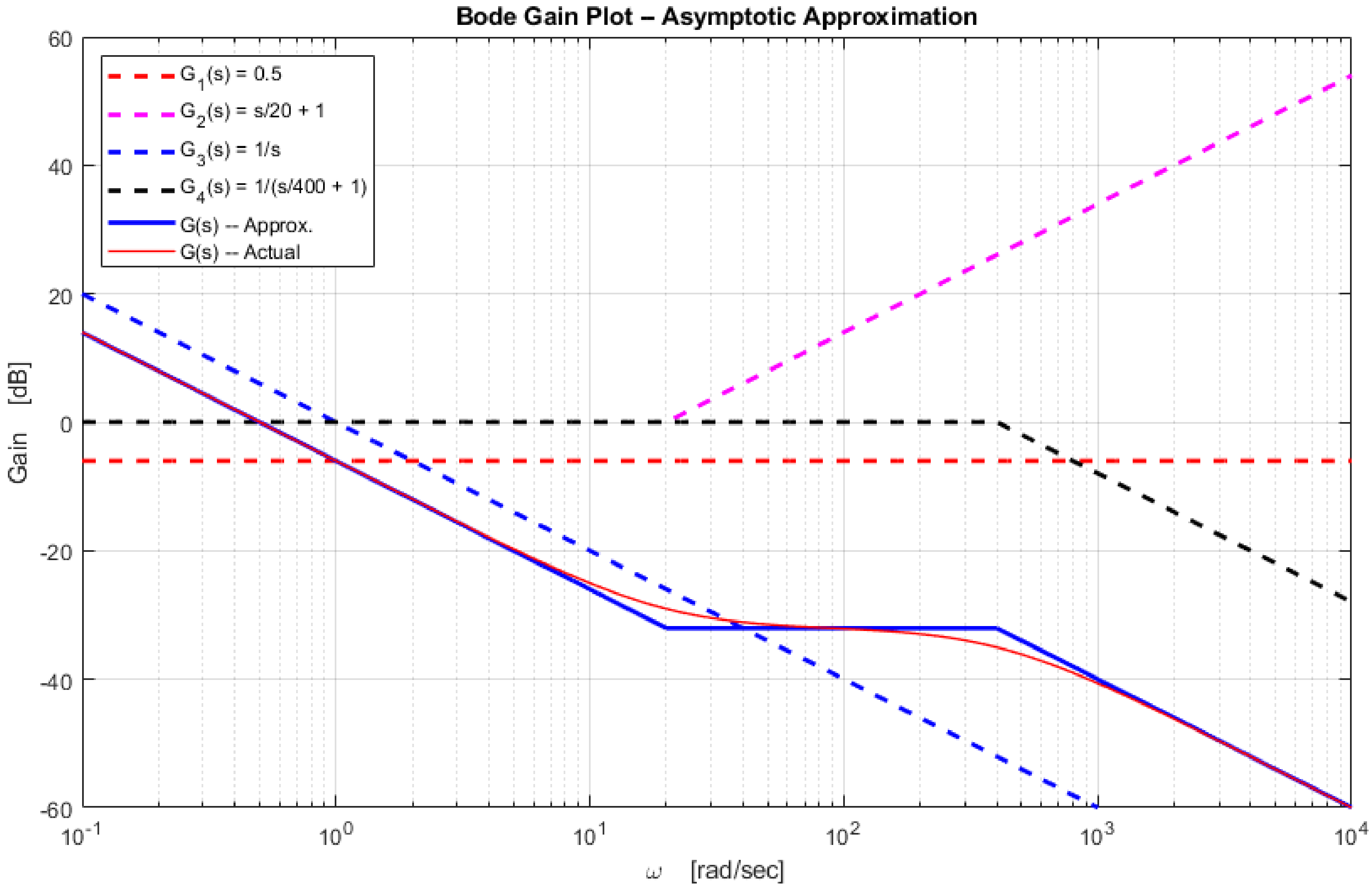
- Put it into ***Bode form***

$$G(s) = \frac{10 \cdot 20 \left(\frac{s}{20} + 1\right)}{s \cdot 400 \left(\frac{s}{400} + 1\right)} = \frac{0.5 \left(\frac{s}{20} + 1\right)}{s \cdot \left(\frac{s}{400} + 1\right)}$$

- Represent as a ***product of factors***

$$G(s) = 0.5 \cdot \left(\frac{s}{20} + 1\right) \cdot \frac{1}{s} \cdot \frac{1}{\left(\frac{s}{400} + 1\right)}$$

Bode Plot Construction – Example



Bode Plot Construction – Example