SECTION 2: BOND GRAPH FUNDAMENTALS

ESE 330 – Modeling & Analysis of Dynamic Systems

Bond Graphs - Introduction

- As engineers, we're interested in ***different types of systems***:
  - Mechanical translational
  - Mechanical rotational
  - Electrical
  - Hydraulic

- Many systems consist of ***subsystems in different domains***, e.g. an electrical motor

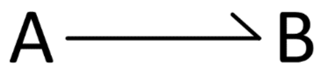
- Common aspect to all systems is the ***flow of energy and power*** between components

- ***Bond graph system models*** exploit this commonality
  - Based on the flow of energy and power
  - Universal – domain-independent
  - Technique used for deriving differential equations from a bond graph model is the same for any type of system

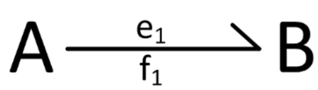
Power and Energy Variables

Bonds and Power Variables

- Systems are made up of ***components***
  - ***Power*** can flow between components
  - We represent this pathway for power to flow with ***bonds***



- **A** and **B** represent ***components***, the line connecting them is a ***bond***
- Quantity on the bond is power
  - Power flow is positive in the direction indicated – arbitrary
- Each bond has two ***power variables*** associated with it
  - ***Effort*** and ***flow***



- ***The product of the power variables is power***

$$\mathcal{P} = e \cdot f$$

Power Variables

- Power variables, e and f, determine the ***power flowing on a bond***
  - The ***rate at which energy flows*** between components

Domain	Effort			Flow			Power
	Quantity	Variable	Units	Quantity	Variable	Units	
General	Effort	e	–	Flow	f	–	$\mathcal{P} = e \cdot f$
Mechanical Translational	Force	F	N	Velocity	v	m/s	$\mathcal{P} = F \cdot v$
Mechanical Rotational	Torque	τ	N-m	Angular velocity	ω	rad/s	$\mathcal{P} = \tau \cdot \omega$
Electrical	Voltage	v	V	Current	i	A	$\mathcal{P} = v \cdot i$
Hydraulic	Pressure	P	Pa (N/m^2)	Flow rate	Q	m^3/s	$\mathcal{P} = P \cdot Q$

Energy Variables

- Bond graph models are energy-based models
- Energy in a system can be:
  - ***Supplied*** by external sources
  - ***Stored*** by system components
  - ***Dissipated*** by system components
  - ***Transformed*** or ***converted*** by system components
- In addition to power variables, we need two more variables to describe energy storage: ***energy variables***
  - ***Momentum***
  - ***Displacement***

Momentum

- ***Momentum*** – the integral of effort

$$p(t) \equiv \int e(t)dt$$

so

$$e(t) = \frac{dp}{dt} = \dot{p}$$

- For mechanical systems:

$$e = F = \frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$$

which, for constant mass, becomes

$$F = m\frac{dv}{dt} = ma$$

- Newton's second law

Displacement

- ***Displacement*** – the integral of flow

$$q(t) \equiv \int f(t)dt$$

so

$$f(t) = \frac{dq}{dt} = \dot{q}$$

- For mechanical systems:

$$q(t) = x(t)$$

$$f(t) = v(t)$$

$$f(t) = \frac{dq}{dt} = v(t) = \frac{dx}{dt}$$

- The definition of velocity

Energy Variables

- ***Displacement*** and ***momentum*** are familiar concepts for mechanical systems
  - All types of systems have analogous ***energy variables***
  - We'll see that these quantities are useful for describing ***energy storage***

Domain	Momentum			Displacement		
	Quantity	Variable	Units	Quantity	Variable	Units
General	Momentum	p	–	Displacement	q	–
Mechanical Translational	Momentum	p	N-s	Displacement	x	m
Mechanical Rotational	Angular momentum	L	N-m-s	Angle	θ	rad
Electrical	Magnetic flux	λ	V-s	Charge	q	C
Hydraulic	Hydraulic momentum	Γ	N-s/m^2	Volume	V	m^3

Energy – Kinetic Energy

- Energy is the integral of power

$$E(t) = \int \mathcal{P}(t)dt = \int e(t)f(t)dt$$

- We can relate effort to momentum

$$e(t) = \frac{dp}{dt}$$

- So, if it is possible to ***express flow as a function of momentum***, $f(p)$, we can express ***energy as a function of momentum***, $E(p)$

$$E(p) = \int \frac{dp}{dt} f(p)dt = \int f(p)dp$$

- This is ***kinetic energy***
  - ***Energy expressed as a function of momentum***

Energy – Potential Energy

- Energy is the integral of power

$$E(t) = \int \mathcal{P}(t)dt = \int e(t)f(t)dt$$

- We can relate flow to displacement

$$f(t) = \frac{dq}{dt}$$

- So, if it is possible to ***express effort as a function of displacement***, $e(q)$, we can express ***energy as a function of displacement***, $E(q)$

$$E(q) = \int e(q)\frac{dq}{dt}dt = \int e(q)dq$$

- This is ***<u>potential energy</u>***
  - ***Energy expressed as a function of displacement***

Energy – Mechanical Translational

- For a mechanical translational system

$$E(t) = \int e(t)f(t)dt = \int F(t)v(t)dt$$

and

$$F(t) = \frac{dp}{dt}\,, \qquad v(p) = \frac{1}{m}p$$

so

$$E(p) = \int \frac{dp}{dt}\frac{1}{m}p\,dt = \frac{1}{m}\int p\,dp = \frac{1}{2m}p^2 = \frac{1}{2m}m^2v^2 = \frac{1}{2}mv^2 = \boldsymbol{K.E.}$$

- We can also express force and energy as a function of displacement

$$v(t) = \frac{dx}{dt}, \qquad F(x) = kx$$

so

$$E(x) = \int kx\frac{dx}{dt}dt = k\int x\,dx = \frac{1}{2}kx^2 = \boldsymbol{P.E.}$$

Energy – Electrical

- For an electrical system

$$E(t) = \int e(t)f(t)dt = \int v(t)\, i(t)dt$$

and

$$v(t) = \frac{d\lambda}{dt}, \qquad i(\lambda) = \frac{1}{L}\lambda$$

so

$$E(\lambda) = \int \frac{d\lambda}{dt}\frac{1}{L}\lambda dt = \frac{1}{L}\int \lambda\, d\lambda = \frac{1}{2L}\lambda^2 = \frac{1}{2L}L^2 i^2 = \frac{1}{2}Li^2 = \boldsymbol{Mag.Energy}$$

- We can also express voltage and energy as a function of charge

$$i(t) = \frac{dq}{dt}, \qquad v(q) = \frac{1}{C}q$$

so

$$E(q) = \int \frac{1}{C}q\frac{dq}{dt}dt = \frac{1}{C}\int q\, dq = \frac{1}{2C}q^2 = \frac{1}{2C}C^2 v^2 = \frac{1}{2}Cv^2 = \boldsymbol{Elect.Energy}$$

Energy – Summary

$$E(t) = \int e(t)\, f(t)\, dt$$

- For some system components, ***flow can be expressed as a function of momentum***
  - These components ***store energy as a function of momentum***
  - This is ***kinetic energy*** or ***magnetic energy***

$$E(p) = \int f(p)\, dp$$

- For other components, ***effort can be expressed as a function of displacement***
  - These components ***store energy as a function of displacement***
  - This is ***potential energy*** or ***electrical energy***

$$E(q) = \int e(q)\, dq$$

One-Port Bond Graph Elements

System Components

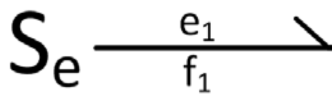
- System components are defined by how they affect energy flow within the system – they can:

    1. ***Supply energy***
    2. ***Store energy***
        a) ***As a function of p – kinetic or magnetic energy***
        b) ***As a function of q – potential or electrical energy***
    3. ***Dissipate energy***
    4. ***Transform or convert energy***

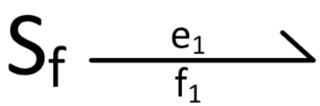
- Different bond graph elements for components in each of these categories
    - Categorized by the number of ***ports*** - bond attachment points

Active One-Port Elements

- External sources that ***supply energy*** to the system
- ***Effort Source***
  - Supplies a specific effort to the system
  - E.g., force source, voltage source, pressure source



- ***Flow Source***
  - Supplies a specific flow to the system
  - E.g., velocity source, current source, flow source



Passive One-Port Elements

- One-port elements categorized by whether they dissipate energy or store kinetic or potential energy
- Three different ***one-port elements***:
    - ***Inertia***
    - ***Capacitor***
    - ***Resistor***
- Same three elements used to model system components in all different energy domains
- Each defined by a ***constitutive law***
    - A defining relation between two physical quantities – two of the four energy and power variables

Inertia

- ***Inertia*** – a component whose constitutive law relates ***flow to momentum***

$$f = \frac{1}{I} p$$

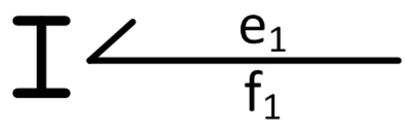
where I is the relevant ***inertia*** of the component

- Inertias ***store energy as a function of momentum***
- A ***kinetic energy*** storage element

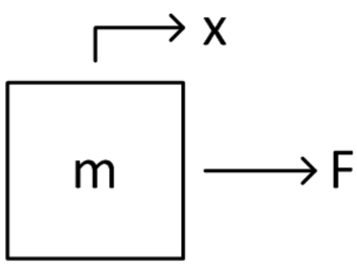
Domain	Inertia Parameter	Symbol	Units
General	Inertia	I	-
Translational	Mass	m	kg
Rotational	Moment of inertia	J	Kg-m^2
Electrical	Inductance	L	H
Hydraulic	Hydraulic inertia	I	Kg/m^4

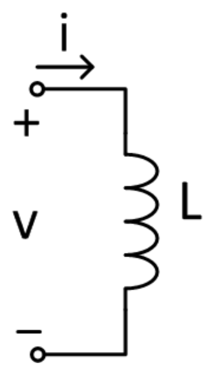
Inertia

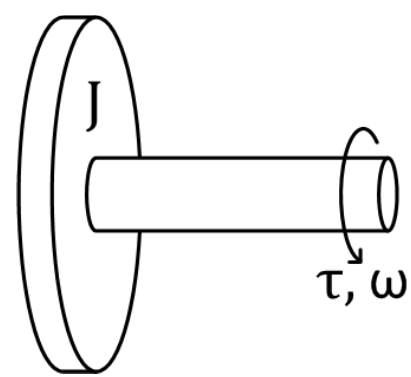
- Bond graph symbol for an inertia:



- Physical components modeled as inertias:







Inertia – Energy Storage

- Constitutive law:

$$f = \frac{1}{I}p$$

- Stored energy:

$$K.E. = E(p) = \int f(p)\,dp$$

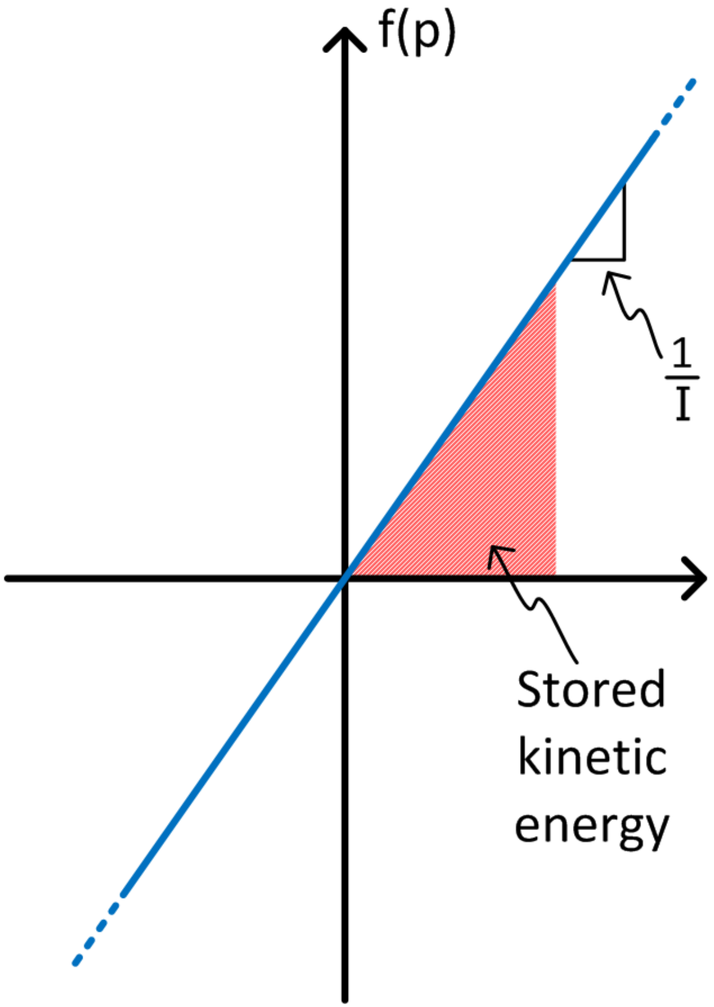
$$K.E. = \frac{1}{I}\int p\,dp = \frac{p^2}{2I}$$

- Mechanical:

$$K.E. = \frac{(mv)^2}{2m} = \frac{1}{2}mv^2$$

- Electrical:

$$M.E. = \frac{(LI)^2}{2L} = \frac{1}{2}LI^2$$



Inertia – Constitutive Law

- Constitutive law for an inertia can be expressed in linear, integral, or derivative form:

$$f = \frac{1}{I} p = \frac{1}{I} \int e \, dt \qquad \text{or} \qquad e = I \frac{df}{dt}$$

Domain	Linear	Integral	Derivative
General	$f = \frac{1}{I} p$	$f = \frac{1}{I} \int e \, dt$	$e = I \frac{df}{dt}$
Translational	$v = \frac{1}{m} p$	$v = \frac{1}{m} \int F \, dt$	$F = m \frac{dv}{dt}$
Rotational	$\omega = \frac{1}{J} L$	$\omega = \frac{1}{J} \int \tau \, dt$	$\tau = J \frac{d\omega}{dt}$
Electrical	$i = \frac{1}{L} \lambda$	$i = \frac{1}{L} \int v \, dt$	$v = L \frac{di}{dt}$
Hydraulic	$Q = \frac{1}{I} \Gamma$	$Q = \frac{1}{I} \int P \, dt$	$P = I \frac{dQ}{dt}$

Capacitors

- ***Capacitor*** – a component whose constitutive law relates ***effort to displacement***

$$e = \frac{1}{C} q$$

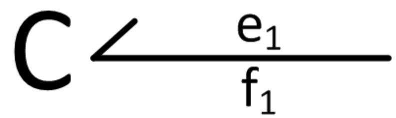
where C is the relevant ***capacitance*** of the component

- Capacitors ***store energy as a function of displacement***
- A ***potential-energy***-storage element

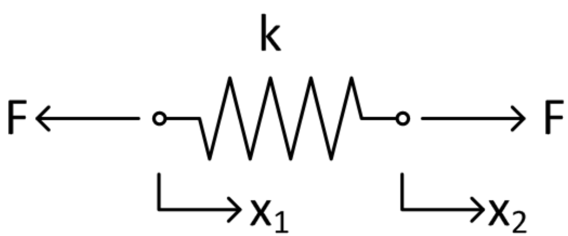
Domain	Capacitance Parameter	Symbol	Units
General	Capacitance	C	-
Translational	Compliance	$1/k$	m/N
Rotational	Rotational compliance	$1/k_\tau$	rad/N-m
Electrical	Capacitance	C	F
Hydraulic	Hydraulic capacitance	C	m^5/N

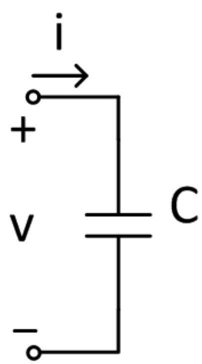
Capacitor

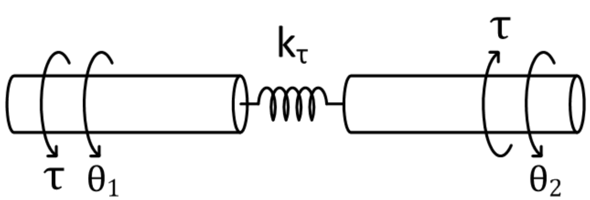
- Bond graph symbol for a capacitor:



- Physical components modeled as capacitors:







- Note that spring constants are the inverse of capacitance or compliance

Capacitor – Constitutive Law

- Constitutive law:

$$e = \frac{1}{C} q$$

- Stored energy:

$$P.E. = E(q) = \int e(q)\, dq$$

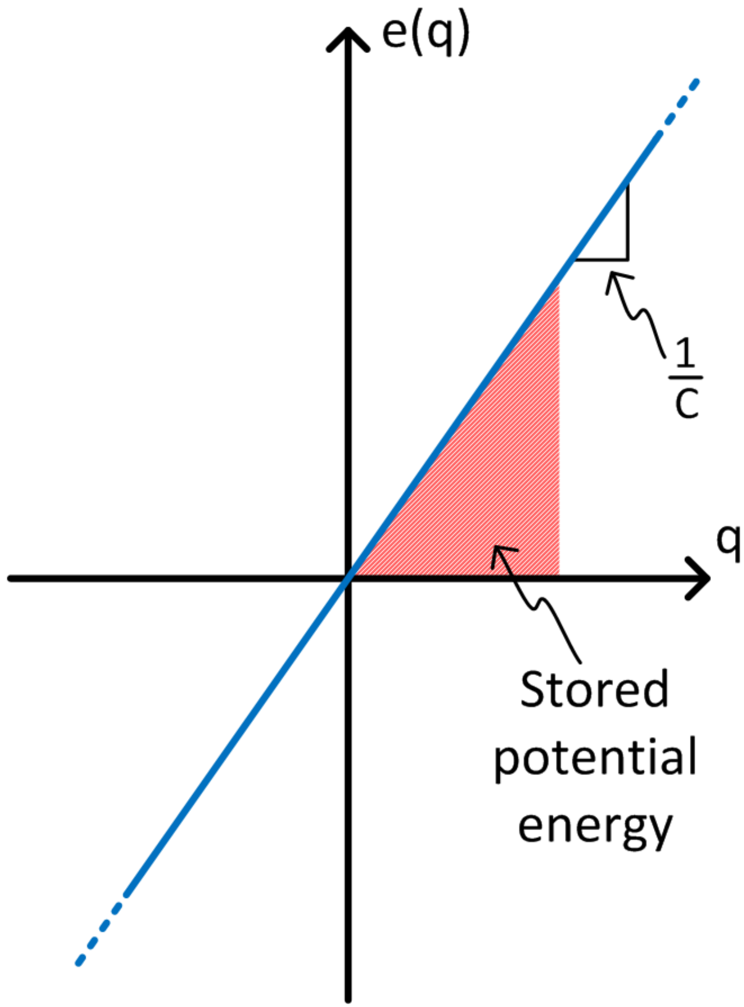
$$P.E. = \frac{1}{C} \int q\, dq = \frac{q^2}{2C}$$

- Mechanical:

$$P.E. = \frac{x^2}{2/k} = \frac{1}{2} k x^2$$

- Electrical:

$$E.E. = \frac{(C \cdot v)^2}{2C} = \frac{1}{2} C v^2$$



Capacitor – Constitutive Law

- Constitutive law for a capacitor can be expressed in linear, integral, or derivative form:

$$e = \frac{1}{C} q = \frac{1}{C} \int f \, dt \qquad \text{or} \qquad f = C \frac{de}{dt}$$

Domain	Linear	Integral	Derivative
General	$e = \frac{1}{C} q$	$e = \frac{1}{C} \int f \, dt$	$f = C \frac{de}{dt}$
Translational	$F = kx$	$F = k \int v \, dt$	$v = \frac{1}{k} \frac{dF}{dt}$
Rotational	$\tau = k_\tau \theta$	$\tau = k_\tau \int \omega \, dt$	$\omega = \frac{1}{k_\tau} \frac{d\tau}{dt}$
Electrical	$v = \frac{1}{C} q$	$v = \frac{1}{c} \int i \, dt$	$i = C \frac{dv}{dt}$
Hydraulic	$P = \frac{1}{C} \mathrm{V}$	$P = \frac{1}{C} \int Q \, dt$	$Q = C \frac{dP}{dt}$

Resistors

- ***Resistor*** – a component whose constitutive law relates ***flow to effort***

$$f = \frac{1}{R}e \quad \text{or} \quad e = R \cdot f$$

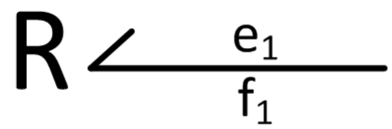
where R is the relevant ***resistance*** of the component

- Resistors ***dissipate energy***
- A ***loss mechanism***

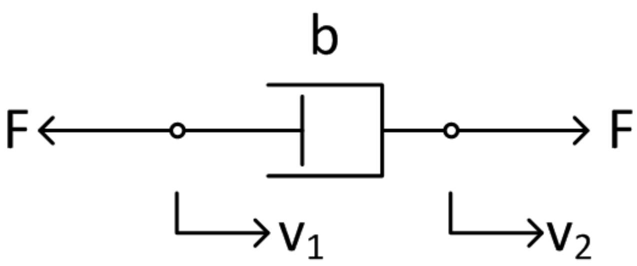
Domain	Resistance Parameter	Symbol	Units
General	Resistance	R	-
Translational	Damping coefficient	b	N-s/m
Rotational	Rotational damping coeff.	b_τ	N-m-s/rad
Electrical	Resistance	R	Ω
Hydraulic	Hydraulic resistance	R	N-s/m^5

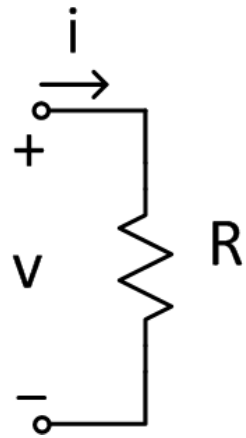
Resistor

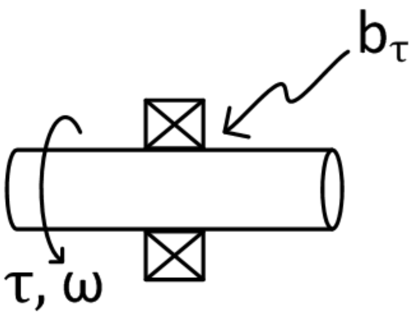
- Bond graph symbol for a resistor:



- Physical components modeled as resistors:







Resistor – Power Dissipation

- Constitutive law:

$$f = \frac{1}{R}e$$

or

$$e = R \cdot f$$

- Power dissipation:

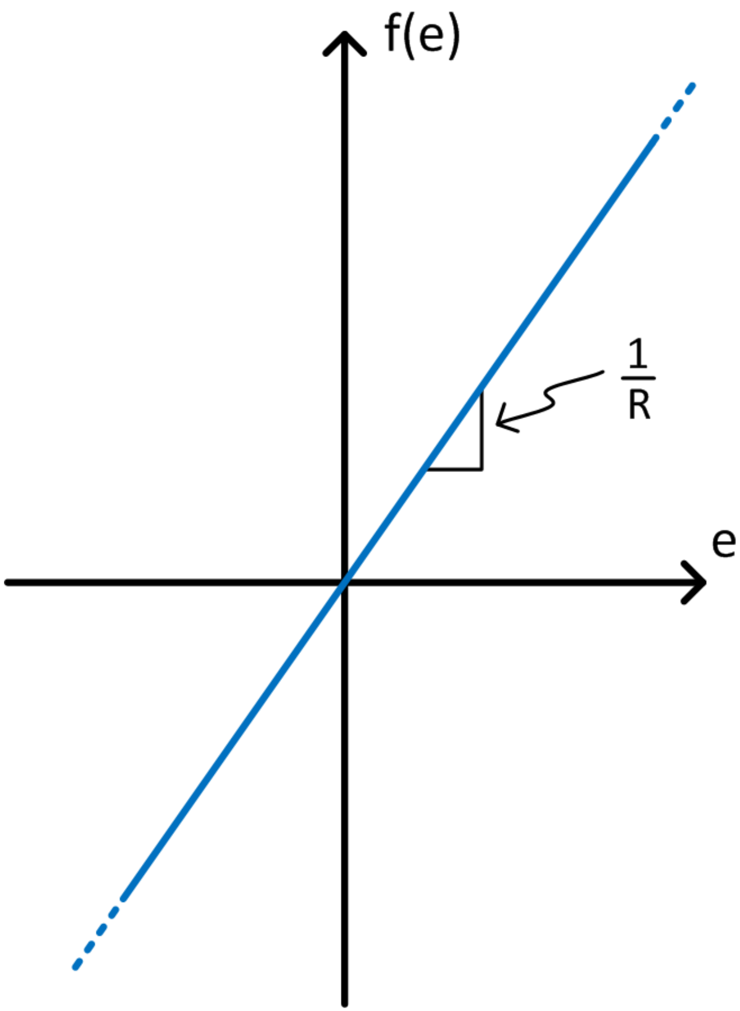
$$\mathcal{P} = e \cdot f = f^2 R = \frac{e^2}{R}$$

- Mechanical:

$$\mathcal{P} = v^2 b = \frac{F^2}{b}$$

- Electrical:

$$\mathcal{P} = i^2 R = \frac{v^2}{R}$$



Resistor – Constitutive Law

- Constitutive law for a resistor can express flow in terms of effort, or vice-versa:

$$f = \frac{1}{R}e \qquad \text{or} \qquad e = R \cdot f$$

Domain	Flow	Effort
General	$f = \frac{1}{R}e$	$e = R \cdot f$
Translational	$v = \frac{1}{b}F$	$F = b \cdot v$
Rotational	$\omega = \frac{1}{b_\tau}\tau$	$\tau = b_\tau \cdot \omega$
Electrical	$i = \frac{1}{R}v$	$v = R \cdot i$
Hydraulic	$Q = \frac{1}{R}P$	$P = b \cdot Q$

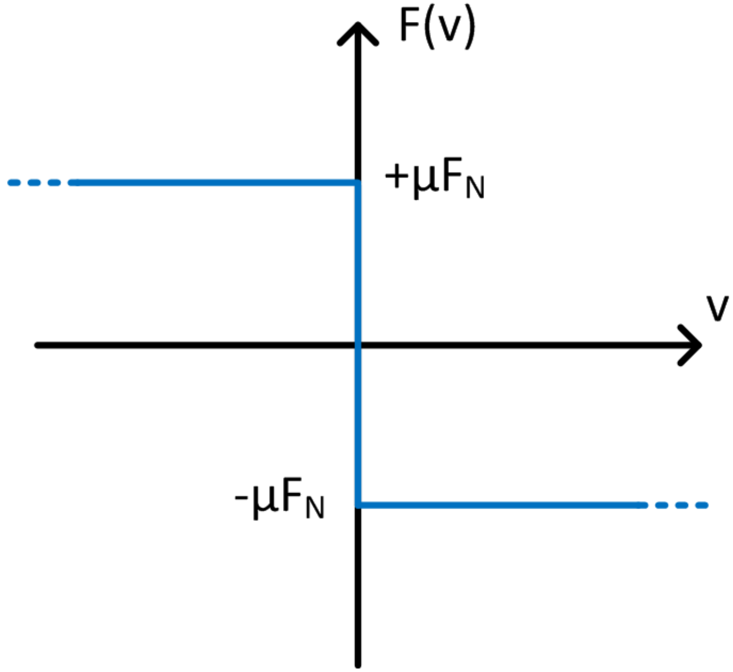
Viscous vs. Coulomb Friction

- We've assumed a specific type of mechanical resistance – ***viscous friction***
  - A ***linear*** resistance
  - Realistic? – Sometimes
- Can we model ***coulomb friction*** as a resistor?

$$F = \mu F_N$$

  - Yes, if the constitutive law relates effort (F) and flow (v)
  - It does – velocity determines ***direction*** of the friction force

$$F = -\mu F_N \cdot sign(v)$$



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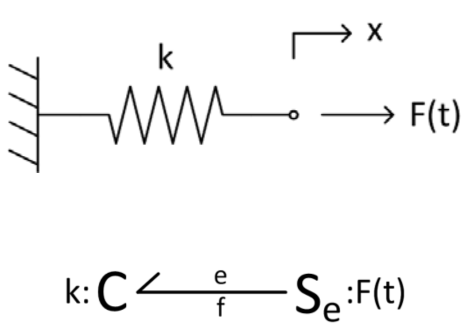
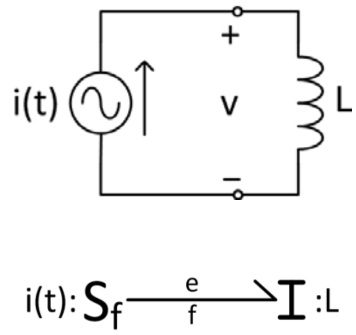
N-Port Bond Graph Elements

Multi-Port Elements - Junctions

- So far, we have ***sources*** and other ***one-port elements***
  - These allow us to model things like this:

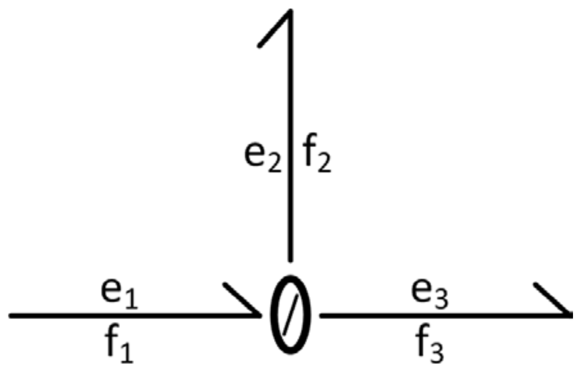




- Want to be able to model multiple interconnected components in a system
  - Need components with more than one port
  - ***Junctions***: ***0-junction*** and ***1-junction***

0-Junctions

- ***0-junction*** – a ***constant effort*** junction
  - All bonds connected to a 0-junction have equal effort
  - ***Power is conserved*** at a 0-junction



- Constant effort:

$$e_1 = e_2 = e_3$$

- Power is conserved:

$$\sum \mathcal{P}_{in} = \sum \mathcal{P}_{out}$$

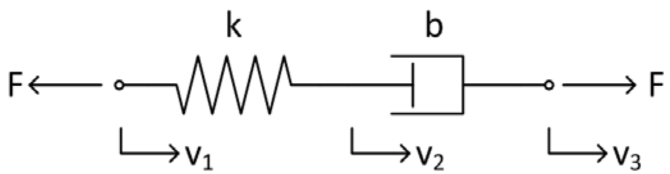
$$e_1 f_1 = e_2 f_2 + e_3 f_3$$

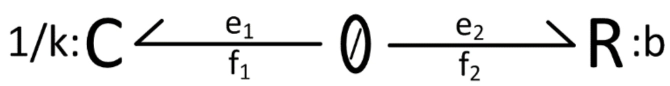
so

$$f_1 = f_2 + f_3$$

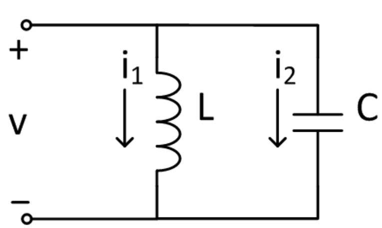
0-Junctions

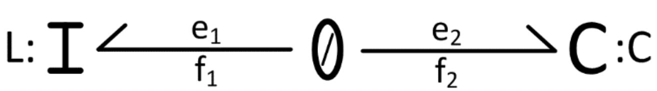
- Constant-effort 0-junction translates to different physical configurations in different domains
- ***Mechanical translational***
  - ***Constant force*** – components connected in ***series***





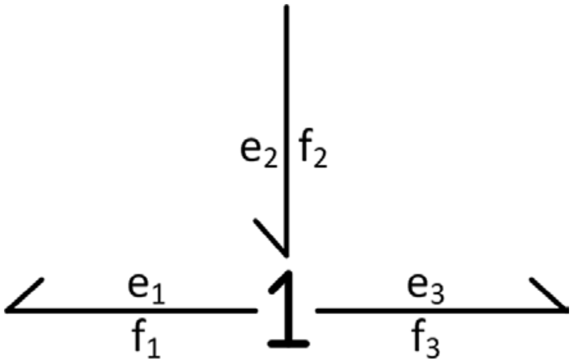
- ***Electrical***
  - ***Constant voltage*** – components connected in ***parallel***





1-Junctions

- ***1-junction*** – a ***constant flow*** junction
  - All bonds connected to a 1-junction have equal flow
  - ***Power is conserved*** at a 1-junction



- Constant flow:

$$f_1 = f_2 = f_3$$

- Power is conserved:

$$\sum \mathcal{P}_{in} = \sum \mathcal{P}_{out}$$

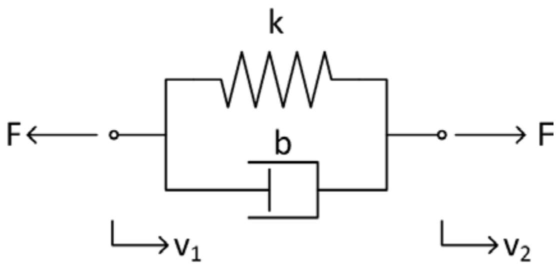
$$e_2 f_2 = e_1 f_1 + e_3 f_3$$

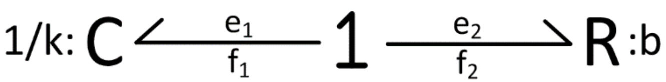
so

$$e_2 = e_1 + e_3$$

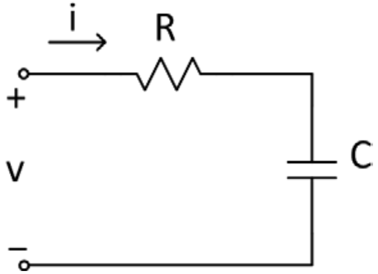
1-Junctions

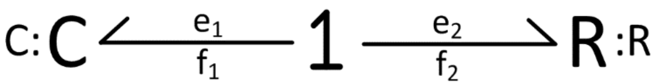
- Constant-flow 1-junction translates to different physical configurations in different domains
- ***Mechanical translational***
  - ***Constant velocity*** – components connected in ***parallel***



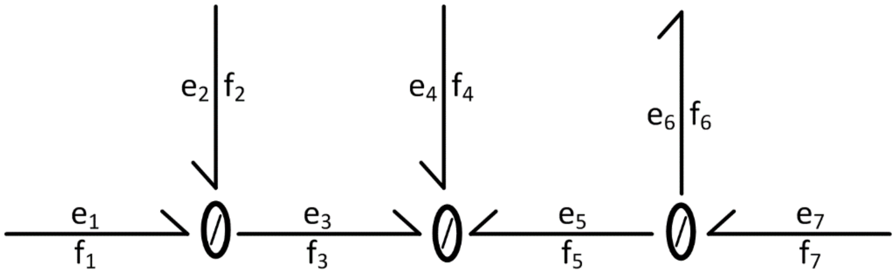


- ***Electrical***
  - ***Constant current*** – components connected in ***series***





Cascaded 0-Junctions



- Equal efforts, flows sum to zero

$$f_1 + f_2 = f_3 \qquad (1)$$

$$f_3 + f_4 + f_5 = 0 \qquad (2)$$

$$f_7 = f_5 + f_6 \qquad (3)$$

- Substitute (2) into (1)

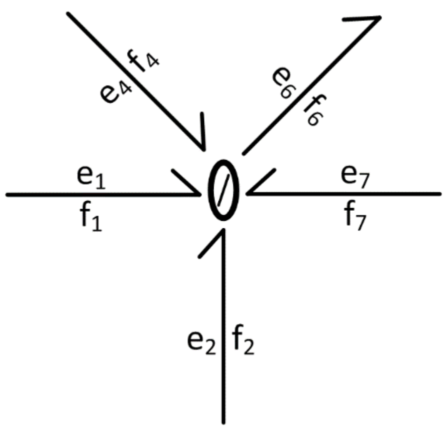
$$f_1 + f_2 = -f_4 - f_5 \qquad (4)$$

- Substitute (3) into (4)

$$f_1 + f_2 = -f_4 + f_6 - f_7$$

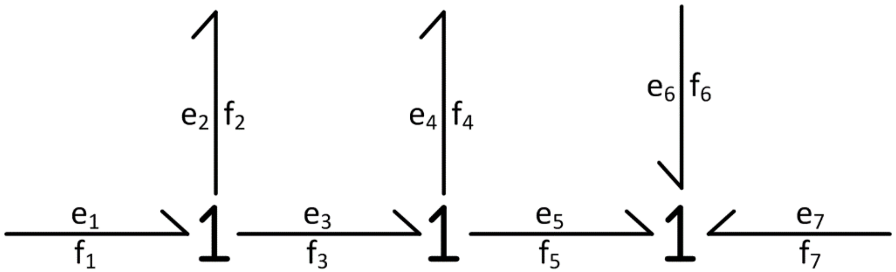
$$f_1 + f_2 + f_4 + f_7 = f_6$$

- Can collapse the cascade to a single 0-junction



- Internal bond directions are irrelevant

Cascaded 1-Junctions



- Equal flow, efforts sum to zero

$$e_1 = e_2 + e_3 \quad (1)$$

$$e_3 = e_4 + e_5 \quad (2)$$

$$e_5 + e_6 + e_7 = 0 \quad (3)$$

- Substitute (2) into (1)

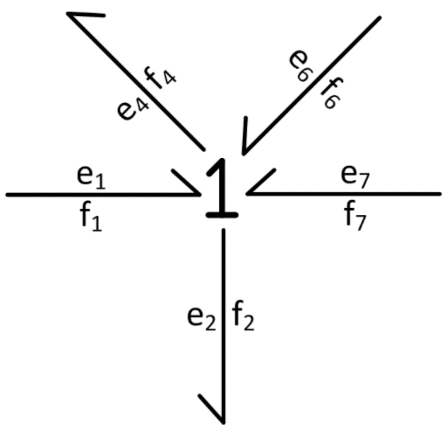
$$e_1 = e_2 + e_4 + e_5 \quad (4)$$

- Substitute (3) into (4)

$$e_1 = e_2 + e_4 - e_6 - e_7$$

$$e_1 + e_6 + e_7 = e_2 + e_4$$

- Can collapse the cascade to a single 1-junction



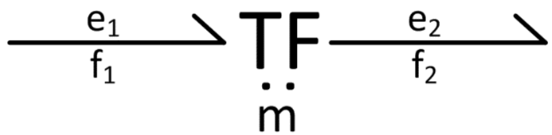
- Internal bond directions are irrelevant

Two-Port Bond Graph Elements

Two-Port Bond Graph Elements

- ***Two-port elements:***
  - ***Transformer***
  - ***Gyrator***
- ***Transmit power***
- Two ports – two bond connection points
- ***Power is conserved***: $\mathcal{P}_{in} = \mathcal{P}_{out}$
  - ***Ideal***, i.e. ***lossless***, elements
- May provide an ***interface between energy domains***
  - E.g. transmission of power between mechanical and electrical subsystems
- Bonds always follow a *through* convention
  - One in, one out

Transformer



- ***<u>Transformers</u>*** – relate effort at one port to effort at the other and flow at one port to flow at the other
  - Efforts and flows related through the ***transformer modulus, m***
  - ***Constitutive law:***

$$e_2 = m e_1 \qquad \text{and} \qquad f_2 = \frac{1}{m} f_1$$

- Power is conserved, so

$$e_1 f_1 = e_2 f_2 = m e_1 \frac{1}{m} f_1 = e_1 f_1$$

Transformer – Mechanical

- Relation of efforts, F_1 and F_2
  - Balance the moments:

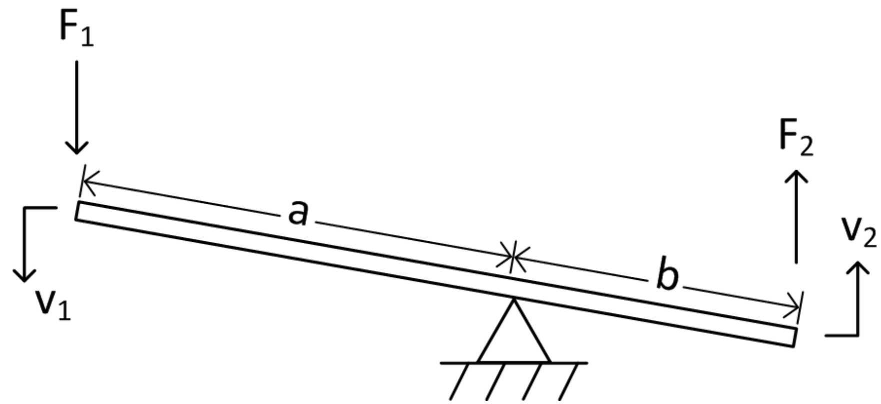
$$aF_1 = bF_2$$

$$F_2 = \frac{a}{b}F_1$$

- Relation of flows, v_1 and v_2
  - Equal angular velocity all along lever arm:

$$\omega = \frac{v_1}{a} = \frac{v_2}{b}$$

$$v_2 = \frac{b}{a}v_1$$



- Bond graph model:

a/b

$e_2 = (a/b)e_1$

- Include effort-to-effort or flow-to-flow relationship

Transformer – Electrical

- Relation of flows, i_1 and i_2
  - Current scales with the ***turns ratio***:

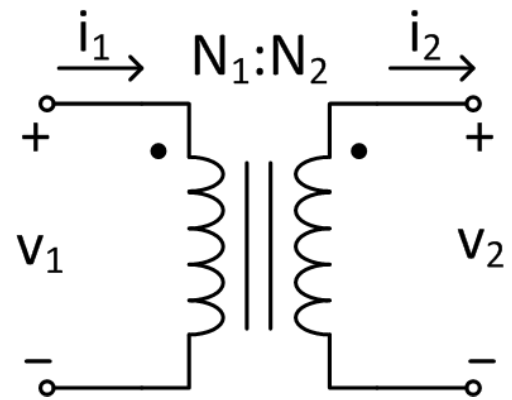
$$i_2 = \frac{N_1}{N_2} i_1$$

- Relation of efforts, v_1 and v_2
  - Voltage scales with the ***inverse of the turns ratio***:

$$v_2 = \frac{N_2}{N_1} v_1$$

- Power is conserved

$$\mathcal{P}_{out} = i_2 v_2 = \frac{N_1}{N_2} i_1 \frac{N_2}{N_1} v_1 = i_1 v_1 = \mathcal{P}_{in}$$



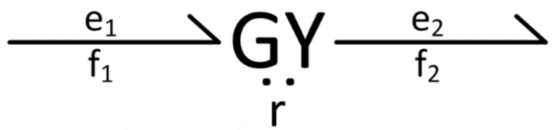
- Bond graph model:

N_1/N_2

$e_2 = (N_2/N_1)e_1$

- Include effort-to-effort or flow-to-flow relationship

Gyrator



- ***Gyrators*** – effort at one port related to flow at the other
  - Efforts and flows related through the ***gyrator modulus, r***
  - ***Constitutive law:***

$$e_2 = r f_1 \qquad \text{and} \qquad f_2 = \frac{1}{r} e_1$$

- Power is conserved so

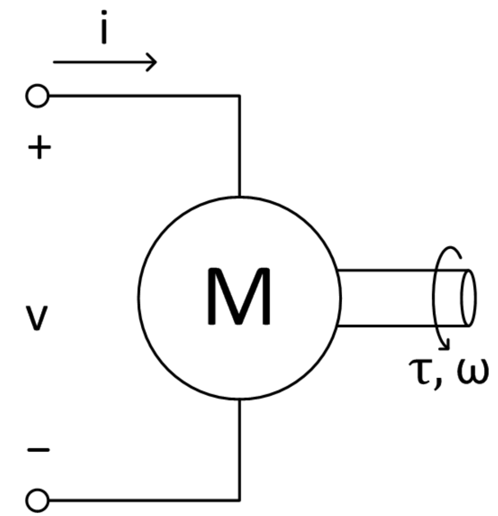
$$e_1 f_1 = e_2 f_2 = r f_1 \frac{1}{r} e_1 = e_1 f_1$$

- Gyrator modulus relates effort and flow – a ***resistance***
  - Really, a ***transresistance***

Gyrator - Example

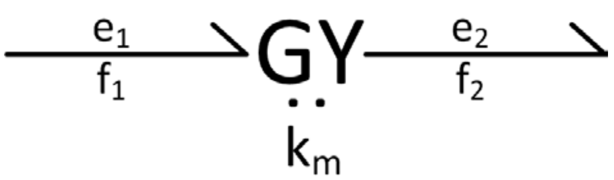
- ***Ideal electric motor***
  - Electrical current, a flow, converted to torque, an effort
- Current and torque related through the ***motor constant***, k_m

$$\tau = k_m i$$



- Power is conserved
  - relationship between voltage and angular velocity is the inverse

$$\omega = \frac{1}{k_m} v$$



$e_2 = k_m f_1$