SECTION 4: MATHEMATICAL MODELING

ESE 330 – Modeling & Analysis of Dynamic Systems

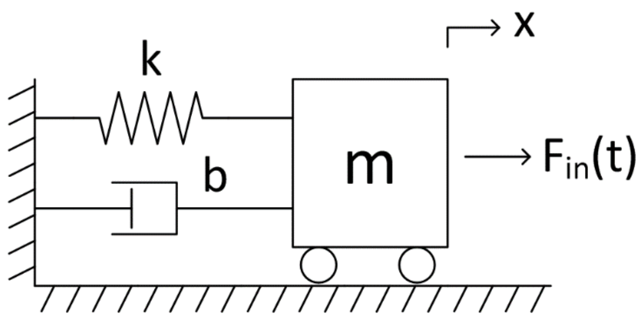
Introduction

In the last section of notes, we saw how to create a bond graph model from a physical system model.

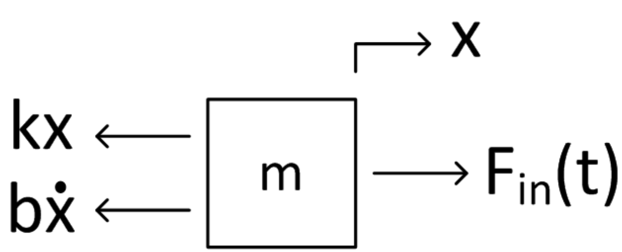
The next step in the modeling process is the creation of a mathematical model

Mathematical Modeling – Introduction

- You're already familiar with some techniques for creating mathematical models for physical systems
- For example:

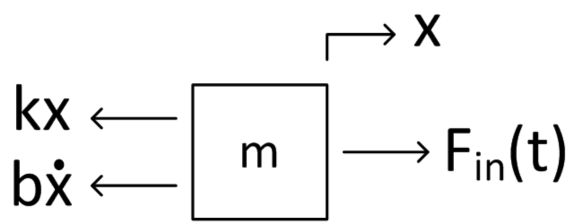


- First, create a free-body diagram:



Mathematical Modeling – Introduction

- Next, apply Newton's 2nd law



$$\Sigma F = ma$$

$$F_{in}(t) - kx - b\dot{x} = m\ddot{x}$$

rearranging:

$$m\ddot{x} + b\dot{x} + kx = F_{in}(t) \qquad (1)$$

- This is a ***mathematical model***
  - A second-order, linear, constant-coefficient, ordinary differential equation

Reduction to a System of 1st-Order ODE's

- Can reduce this 2nd-order ODE to a system of two 1st-order ODE's
- We know that

$$\dot{x} = v \quad (2)$$

and

$$\ddot{x} = a = \dot{v} \quad (3)$$

- Using (2) and (3), rewrite (1), the original ODE

$$m\dot{v} + bv + kx = F_{in}(t)$$

where (4)

$$v = \dot{x}$$

Reduction to a System of 1st-Order ODE's

- Equations (4) is a system of first-order ODE's that is equivalent to (1)
- Rearranging (4):

$$\dot{v} = -\frac{k}{m}x - \frac{b}{m}v + \frac{1}{m}F_{in}(t)$$
$$\dot{x} = v \qquad (5)$$

- These equations can be put into matrix form :

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F_{in}(t) \qquad (6)$$

Reduction to a System of 1st-Order ODE's

- Let's say we want to consider the ***displacement*** of the mass as the ***output*** of the system
- We can add an ***output equation*** to the mathematical model

$$y = x \tag{7}$$

- We can rewrite (7) in a matrix form similar to (6):

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + [0] F_{in}(t) \tag{8}$$

Mathematical Model

- Together, (6) and (8) comprise the mathematical model for our mechanical system:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F_{in}(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \qquad (9)$$

- Note that $\dot{x}$, $\dot{v}$, x, v, and y are all ***functions of time***
  - The (t) is dropped to simplify the notation
  - The convention used here is to only include the (t) for ***inputs***, e.g. $F_{in}(t)$

State-Space Representation

- The system model of (9) is the ***state-space representation*** of the system, or the ***state-variable equations*** for the system
- Can be expressed in generic form as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{10}$$

where

- $\mathbf{x}$: the state vector
- $\dot{\mathbf{x}}$: derivative of the state
- $\mathbf{u}$: vector of inputs
- $\mathbf{y}$: vector of outputs
- $\mathbf{A}$: system matrix
- $\mathbf{B}$: input matrix
- $\mathbf{C}$: output matrix
- $\mathbf{D}$: feed-through matrix

MIMO vs. SISO Systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \qquad (10)$$

- Note that the state-space model (10) allows for ***vectors of inputs and outputs***, $\mathbf{u}$ and $\mathbf{y}$
- ***Multi-input, multi-output (MIMO) systems***
  - $\mathbf{u}$ and $\mathbf{y}$ will be ***vectors***
- ***Single-input, single-output (SISO) systems***
  - u and y will be ***scalars***
- In this course, we'll mostly focus on SISO systems
  - For now, we'll assume the more general MIMO case

System State and State Variables

- The vector $\mathbf{x}$ is the ***state vector***
  - Elements of $\mathbf{x}$ are the ***state variables*** of the system
- The ***state*** of the system is a complete description of the current condition of the system
  - From our energy-based perspective, the state describes all of the energy in a system, i.e. where it is stored, at a given point in time
- The ***state variables*** are *a* (not *the*) minimum set of system variables required to completely describe the ***state*** of a system

State Variables are Not Unique

- The state vector, i.e. the choice of state variables, for a system is ***not unique***
  - In this example, we have chosen displacement and velocity as the state variables, i.e.

$$\mathbf{x} = \begin{bmatrix} x \\ v \end{bmatrix}$$

  - Could have chosen other quantities – later, we will
  - State variables need not even have direct physical significance
- Different state-space representations for the same system are related by ***similarity transforms***
  - Beyond the scope of this class

The Feed-Through Matrix

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \qquad (10)$$

- $\mathbf{D}$ is the feed-through or feed-forward matrix
  - Very often zero for physical systems, as in our example
- Non-zero $\mathbf{D}$ implies that the input affects the output instantaneously
  - There exists a direct feed-through path from the input to the output

State-Space Vector and Matrix Dimensions

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \qquad (10)$$

- Assume the state space model of (10) represents an n^{th}-order, m-input, p-output MIMO system
- The ***state vector*** is an $n \times 1$ column vector
- The system has m inputs, so the ***input vector*** is an $m \times 1$ column vector
- There are p outputs, so the ***output vector*** is a $p \times 1$ column vector

State-Space Vector and Matrix Dimensions

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \qquad (10)$$

- If $\mathbf{x}$ is $n \times 1$, then its derivative, $\dot{\mathbf{x}}$, is also $n \times 1$
- The product $\mathbf{Ax}$ must have the same dimensions as $\dot{\mathbf{x}}$, $n \times 1$
  - The system matrix, $\mathbf{A}$, is a square $n \times n$ matrix
- The product $\mathbf{Bu}$ must also be $n \times 1$
  - The vector of inputs, $\mathbf{u}$, is $m \times 1$, so $\mathbf{B}$ is $n \times m$

State-Space Vector and Matrix Dimensions

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \qquad (10)$$

- The vector of p outputs, $\mathbf{y}$, is $p \times 1$
- The product $\mathbf{Cx}$ must also have dimension $p \times 1$
  - $\mathbf{x}$ is $n \times 1$, so $\mathbf{C}$ must be $p \times n$

- The product $\mathbf{Du}$ must also have the same dimension as $\mathbf{y}$, $p \times 1$
  - The vector of inputs, $\mathbf{u}$, is $m \times 1$, so $\mathbf{D}$ is $p \times m$

State-Space Vector and Matrix Dimensions

- For an m-input, p-output, MIMO system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Term	Dimension
$\mathbf{u}$	$m \times 1$
$\mathbf{y}$	$p \times 1$
$\mathbf{x}$	$n \times 1$
$\dot{\mathbf{x}}$	$n \times 1$

Term	Dimension
$\mathbf{A}$	$n \times n$
$\mathbf{B}$	$n \times m$
$\mathbf{C}$	$p \times n$
$\boldsymbol{D}$	$p \times m$

State-Space Vector and Matrix Dimensions

- For SISO system, u and y, as well as D, are scalars:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

Term	Dimension
u	1×1
y	1×1
$\mathbf{x}$	$n \times 1$
$\dot{\mathbf{x}}$	$n \times 1$

Term	Dimension
$\mathbf{A}$	$n \times n$
$\mathbf{B}$	$n \times 1$
$\mathbf{C}$	$1 \times n$
D	1×1

State-Space Model Explained

- Remember, our reason for modeling a system is to enable the ***analysis of its dynamic behavior***

- Basic idea of the state space model:
  - If the current state of a system is known, and the current and future values of the inputs are known, then the trajectory of the system (i.e. the time-evolution of its state variables) can be determined

  - Don't need explicit knowledge of the history of the system or its inputs – no past information
    - All history is accounted for in the current value of the state

State-Space Model – Physical Significance

- Consider the physical meaning of the state-space system model

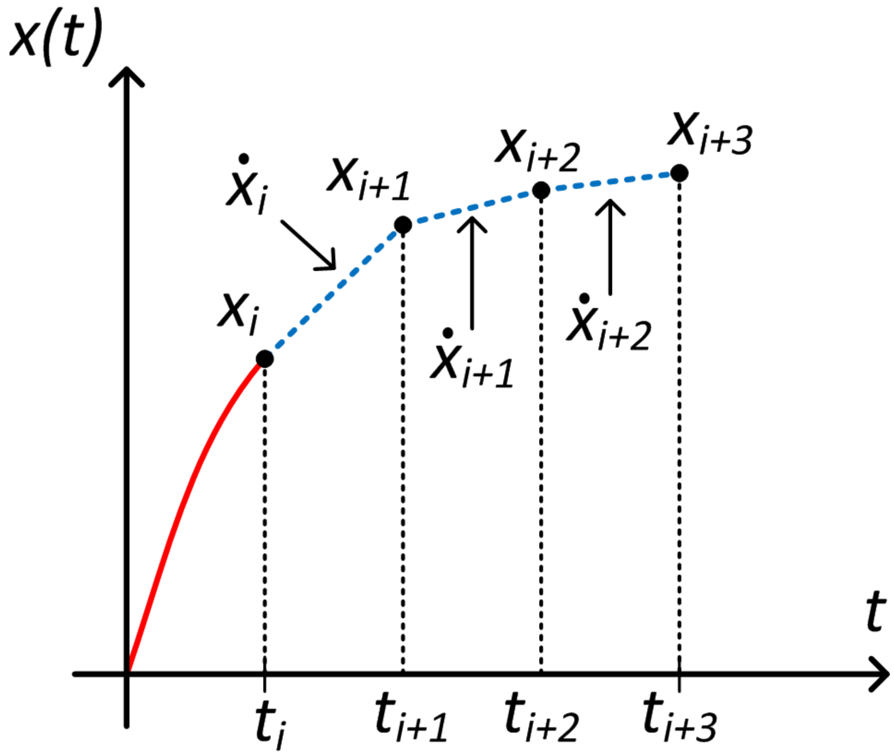
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- The ***time derivative of a system's state variables*** can be expressed as a linear combination of the current state variables and the current inputs

- The ***outputs of a system*** can be expressed as a linear combination of the current state and the current inputs

State-Space Model – Utility

- Again, our goal is to analyze a system's time-domain behavior – the time-evolution of its state variables
- Knowledge of the current state variables, as well as the current rate of change of those state variables, allows us to do this



Where We're Going

- In the previous example, we derived the state-space model for a mechanical system by applying Newton's 2^{nd} law
  - For an electrical system we could have applied Kirchhoff's and Ohm's laws
  - Can always derive a mathematical model by applying domain-specific laws to the physical model

- Our approach will be to ***derive state equations from bond-graph system models***

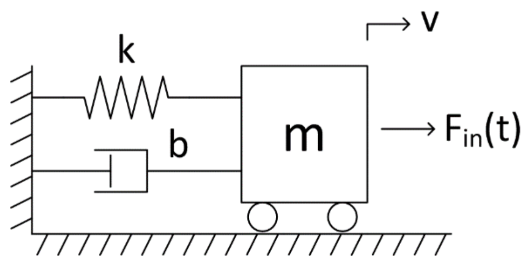
State Equations from Bond-Graph Models

- Bond graphs are ***energy-based*** models
  - Our choice of ***state variables*** will be those that describe the ***storage of energy*** within a system at a given instant in time
- State variables will be ***energy variables*** of the ***independent energy-storage elements*** in a system
  - ***Displacements of capacitors***
  - ***Momenta of inertias***
- Only independent I's and C's
  - State variables represent a minimum set of system variables needed to completely describe the state

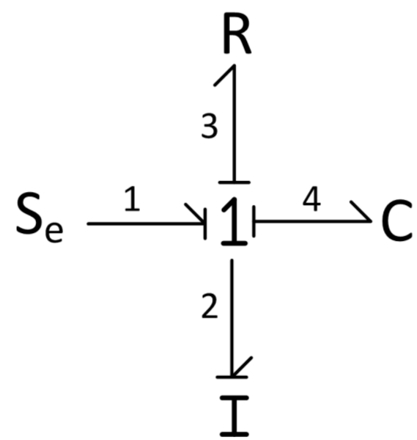
State Equation Derivation

Deriving State Equations from Bond Graphs

- Start with the same mechanical ***system model***:



- The ***computational bond graph***:



- Two independent energy-storage elements
  - ***State variables*** will be the ***energy variables*** associated with these two elements:

$$\mathbf{x} = \begin{bmatrix} p_2 \\ q_4 \end{bmatrix}$$

State Equation Derivation – State Variables

- State equation will be of the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

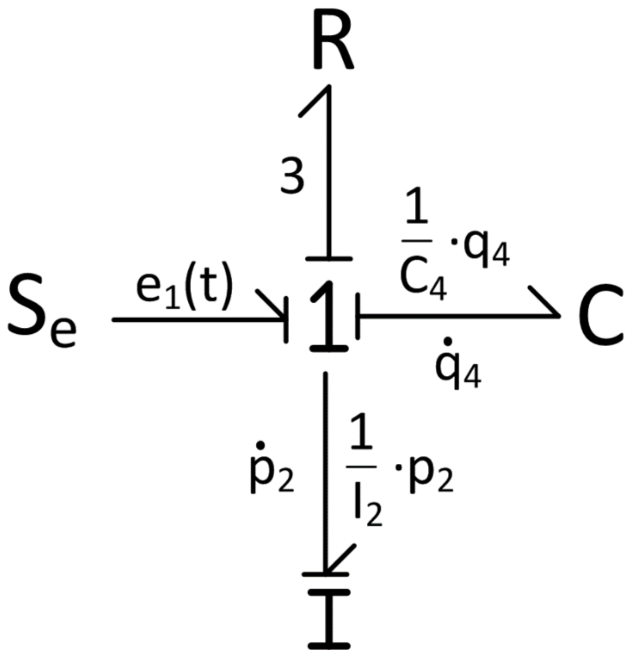
$$\begin{bmatrix} \dot{p}_2 \\ \dot{q}_4 \end{bmatrix} = \mathbf{A} \begin{bmatrix} p_2 \\ q_4 \end{bmatrix} + \mathbf{B} e_1(t)$$

- In general, state variables will be momenta and displacements
  - Their derivatives will be efforts and flows, respectively
  - For this example:

$$\begin{bmatrix} \dot{p}_2 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} e_2 \\ f_4 \end{bmatrix}$$

State Equation Derivation – Preparation

- ***Annotate*** the computational bond graph with ***state variable derivatives***
  - Efforts on the independent Inertias and the flows on the independent Capacitors
- Apply constitutive laws to annotate the ***other power variables*** on the I's and C's
- Annotate the ***known source power variables*** and indicate as functions of time



State Equation Derivation – Procedure

- **Objective**: derive a set of n equations, each expressing a state variable derivative as a linear combination of state variables and inputs
  - Determine the $\mathbf{A}$ and $\mathbf{B}$ matrices

- First, choose a state variable and write its derivative as an effort or flow:

$$\dot{p}_2 = e_2 \tag{1}$$

- Next, use the causality assigned to the bond graph to work from (1) to a state equation
  - Express $\dot{p}_2$ as a linear combination of states and inputs
  - Will ultimately relate an effort or flow to a state variable by applying a constitutive relationship for an energy-storage element

State Equation Derivation

- e_2 is an effort on a 1-jct
  - Caused by e_1, e_3, and e_4, and e_1 is known, so

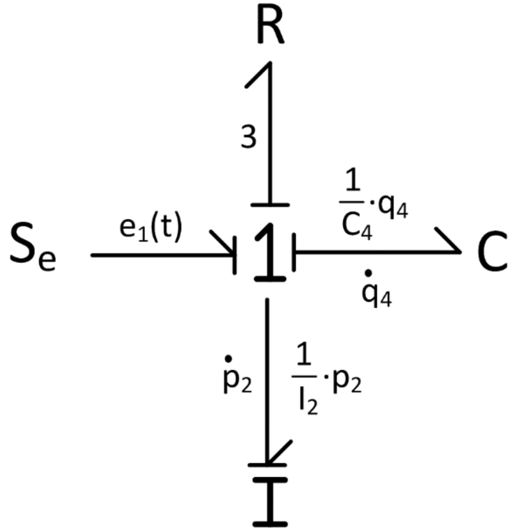
$$\dot{p}_2 = e_2 = e_1(t) - e_3 - e_4 \qquad (2)$$

- Relate e_3 to f_3 using the const. law for the resistor

$$e_3 = R_3 f_3 \qquad (3)$$

- f_3 is the flow on a 1-jct, set by f_2, related to s.v. p_2 by the const. law for the inertia

$$f_3 = f_2 = \frac{1}{I_2} p_2 \qquad (4)$$



State Equation Derivation – Procedure

- Substituting (4) into (3)

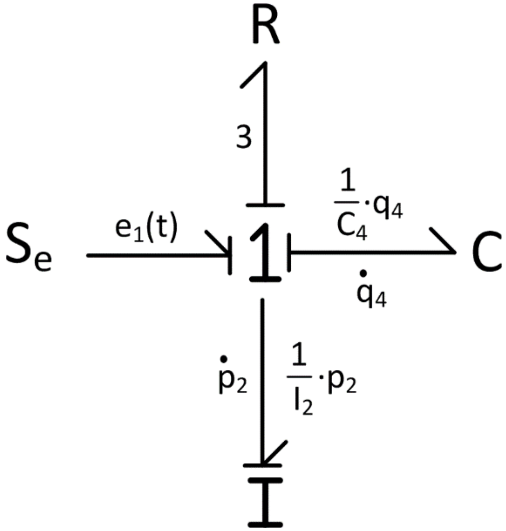
$$e_3 = \frac{R_3}{I_2} p_2 \qquad (5)$$

- And substituting (5) back into (2)

$$\dot{p}_2 = e_1(t) - \frac{R_3}{I_2} p_2 - e_4 \qquad (6)$$

- Still need to eliminate e_4
  - e_4 related to state variable q_4 through constitutive law for the capacitor

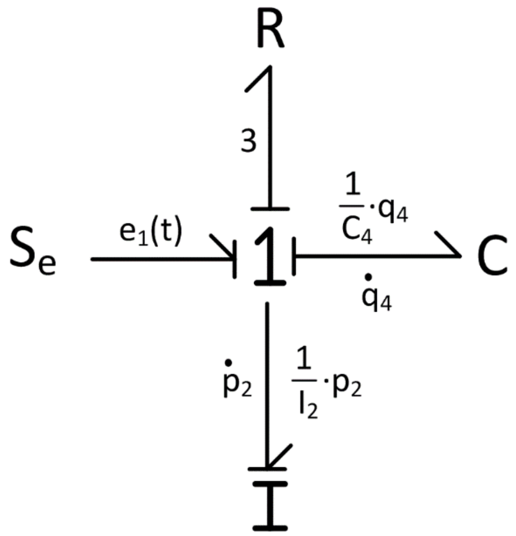
$$e_4 = \frac{1}{C_4} q_4 \qquad (7)$$



State Equation Derivation

- Substituting (7) into (6) yields the first of two ***state equations***

$$\dot{p}_2 = -\frac{R_3}{I_2}p_2 - \frac{1}{C_4}q_4 + e_1(t) \qquad (8)$$



- Next, follow a similar procedure for q_4

$$\dot{q}_4 = f_4 \qquad (9)$$

- f_4 is the flow on a 1-jct, set by f_2, related to state variable p_2 by the const. law for the inertia

$$f_4 = f_2 = \frac{1}{I_2}p_2 \qquad (10)$$

State Equation Derivation

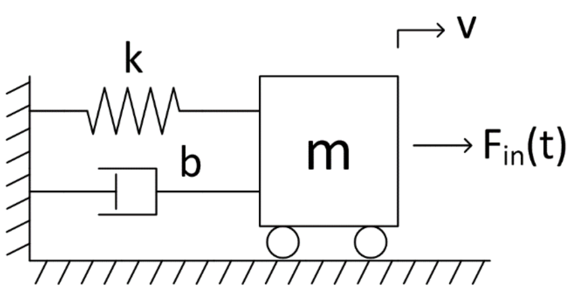
- Substituting (10) into (9) yields the second of two ***state equations***

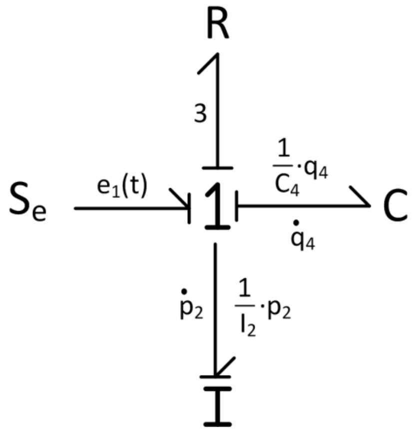
$$\boxed{\dot{q}_4 = \frac{1}{I_2} p_2} \tag{11}$$

- Combine (8) and (11) into the ***state-variable model*** for our system in matrix form

$$\begin{bmatrix} \dot{p}_2 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -\frac{R_3}{I_2} & -\frac{1}{C_4} \\ \frac{1}{I_2} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_1(t) \tag{12}$$

State Equation Derivation



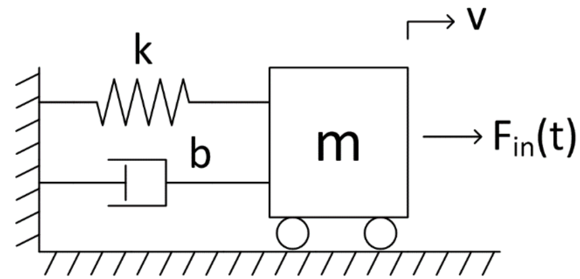


- Can now replace the computational bond graph parameters in (12) with physical system parameters

$$\begin{bmatrix}\dot{p}\\ \dot{x}\end{bmatrix} = \begin{bmatrix}-\frac{b}{m} & -k\\ \frac{1}{m} & 0\end{bmatrix}\begin{bmatrix}p\\ x\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}F_{in}(t) \qquad (13)$$

State Equation Derivation – Output Equation

- Can also define an ***output equation*** as part of our state-space model
- Suppose we want to consider the ***velocity of the mass*** as our output



  - Constitutive relation relates an inertia's flow to its momentum:

$$f_2 = v = \frac{1}{I_2} p_2 = \frac{1}{m} p \qquad (14)$$

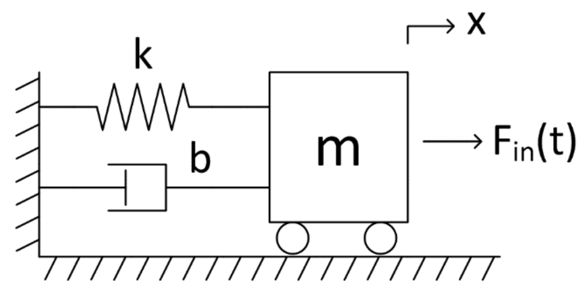
- The output equation would be:

$$y = [1/m \quad 0] \begin{bmatrix} p \\ x \end{bmatrix} \qquad (15)$$

- Equations (13) and (15) comprise the complete state-space system model

State Equation Derivation – Output Equation

- Perhaps, instead, we want to consider the ***displacement of the mass*** as our output
  - Same as spring displacement – a state variable



- State-space model, including output equation, becomes:

$$\begin{bmatrix} \dot{p} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} p \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ x \end{bmatrix} \tag{16}$$

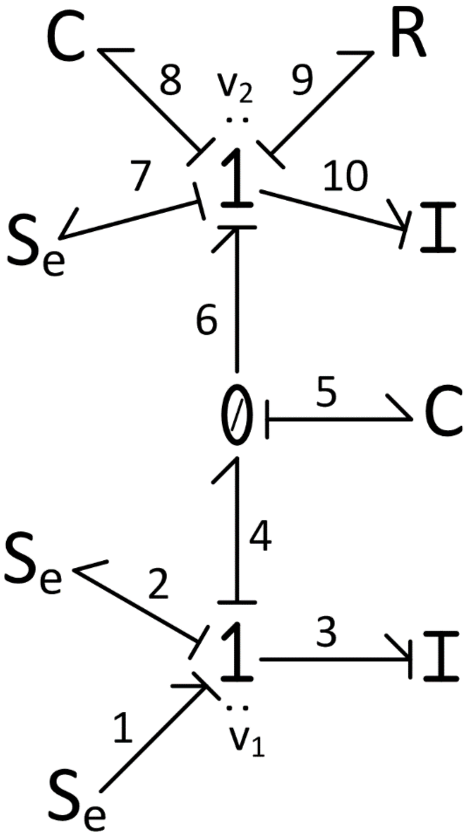
State Equation Derivation – Causality

- In this example, assignment of causality yielded the simplest result:
  - ***All energy-storage elements ended up in integral causality*** – all were independent
  - ***No resistors had their causality arbitrarily assigned***
- Lack of derivative causality and/or algebraic loops (resistor fields) results in straightforward state equation derivation
  - Unfortunately, the inverse is also true
- Next, we'll look at two more examples without derivative causality or algebraic loops

State Equation Derivation – Example 1

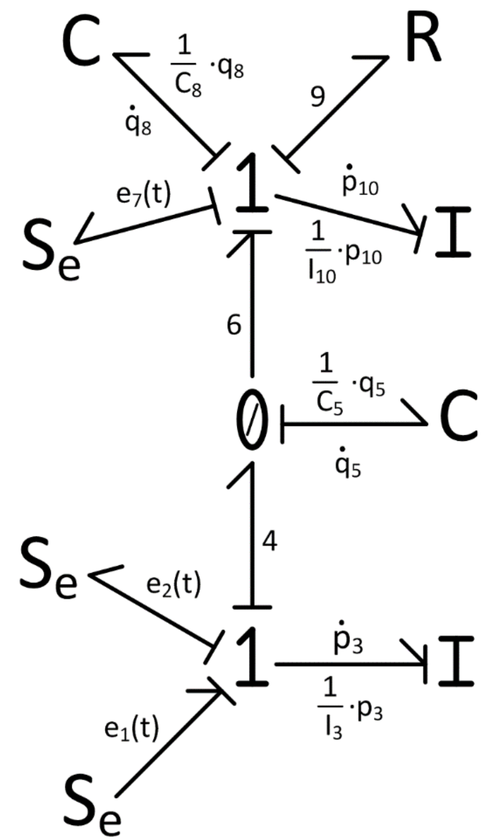
- Consider the mechanical example from Section 3
- Four independent energy-storage elements
  - Fourth-order system
  - Four state variables:

$$\mathbf{x} = \begin{bmatrix} p_3 \\ q_5 \\ q_8 \\ p_{10} \end{bmatrix}$$



State Equation Derivation – Example 1

- Annotate the bond graph:
  - State variable derivatives
    - Efforts on independent inertias
    - Flows on independent capacitors
  - Use constitutive laws and state variables to express:
    - Flows on independent inertias
    - Efforts on independent capacitors
  - Known source quantities



State Equation Derivation – Example 1

- Choose a state variable derivative and express it as an effort or a flow

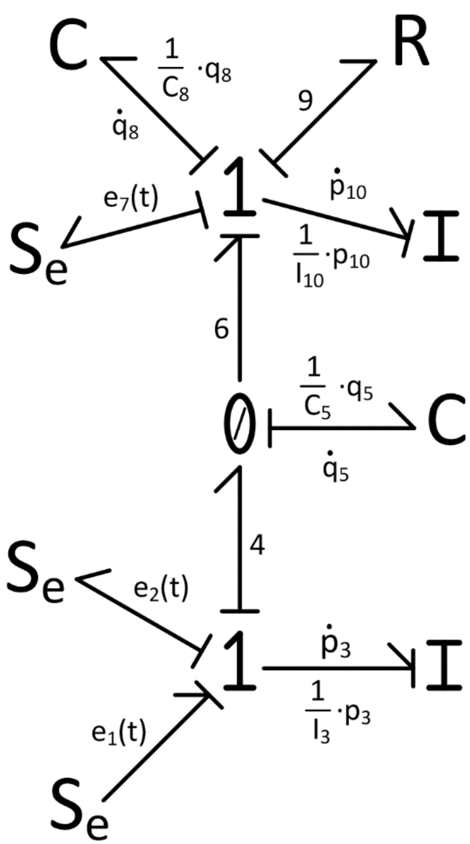
$$\dot{p}_3 = e_3 = e_1(t) - e_2(t) - e_4 \qquad (1)$$

- Known source efforts can remain
  - Need to eliminate e_4
  - Effort on a 0-jct, set by e_5

$$e_4 = e_5 = \frac{1}{C_5} q_5 \qquad (2)$$

- Substituting (2) into (1) yields the ***first of four state equations***

$$\boxed{\dot{p}_3 = -\frac{1}{C_5} q_5 + e_1(t) - e_2(t)} \qquad (3)$$



State Equation Derivation – Example 1

- Move on to the next state variable

$$\dot{q}_5 = f_5 = f_4 - f_6 \qquad (4)$$

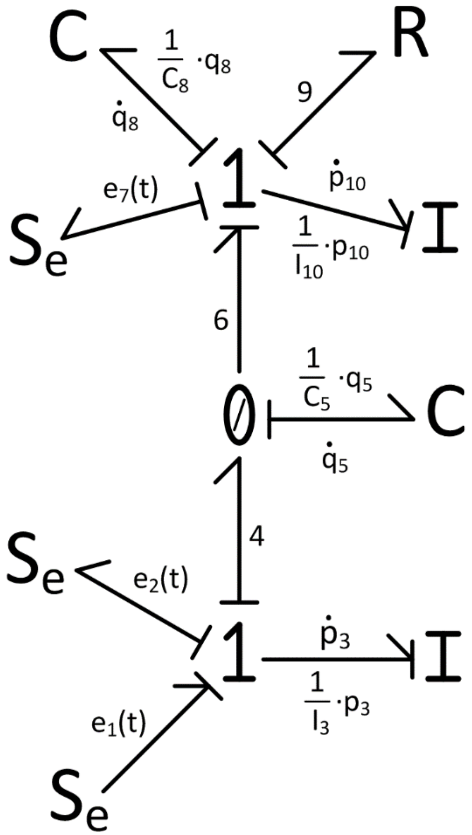
- f_4 and f_6 are both flows on 1-jct's set by f_3 and f_{10}, respectively

$$f_4 = f_3 = \frac{1}{I_3} p_3 \qquad (5)$$

$$f_6 = f_{10} = \frac{1}{I_{10}} p_{10} \qquad (6)$$

- Substituting (6) and (5) into (4) yields the ***second state equation***

$$\dot{q}_5 = \frac{1}{I_3} p_3 - \frac{1}{I_{10}} p_{10} \qquad (7)$$



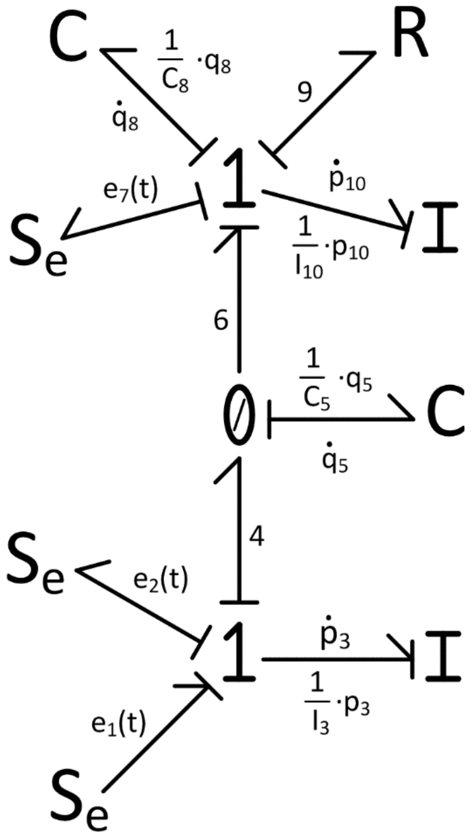
State Equation Derivation – Example 1

- Move on to $\dot{q}_8$

$$\dot{q}_8 = f_8 = f_{10} = \frac{1}{I_{10}} p_{10} \qquad (8)$$

which gives the ***third state equation***

$$\boxed{\dot{q}_8 = \frac{1}{I_{10}} p_{10}} \qquad (9)$$



- Finally, derive the equation for $\dot{p}_{10}$

$$\dot{p}_{10} = e_{10} = e_6 - e_7(t) - e_8 - e_9 \qquad (10)$$

- e_6 is the effort on a 0-jct, set by e_5

$$e_6 = e_5 = \frac{1}{C_5} q_5 \qquad (11)$$

State Equation Derivation – Example 1

- e_8 is related to state variable q_8

$$e_8 = \frac{1}{C_8} q_8 \tag{12}$$

- e_9 can be related to f_9 using the constitutive law for resistor R_9

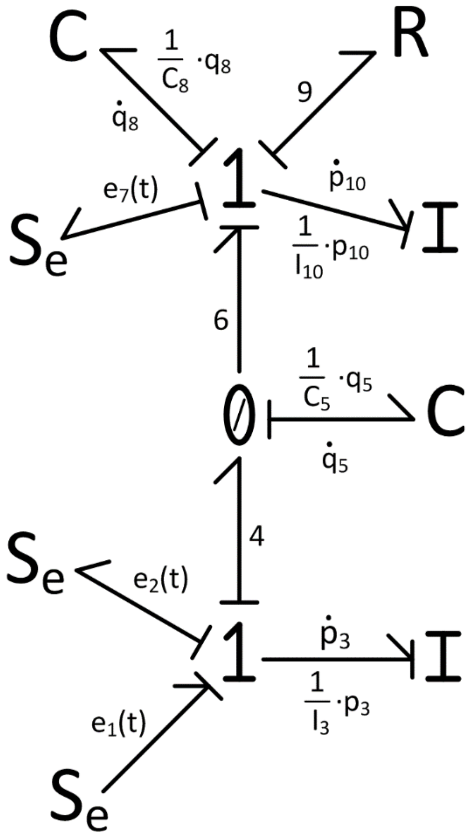
$$e_9 = R_9 f_9 \tag{13}$$

- And, f_9 is the flow on a 1-jct, set by f_{10}

$$e_9 = R_9 f_{10} = R_9 \frac{1}{I_{10}} p_{10} \tag{14}$$

- Substituting (11), (12), and (14) into (10) yields the ***final state equation***

$$\boxed{\dot{p}_{10} = \frac{1}{C_5} q_5 - \frac{1}{C_8} q_8 - \frac{R_9}{I_{10}} p_{10} - e_7(t)} \tag{15}$$



State Equation Derivation – Example 1

- Combine the state equations into matrix form

$$\begin{bmatrix} \dot{p}_3 \\ \dot{q}_5 \\ \dot{q}_8 \\ \dot{p}_{10} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C_5} & 0 & 0 \\ \frac{1}{I_3} & 0 & 0 & -\frac{1}{I_{10}} \\ 0 & 0 & 0 & \frac{1}{I_{10}} \\ 0 & \frac{1}{C_5} & -\frac{1}{C_8} & -\frac{R_9}{I_{10}} \end{bmatrix} \begin{bmatrix} p_3 \\ q_5 \\ q_8 \\ p_{10} \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_7(t) \end{bmatrix} \qquad (16)$$

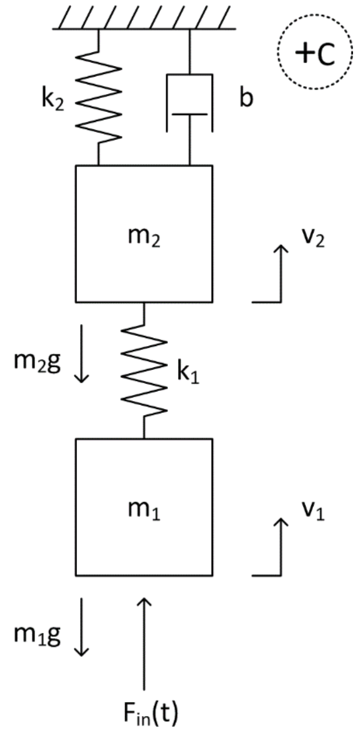
State Equation Derivation – Example 1

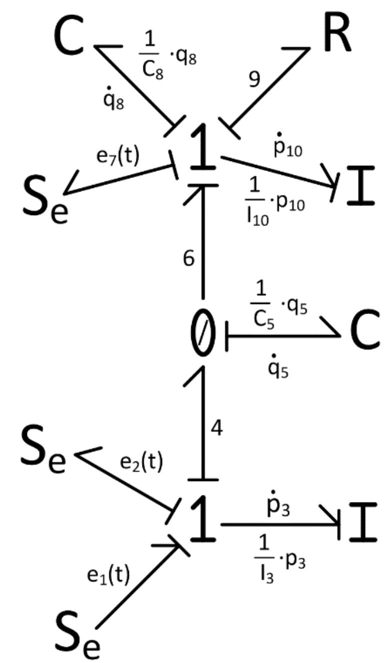
- Let the ***position of each mass*** to be our outputs
  - Two outputs
- Displacement of m_2 (I_{10}) is the displacement of the upper spring

$$x_2 = q_8 \tag{17}$$

- Displacement of m_1 is the sum of the spring displacements

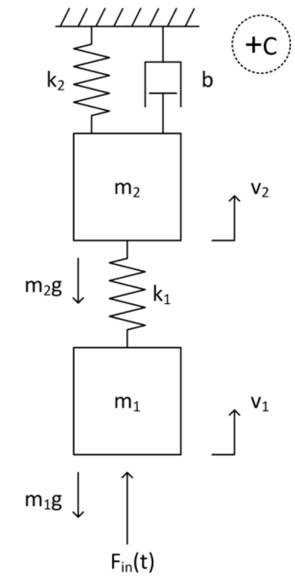
$$x_1 = q_5 + q_8 \tag{18}$$





State Equation Derivation – Example 1

- Combine (17) and (18) into our output equation
  - Multiple outputs, so $\mathbf{C}$ will be a ***matrix***
- Complete state-space model, including output equation:

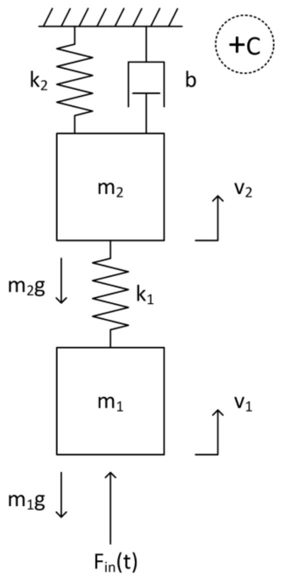


$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p}_3 \\ \dot{q}_5 \\ \dot{q}_8 \\ \dot{p}_{10} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C_5} & 0 & 0 \\ \frac{1}{I_3} & 0 & 0 & -\frac{1}{I_{10}} \\ 0 & 0 & 0 & \frac{1}{I_{10}} \\ 0 & \frac{1}{C_5} & -\frac{1}{C_8} & -\frac{R_9}{I_{10}} \end{bmatrix} \begin{bmatrix} p_3 \\ q_5 \\ q_8 \\ p_{10} \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_7(t) \end{bmatrix} \quad (21)$$

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_3 \\ q_5 \\ q_8 \\ p_{10} \end{bmatrix}$$

State Equation Derivation – Example 1

- Can rewrite our state-space model, substituting in physical parameters
  - q_1 and q_2 are the displacements of springs k_1 and k_2, respectively

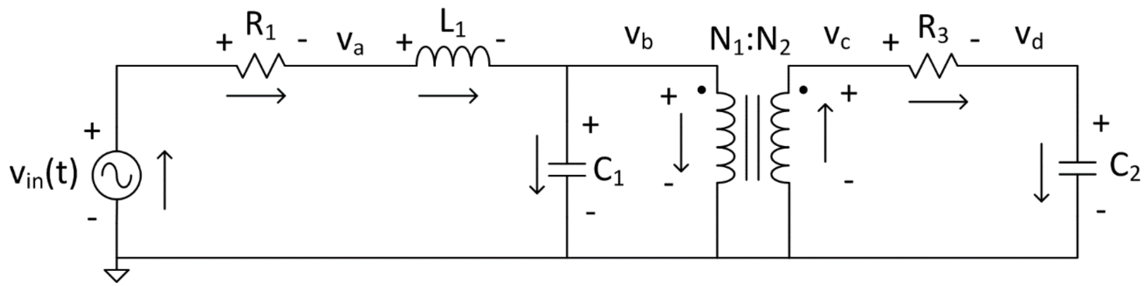


$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 & -\frac{1}{m_2} \\ 0 & 0 & 0 & \frac{1}{m_2} \\ 0 & k_1 & -k_2 & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ q_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} F_{in}(t) \\ m_1 g \\ m_2 g \end{bmatrix} \tag{21}$$

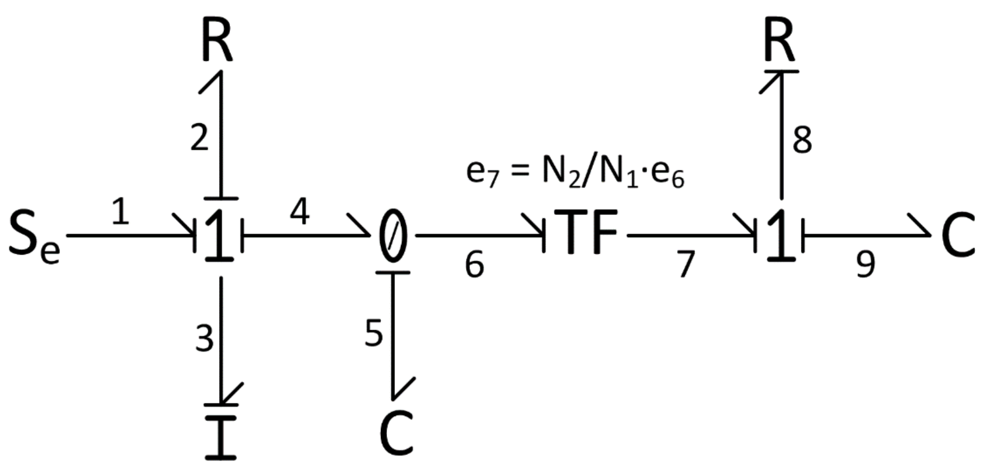
$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ q_2 \\ p_2 \end{bmatrix}$$

State Equation Derivation – Example 2

- A slightly modified version of the electrical circuit from Section 3:



- The computational bond graph for this circuit:

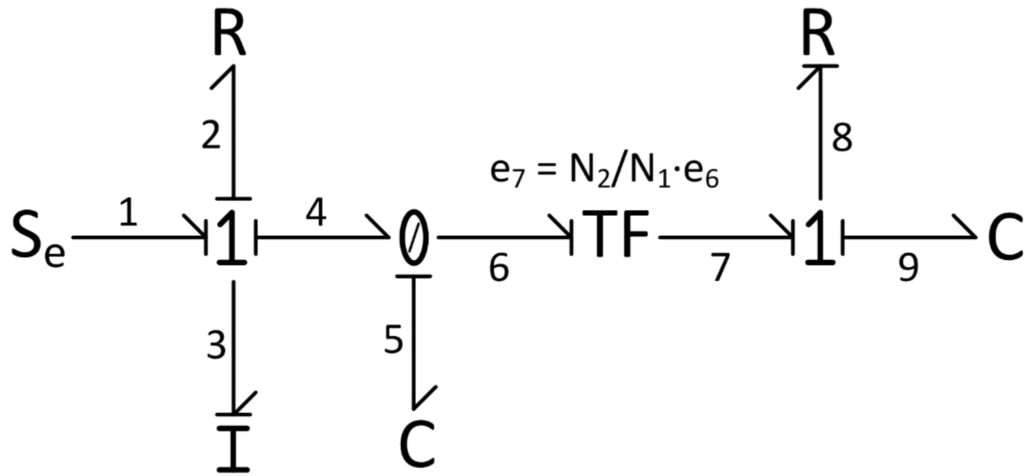


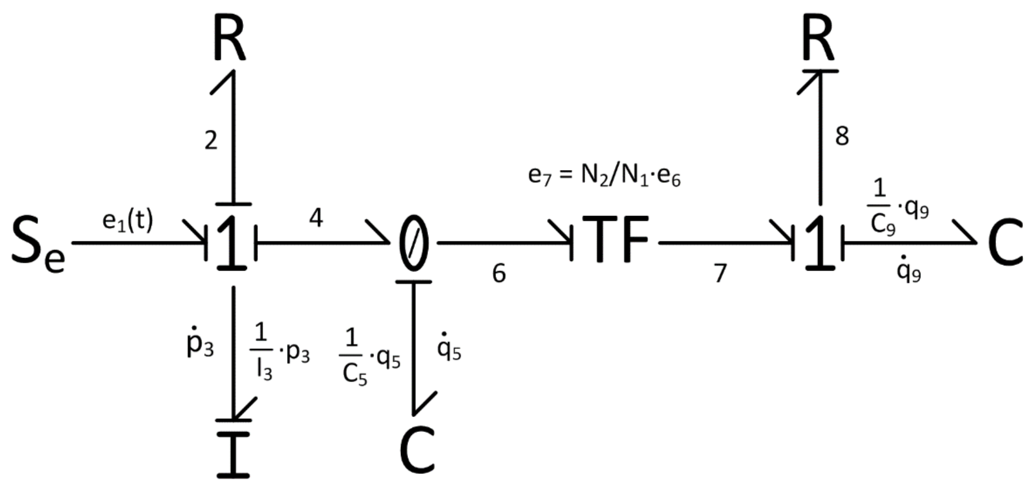
State Equation Derivation – Example 2

- Three independent energy-storage elements
  - Third order
- State variables:

$$\mathbf{x} = \begin{bmatrix} p_3 \\ q_5 \\ q_9 \end{bmatrix}$$

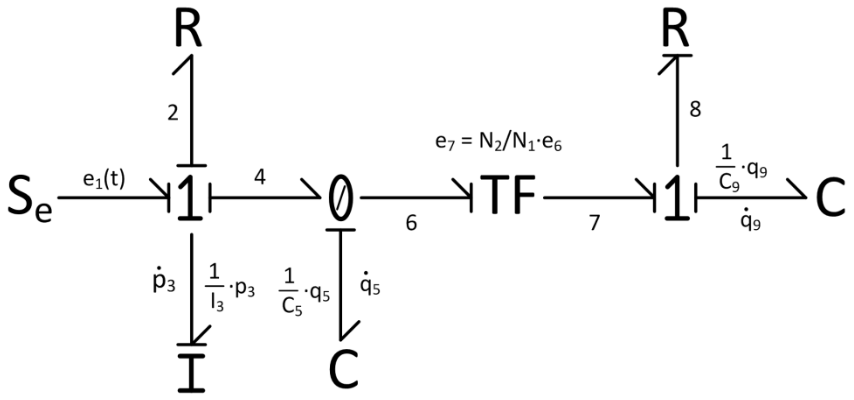
- Annotate the computational bond graph





State Equation Derivation – Example 2

- Begin with equation for $\dot{p}_3$



$$\dot{p}_3 = e_3 = e_1(t) - e_2 - e_4 \qquad (1)$$

$$e_2 = R_2 f_2 = R_2 f_3 = R_2 \frac{1}{I_3} p_3 \qquad (2)$$

- e_4 is the effort on a 0-jct, set by the effort on C_5

$$e_4 = e_5 = \frac{1}{C_5} q_5 \qquad (3)$$

- Substituting (2) and (3) into (1) gives the ***first of three state equations***

$$\boxed{\dot{p}_3 = -\frac{R_2}{I_3} p_3 - \frac{1}{C_5} q_5 + e_1(t)} \qquad (4)$$

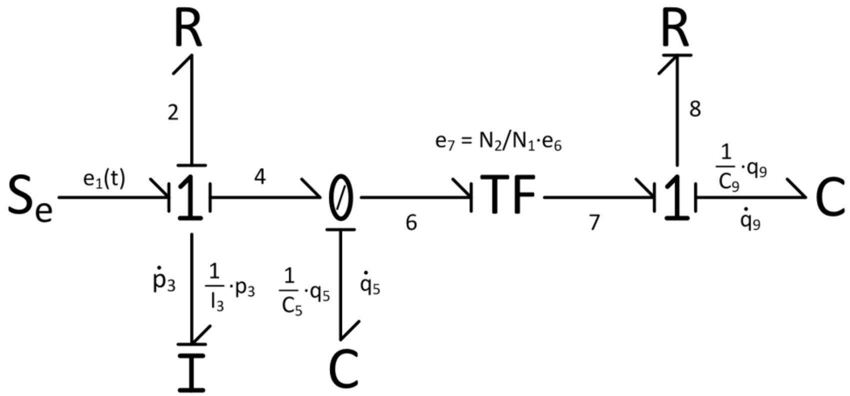
State Equation Derivation – Example 2

- Next, move on to $\dot{q}_5$

$$\dot{q}_5 = f_5 = f_4 - f_6 \qquad (5)$$

- f_4 is set by f_3

$$f_4 = f_3 = \frac{1}{I_3} p_3 \qquad (6)$$



- The transformer modulus relates f_6 to f_7, which is the flow on a 1-jct, set by f_8

$$f_6 = \frac{N_2}{N_1} f_7 = \frac{N_2}{N_1} f_8 = \frac{N_2}{N_1} \frac{1}{R_8} e_8 \qquad (7)$$

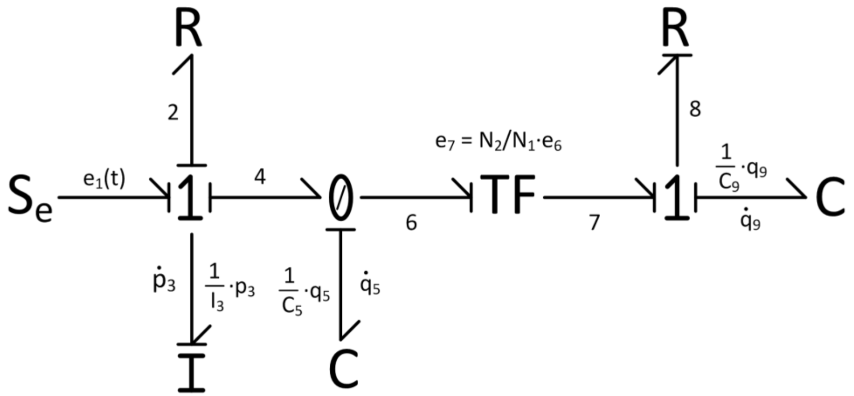
- e_8 is algebraically related to e_7 and e_9

$$e_8 = e_7 - e_9 = e_7 - \frac{1}{C_9} q_9 \qquad (8)$$

State Equation Derivation – Example 2

- The transformer relates e_7 back to e_6, which is set by e_5

$$e_7 = \frac{N_2}{N_1} e_6 = \frac{N_2}{N_1} e_5 = \frac{N_2}{N_1} \frac{1}{C_5} q_5 \qquad (9)$$



- Substituting (9) into (8) gives

$$e_8 = \frac{N_2}{N_1} \frac{1}{C_5} q_5 - \frac{1}{C_9} q_9 \qquad (10)$$

- Equation (10) can be substituted into (7)

$$f_6 = \frac{N_2}{N_1} \frac{1}{R_8} \left(\frac{N_2}{N_1} \frac{1}{C_5} q_5 - \frac{1}{C_9} q_9 \right) \qquad (11)$$

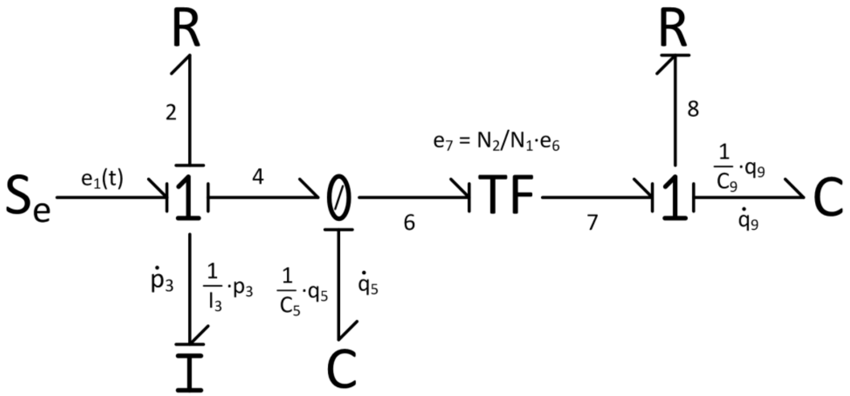
- Using (11) and (6) in (5) gives us our ***second state equation***

$$\boxed{\dot{q}_5 = \frac{1}{I_3} p_3 - \left(\frac{N_2}{N_1} \right)^2 \frac{1}{R_8 C_5} q_5 + \frac{N_2}{N_1} \frac{1}{R_8 C_9} q_9} \qquad (12)$$

State Equation Derivation – Example 2

- Finally, derive the equation for $\dot{q}_9$

$$\dot{q}_9 = f_9 \qquad (13)$$



- f_9 is the flow on a 1-jct, which is set by f_8

$$f_9 = f_8 = \frac{1}{R_8} e_8 \qquad (14)$$

- Substituting (10) into (14)

$$f_9 = \frac{1}{R_8}\left(\frac{N_2}{N_1}\frac{1}{C_5} q_5 - \frac{1}{C_9} q_9\right) \qquad (15)$$

- Substituting (15) in (13) gives us our ***third state equation***

$$\boxed{\dot{q}_9 = \frac{N_2}{N_1}\frac{1}{R_8 C_5} q_5 - \frac{1}{R_8 C_9} q_9} \qquad (16)$$

State Equation Derivation – Example 2

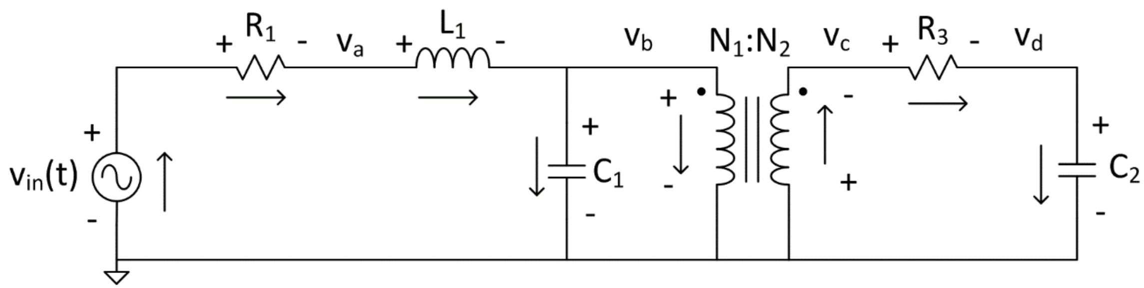
- Combine the state equations in matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p}_3 \\ \dot{q}_5 \\ \dot{q}_9 \end{bmatrix} = \begin{bmatrix} -\frac{R_2}{I_3} & -\frac{1}{C_5} & 0 \\ \frac{1}{I_3} & -\left(\frac{N_2}{N_1}\right)^2 \frac{1}{R_8 C_5} & \frac{N_2}{N_1}\frac{1}{R_8 C_9} \\ 0 & \frac{N_2}{N_1}\frac{1}{R_8 C_5} & -\frac{1}{R_8 C_9} \end{bmatrix} \begin{bmatrix} p_3 \\ q_5 \\ q_9 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e_1(t) \tag{17}$$

- Replacing computational bond graph parameters with physical parameters

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & -\frac{1}{C_1} & 0 \\ \frac{1}{L_1} & -\left(\frac{N_2}{N_1}\right)^2 \frac{1}{R_3 C_1} & \frac{N_2}{N_1}\frac{1}{R_3 C_2} \\ 0 & \frac{N_2}{N_1}\frac{1}{R_3 C_1} & -\frac{1}{R_3 C_2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_{in}(t) \tag{18}$$

State Equation Derivation – Example 2

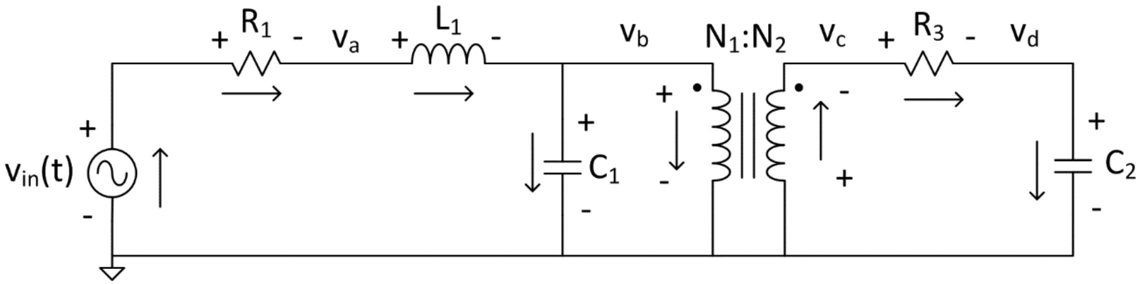


- Choosing the voltage across C_2 as our output, the complete state-space system representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & -\frac{1}{C_1} & 0 \\ \frac{1}{L_1} & -\left(\frac{N_2}{N_1}\right)^2 \frac{1}{R_3 C_1} & \frac{N_2}{N_1}\frac{1}{R_3 C_2} \\ 0 & \frac{N_2}{N_1}\frac{1}{R_3 C_1} & -\frac{1}{R_3 C_2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_{in}(t) \tag{19}$$

$$y = v_d = \begin{bmatrix} 0 & 0 & 1/C_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ q_1 \\ q_2 \end{bmatrix}$$

State Equation Derivation – Example 2



- Instead let the ***voltage across L_1*** be the system output
  - That is, the effort associated with L_1
  - Effort is the time derivative of momentum, so

$$y = v_{L1} = v_a - v_b = \dot{\lambda}_1 \quad (20)$$

- The output equation can be extracted from (19)

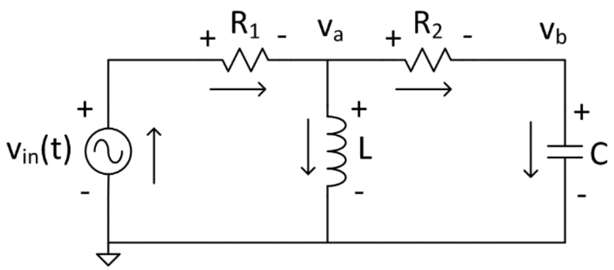
$$y = v_{L1} = \begin{bmatrix} -\frac{R_1}{L_1} & -\frac{1}{C_1} & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ q_1 \\ q_2 \end{bmatrix} + v_{in}(t) \quad (21)$$

- Note that, in this case, the feed-through term, D, is non-zero

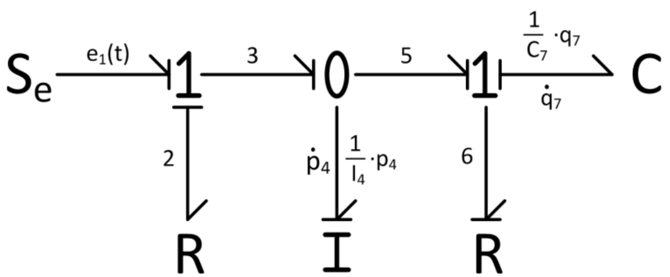
Algebraic Loops or Resistor Fields

Algebraic Loops – Example 1

- Consider the following electrical circuit



- Causality assignment is completed by arbitrarily assigning the causality of resistor R_2 (or R_6)
  - System contains an ***algebraic loop*** (resistor field)

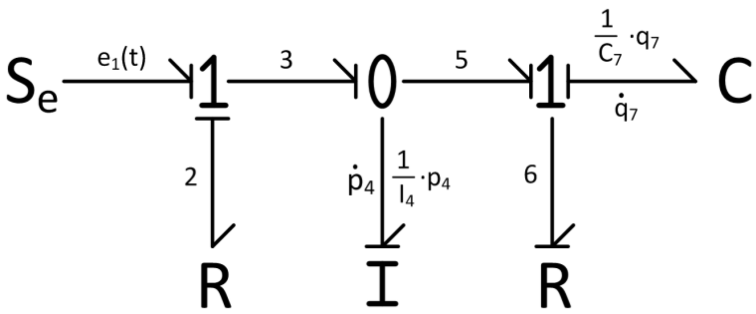


- Presence of the algebraic loop will complicate the state equation derivation a bit

Algebraic Loops – Example 1

- Second-order system
  - State variables are:

$$\mathbf{x} = \begin{bmatrix} p_4 \\ q_7 \end{bmatrix}$$



- Begin deriving equations as usual

$$\dot{p}_4 = e_4 = e_3 = e_1(t) - e_2 = e_1(t) - R_2 f_2 \qquad (1)$$

$$f_2 = f_3 = \frac{1}{I_4} p_4 + f_5 = \frac{1}{I_4} p_4 + f_6 \qquad (2)$$

$$f_6 = \frac{1}{R_6} e_6 = \frac{1}{R_6}\left(e_5 - \frac{1}{C_7} q_7\right) \qquad (3)$$

$$f_6 = \frac{1}{R_6}\left(e_3 - \frac{1}{C_7} q_7\right) \qquad (4)$$

- e_3 has reentered the formulation, and we're back where we started in (1)
  - An ***algebraic loop***

Algebraic Loops – Procedure

1. The ***output*** of the resistor whose causality was arbitrarily assigned – e_2 in this case, though f_6 would work equally well – is the ***auxiliary variable***

2. Derive an expression relating the ***auxiliary variable*** to the ***state variables***, ***inputs***, and to ***itself***

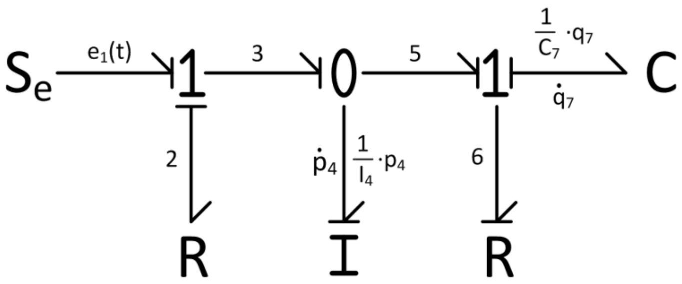
3. Proceed with the ***state equation derivation*** as usual, but leave the auxiliary variable in the formulation along with state variables and inputs

4. ***Substitute*** the result from step 2 into the result from step 3

- One auxiliary variable for each algebraic loop present
  - Multiple loops require solution of a system of equations
- Apply this procedure first, whenever causality assignment involves an arbitrary assignment of resistor causality

Algebraic Loops – Example 1

- Follow causality to derive an expression for auxiliary variable e_2



$$e_2 = R_2 f_2 = R_2 f_3 = R_2\left(\frac{1}{I_4}p_4 + f_5\right) \qquad (5)$$

$$f_5 = f_6 = \frac{1}{R_6}e_6 = \frac{1}{R_6}\left(e_5 - \frac{1}{C_7}q_7\right) \qquad (6)$$

$$e_5 = e_3 = e_1(t) - e_2 \qquad (7)$$

- e_2 is the aux. variable, so it can remain in the expression
- Substituting (7) into (6) into (5)

$$e_2 = \frac{R_2}{I_4}p_4 + \frac{R_2}{R_6}e_1(t) - \frac{R_2}{R_6}e_2 - \frac{R_2}{R_6 C_7}q_7 \qquad (8)$$

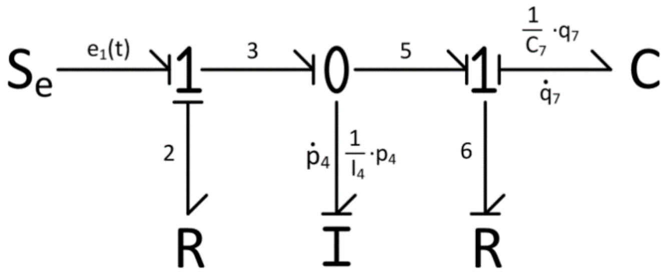
$$e_2\frac{R_2+R_6}{R_6} = \frac{R_2}{I_4}p_4 - \frac{R_2}{R_6 C_7}q_7 + \frac{R_2}{R_6}e_1(t) \qquad (9)$$

Algebraic Loops – Example 1

- Solve (9) for e_2

$$e_2 = \frac{R_2 R_6}{R_2+R_6}\frac{1}{I_4}p_4 - \frac{R_2}{(R_2+R_6)C_7}q_7 + \frac{R_2}{R_2+R_6}e_1(t) \quad (10)$$

- Now, whenever e_2 appears in the formulation, substitute in the expression in (10)



- Going back to (1), we had

$$\dot{p}_4 = e_1(t) - e_2 \quad (1)$$

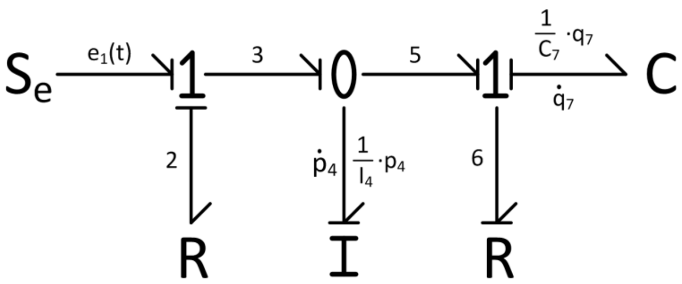
- Substituting in (10) yields the ***first state equation***

$$\dot{p}_4 = -\frac{R_2 R_6}{R_2+R_6}\frac{1}{I_4}p_4 + \frac{R_2}{(R_2+R_6)C_7}q_7 + \frac{R_6}{R_2+R_6}e_1(t) \quad (11)$$

Algebraic Loops – Example 1

- Moving on to $\dot{q}_7$

$$\dot{q}_7 = f_7 = f_6 \tag{12}$$



- We already have an expression for f_6 in (6) and (7)

$$\dot{q}_7 = \frac{1}{R_6} e_1(t) - \frac{1}{R_6} e_2 - \frac{1}{R_6 C_7} q_7 \tag{13}$$

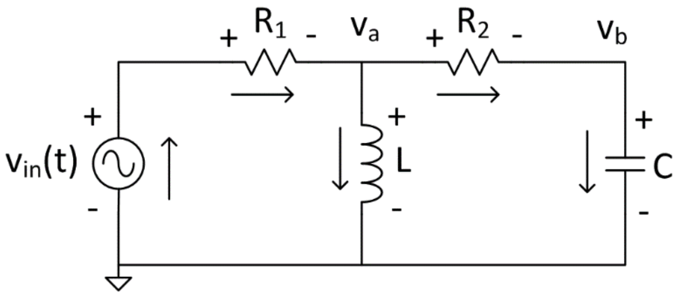
- Substituting in (10) to eliminate e_2

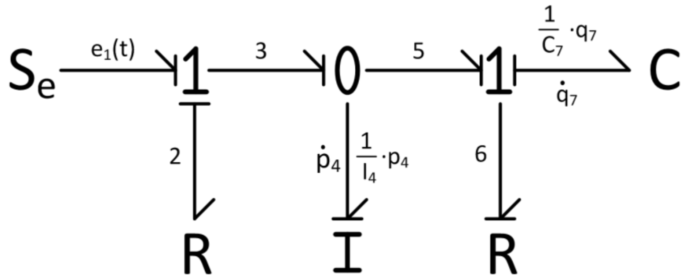
$$\dot{q}_7 = \frac{1}{R_6} e_1(t) - \frac{R_2}{R_2 + R_6}\frac{1}{I_4} p_4 + \frac{R_2}{(R_2 + R_6)R_6 C_7} q_7 - \frac{R_2}{(R_2 + R_6)R_6} e_1(t) - \frac{1}{R_6 C_7} q_7$$

- Rearranging gives the ***second state equation***

$$\boxed{\dot{q}_7 = -\frac{R_2}{R_2+R_6}\frac{1}{I_4} p_4 - \frac{1}{(R_2+R_6)C_7} q_7 + \frac{1}{R_2+R_6} e_1(t)} \tag{14}$$

Algebraic Loops – Example 1

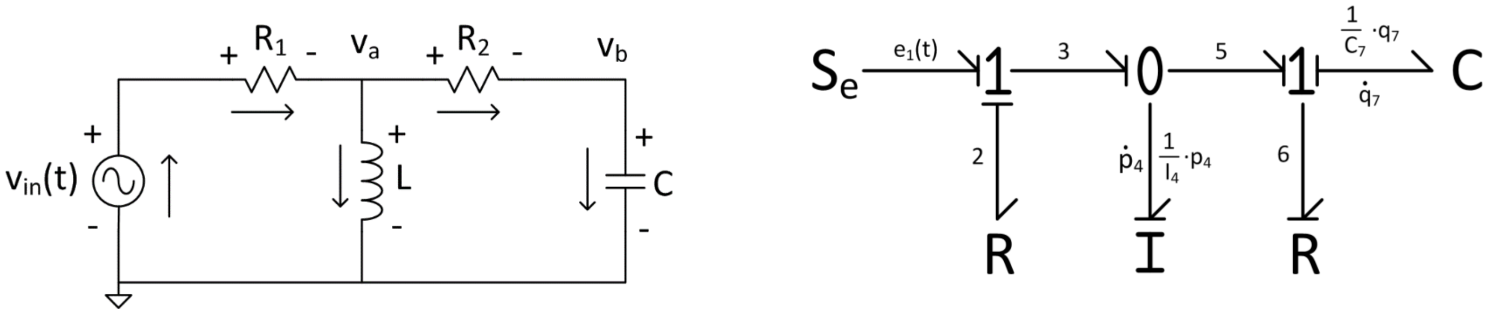




- Assembling (11) and (14) in matrix form gives our state variable system model

$$\begin{bmatrix} \dot{p}_4 \\ \dot{q}_7 \end{bmatrix} = \begin{bmatrix} -\frac{R_2 R_6}{R_2+R_6}\frac{1}{I_4} & \frac{R_2}{(R_2+R_6)C_7} \\ -\frac{R_2}{R_2+R_6}\frac{1}{I_4} & -\frac{1}{(R_2+R_6)C_7} \end{bmatrix} \begin{bmatrix} p_4 \\ q_7 \end{bmatrix} + \begin{bmatrix} \frac{R_6}{R_2+R_6} \\ \frac{1}{R_2+R_6} \end{bmatrix} e_1(t) \qquad (15)$$

Algebraic Loops – Example 1



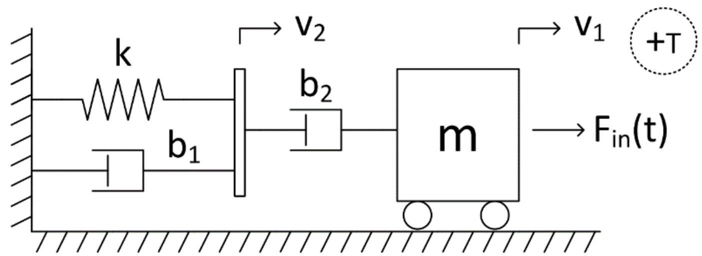
- Substitute in physical parameters and define an output equation for the voltage across the capacitor, v_b

$$\begin{bmatrix}\dot{\lambda}\\ \dot{q}\end{bmatrix} = \begin{bmatrix} -\frac{R_1R_2}{R_1+R_2}\frac{1}{L} & \frac{R_1}{(R_1+R_2)C} \\ -\frac{R_1}{R_1+R_2}\frac{1}{L} & -\frac{1}{(R_1+R_2)C}\end{bmatrix}\begin{bmatrix}\lambda\\ q\end{bmatrix} + \begin{bmatrix}\frac{R_2}{R_1+R_2}\\ \frac{1}{R_1+R_2}\end{bmatrix} v_{in}(t) \tag{16}$$

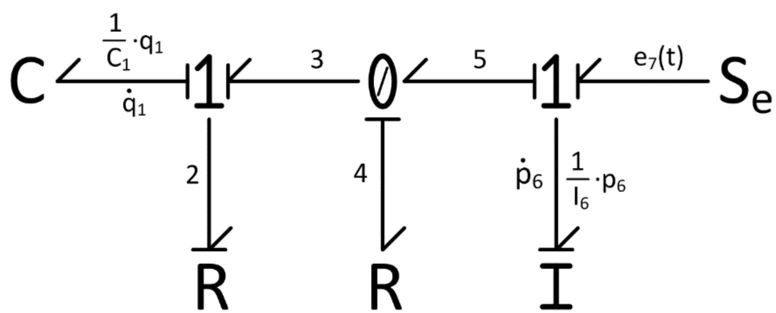
$$y = \begin{bmatrix}0 & 1/C\end{bmatrix}\begin{bmatrix}\lambda\\ q\end{bmatrix}$$

Algebraic Loops – Example 2

- Next, consider a mechanical system



- Causality assignment is completed by arbitrarily assigning the causality of resistor R_2 (or R_4)



- A very similar bond graph to the electrical circuit in the previous example

Algebraic Loops – Example 2

- A ***second-order system*** with state variables:

$$\mathbf{x} = \begin{bmatrix} q_1 \\ p_6 \end{bmatrix}$$

- A ***second-order system*** with state variables:

$$\mathbf{x} = \begin{bmatrix} q_1 \\ p_6 \end{bmatrix}$$

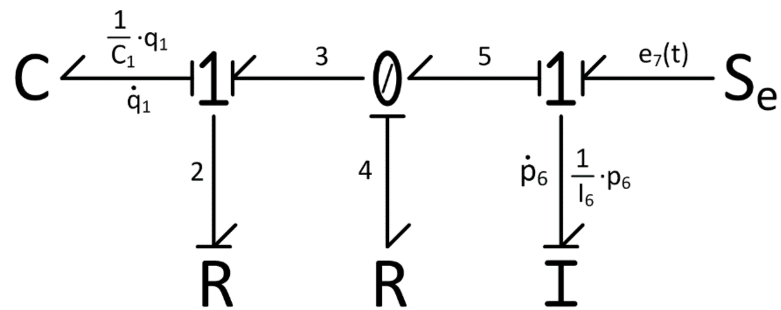
- An ***algebraic loop*** is present, so we'll immediately go to the procedure outlined in the previous example
- ***Auxiliary variable*** is f_2
  - Express f_2 in terms of state variables, inputs, and itself

$$f_2 = \frac{1}{R_2} e_2 = \frac{1}{R_2}(e_3 - e_1) = \frac{1}{R_2}\left(e_4 - \frac{1}{C_1} q_1\right) \quad (1)$$

$$e_4 = R_4 f_4 = R_4(f_5 - f_3) = R_4\left(\frac{1}{I_6} p_6 - f_2\right) \quad (2)$$

- f_2 is the auxiliary variable, so it can remain in the expression

Algebraic Loops – Example 2



- Substitute (2) into (1)

$$f_2 = \frac{R_4}{R_2}\frac{1}{I_6}p_6 - \frac{R_4}{R_2}f_2 - \frac{1}{R_2 C_1}q_1 \qquad (3)$$

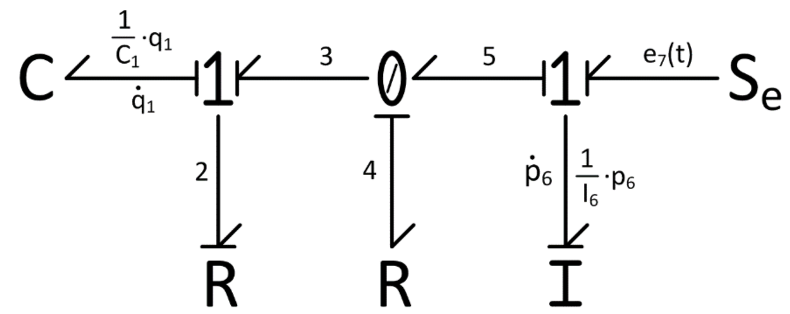
- Then solve for f_2

$$f_2\left(\frac{R_2+R_4}{R_2}\right) = -\frac{1}{R_2 C_1}q_1 + \frac{R_4}{R_2}\frac{1}{I_6}p_6 \qquad (4)$$

$$f_2 = -\frac{1}{(R_2+R_4)C_1}q_1 + \frac{R_4}{R_2+R_4}\frac{1}{I_6}p_6 \qquad (5)$$

- Now, Proceed with the state equation derivation
  - Whenever the auxiliary variable, f_2, appears in the formulation, it will be replaced with (5)

Algebraic Loops – Example 2



- Start with $\dot{q}_1$

$$\dot{q}_1 = f_1 = f2 \tag{6}$$

- Substituting (5) into (6) gives the ***first state equation***

$$\boxed{\dot{q}_1 = -\frac{1}{(R_2+R_4)C_1} q_1 + \frac{R_4}{R_2+R_4}\frac{1}{I_6} p_6} \tag{7}$$

- Moving on to $\dot{p}_6$

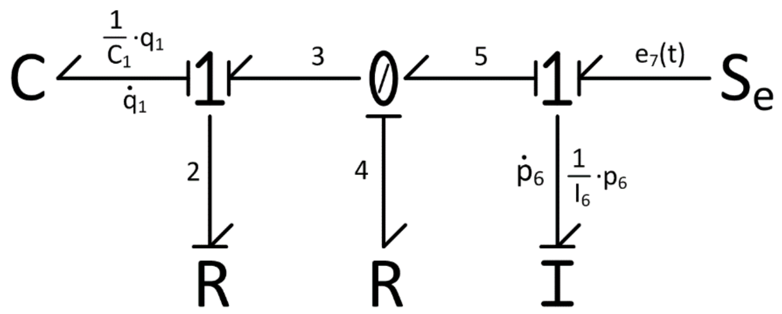
$$\dot{p}_6 = e_6 = e_7(t) - e_5 = e_7(t) - e_4 \tag{8}$$

$$e_4 = R_4 f_4 = R_4(f_5 - f_3) = R_4\left(\frac{1}{I_6} p_6 - f_2\right) \tag{9}$$

Algebraic Loops – Example 2

- Substitute (9) into (8)

$$\dot{p}_6 = e_7(t) - \frac{R_4}{I_6} p_6 + R_4 f_2 \qquad (10)$$



- Substituting (5) in for f_2 gives the ***second state equation***

$$\dot{p}_6 = e_7(t) - \frac{R_4}{I_6} p_6 + R_4 \left(-\frac{1}{(R_2+R_4)C_1} q_1 + \frac{R_4}{R_2+R_4} \frac{1}{I_6} p_6 \right) \qquad (11)$$

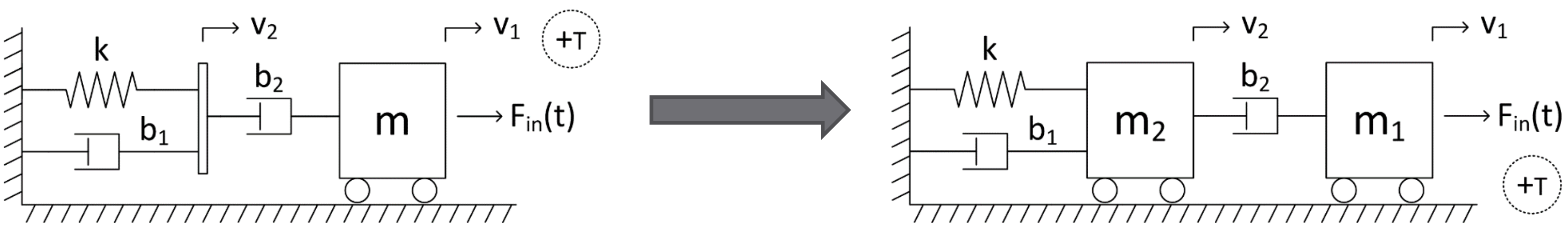
$$\boxed{\dot{p}_6 = -\frac{R_4}{(R_2+R_4)C_1} q_1 - \frac{R_2 R_4}{R_2+R_4} \frac{1}{I_6} p_6 + e_7(t)} \qquad (12)$$

- In matrix form:

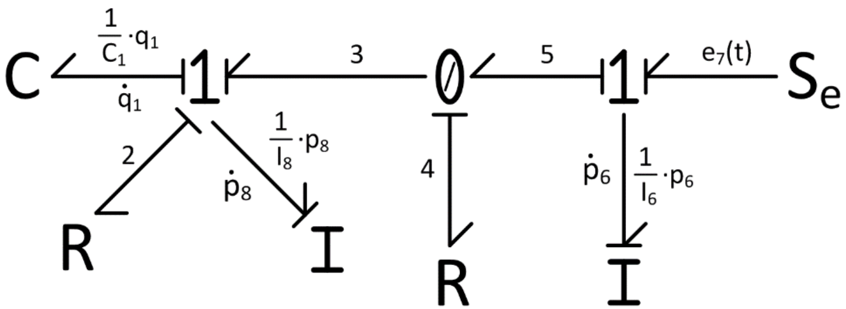
$$\begin{bmatrix} \dot{q}_1 \\ \dot{p}_6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{(R_2+R_4)C_1} & \frac{R_4}{R_2+R_4} \frac{1}{I_6} \\ -\frac{R_4}{(R_2+R_4)C_1} & -\frac{R_2 R_4}{R_2+R_4} \frac{1}{I_6} \end{bmatrix} \begin{bmatrix} q_1 \\ p_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_7(t) \qquad (13)$$

Algebraic Loops – Example 2

- Note that the origin of the algebraic loop in this example was a ***modeling assumption***
  - The connection point between the spring and dampers was considered ***massless***
  - Instead we could ***account for the mass*** of this junction



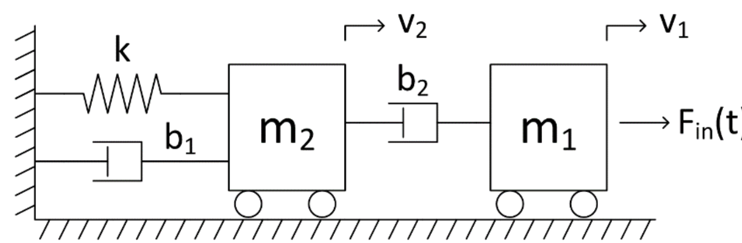
- Now, there are no arbitrary causality assignments and ***no algebraic loops***



- State equation derivation will be greatly simplified

Algebraic Loops – Example 2

- System is now ***third-order***, due to the additional independent energy-storage element



- State equation, after replacing physical parameters:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{m_2} \\ 0 & -\frac{b_2}{m_1} & \frac{b_2}{m_1} \\ -k & \frac{b_2}{m_1} & -\frac{b_1+b_2}{m_2} \end{bmatrix} \begin{bmatrix} x_2 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} F_{in}(t) \qquad (14)$$

- Looks very different from the original second order model, but if $m_2 \ll m_1$, their behaviors are nearly identical

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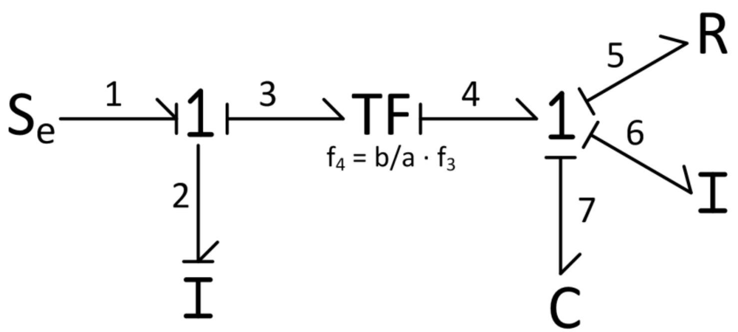
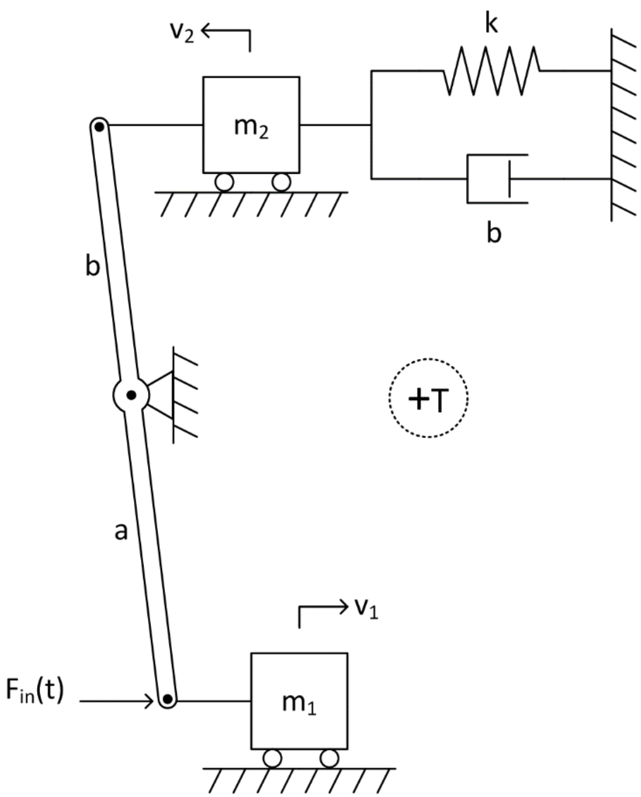
Derivative Causality

Derivative Causality – Example

- Consider the mechanical system from Section 3
- The computational bond graph:



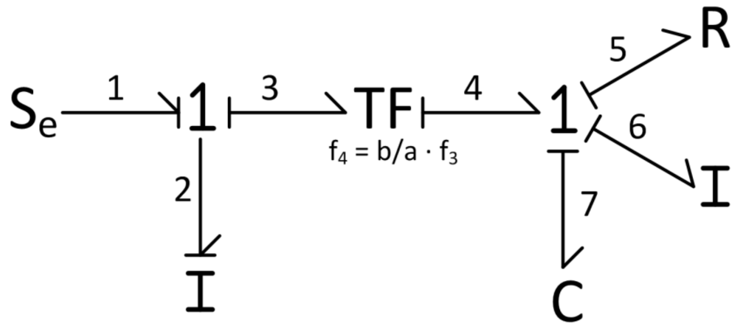


- Two ***independent*** energy-storage elements
  - Second-order
  - State variables are:

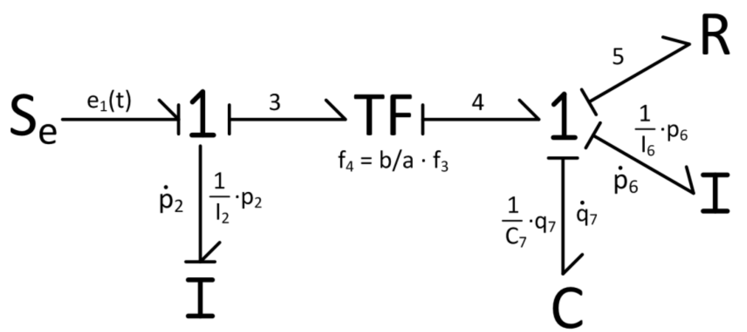
$$\mathbf{x} = \begin{bmatrix} p_2 \\ q_7 \end{bmatrix}$$

Derivative Causality – Example

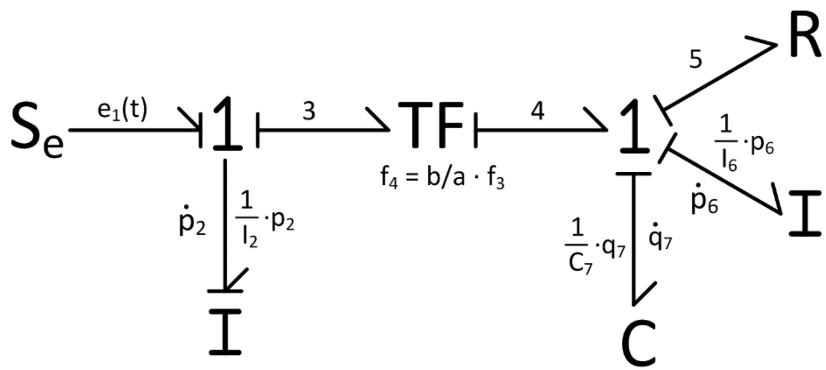
- I_6 is in ***derivative Causality***
  - ***Not independent***
  - ***Does not contribute a state***
  - Its energy variable p_6 (would be a q for an C-element) is ***algebraically related*** to the state variables



- Annotate the bond graph
  - Include the energy variable annotation for the dependent inertia



Derivative Causality – Example



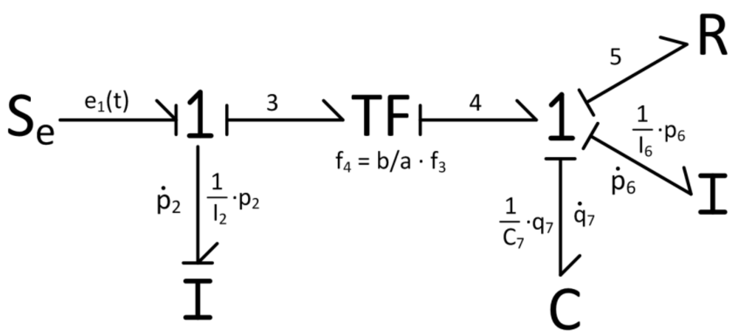
- p_6 is not a state variable
  - State equation derivation requires first determining the algebraic relationship between p_6 and the state variables, p_2 and q_7
  - When p_6 or $\dot{p}_6$ enters the formulation, substitute in this relationship or its derivative

Derivative Causality – Procedure

1. For the dependent energy-storage element, ***apply the constitutive law 'backwards'*** - i.e. express the energy variable as a function of a power variable
   a. Inertia: express momentum as a function of flow
   b. Capacitor: express displacement as a function of effort
2. ***Follow causality*** to relate that power variable to the state variables and inputs
3. ***Substitute*** the expression from step 2 into that from step 1
4. When the energy variable (or its derivative) enters the formulation, ***substitute*** in the expression from step 3

Derivative Causality – Example

- Apply the constitutive law for I_6 'backwards'
  - Express p_6 as a function of f_6



$$p_6 = I_6 f_6 \qquad (1)$$

- Follow causality to express f_6 in terms of state variables and inputs

$$f_6 = f_4 = \frac{b}{a} f_3 = \frac{b}{a}\frac{1}{I_2} p_2 \qquad (2)$$

- Substituting (2) into (1)

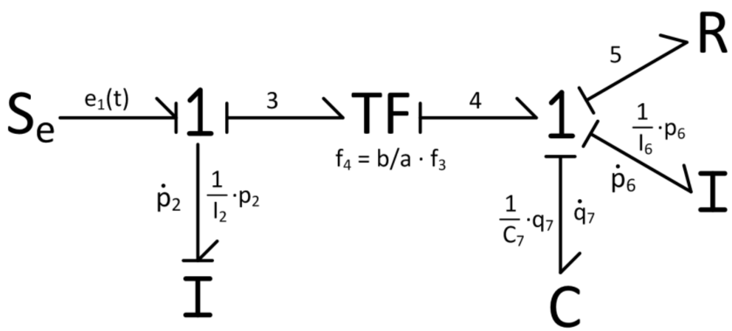
$$p_6 = \frac{b}{a}\frac{I_6}{I_2} p_2 \qquad (3)$$

- Now proceed with derivation, using (3) when needed

Derivative Causality – Example

- Begin state equation derivation with $\dot{p}_2$

$$\dot{p}_2 = e_2 = e_1(t) - e_3 \qquad (4)$$



- TF relates e_3 to e_4

$$\dot{p}_2 = e_1(t) - \frac{b}{a}e_4 = e_1(t) - \frac{b}{a}\left(e_5 + \dot{p}_6 + \frac{1}{C_7}q_7\right) \qquad (5)$$

$$e_5 = R_5 f_5 = R_5 f_4 = R_5\frac{b}{a}f_3 = R_5\frac{b}{a}\frac{1}{I_2}p_2 \qquad (6)$$

- Substituting (6) into (5)

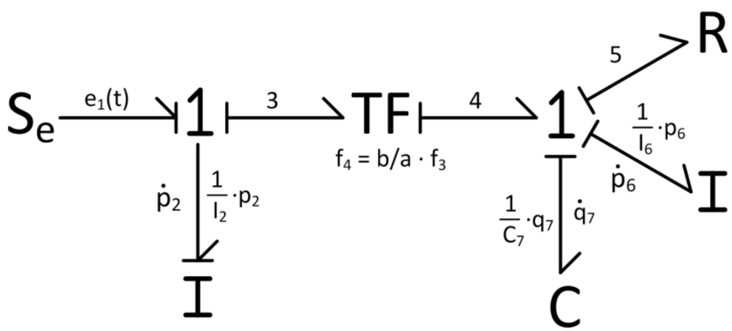
$$\dot{p}_2 = e_1(t) - \frac{b}{a}\left(\frac{b}{a}\frac{R_5}{I_2}p_2 + \dot{p}_6 + \frac{1}{C_7}q_7\right)$$

$$\dot{p}_2 = e_1(t) - \left(\frac{b}{a}\right)^2\frac{R_5}{I_2}p_2 - \frac{b}{a}\dot{p}_6 - \frac{b}{a}\frac{1}{C_7}q_7 \qquad (7)$$

Derivative Causality – Example

- A $\dot{p}_6$ term has entered the formulation
  - Differentiate (3)

$$\dot{p}_6 = \frac{b}{a}\frac{I_6}{I_2}\dot{p}_2 \qquad (8)$$



- Substitute (8) into (7)

$$\dot{p}_2 = e_1(t) - \left(\frac{b}{a}\right)^2 \frac{R_5}{I_2} p_2 - \left(\frac{b}{a}\right)^2 \frac{I_6}{I_2}\dot{p}_2 - \frac{b}{a}\frac{1}{C_7} q_7 \qquad (9)$$

- Solve (9) for $\dot{p}_2$

$$\dot{p}_2\left(\frac{I_2 + (b/a)^2 I_6}{I_2}\right) = e_1(t) - \left(\frac{b}{a}\right)^2 \frac{R_5}{I_2} p_2 - \frac{b}{a}\frac{1}{C_7} q_7 \qquad (10)$$

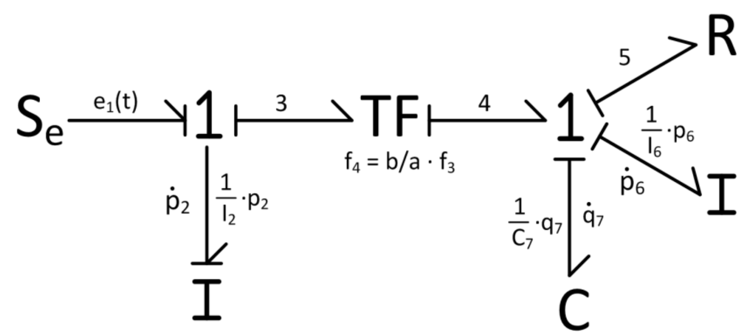
Derivative Causality – Example

- Rearranging (10) gives the ***first of two state equations***:

$$\dot{p}_2 = -\frac{\left(\frac{b}{a}\right)^2 R_5}{I_2+\left(\frac{b}{a}\right)^2 I_6} p_2 - \frac{\left(\frac{b}{a}\right) I_2}{\left(I_2+\left(\frac{b}{a}\right)^2 I_6\right) C_7} q_7 + \frac{I_2}{I_2+\left(\frac{b}{a}\right)^2 I_6} e_1(t) \qquad (11)$$

- Next, move on to $\dot{q}_7$

$$\dot{q}_7 = f_7 = f_4 = \frac{b}{a} f_3 \qquad (12)$$



- The ***second state equation***:

$$\dot{q}_7 = \frac{b}{a}\frac{1}{I_2} p_2 \qquad (9)$$

Derivative Causality – Example

- The state-space system model:

$$\begin{bmatrix} \dot{p}_2 \\ \dot{q}_7 \end{bmatrix} = \begin{bmatrix} -\dfrac{\left(\frac{b}{a}\right)^2 R_5}{I_2 + \left(\frac{b}{a}\right)^2 I_6} & -\dfrac{\left(\frac{b}{a}\right) I_2}{\left(I_2 + \left(\frac{b}{a}\right)^2 I_6\right) C_7} \\ \dfrac{b}{a}\dfrac{1}{I_2} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ q_7 \end{bmatrix} + \begin{bmatrix} \dfrac{I_2}{I_2 + \left(\frac{b}{a}\right)^2 I_6} \\ 0 \end{bmatrix} e_1(t)$$

- With physical parameters:

$$\begin{bmatrix} \dot{p}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{\left(\frac{b}{a}\right)^2 b}{m_1 + \left(\frac{b}{a}\right)^2 m_2} & -\dfrac{\left(\frac{b}{a}\right) m_1 k}{\left(m_1 + \left(\frac{b}{a}\right)^2 m_2\right)} \\ \dfrac{b}{a}\dfrac{1}{m_1} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{m_1}{m_1 + \left(\frac{b}{a}\right)^2 m_2} \\ 0 \end{bmatrix} F_{in}(t)$$

Derivative Causality – Example

- Derivative causality in this case resulted from a ***modeling decision***
  - The lever was assumed to be rigid
  - Adding some compliance to the lever arm eliminates derivative causality (see Section 3 notes)
    - Increases system model to ***fourth-order***
    - Equation derivation simplified at the cost of model complexity

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- In general, derive an expression for the energy variable of each energy-storage element in derivative causality
  - Multiple elements in derivative-causality will require solution of a system of equations