SECTION 5: LAPLACE TRANSFORMS

ESE 330 – Modeling & Analysis of Dynamic Systems

Introduction – Transforms

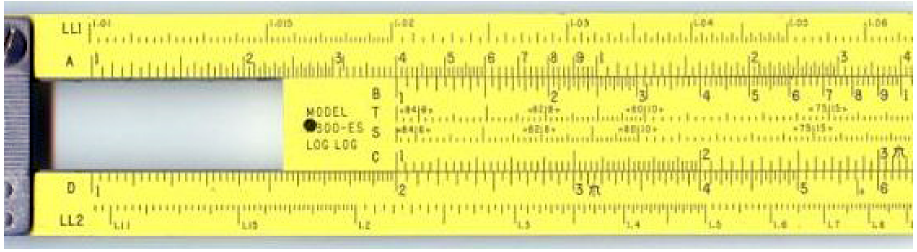
This section of notes contains an introduction to Laplace transforms. This should mostly be a review of material covered in your differential equations course.

Transforms

- **What is a transform?**
  - A ***mapping*** of a mathematical function from one ***domain*** to another
  - A change in ***perspective*** not a change of the function
- **Why use transforms?**
  - Some mathematical problems are difficult to solve in their natural domain
    - Transform to and solve in a new domain, where the problem is simplified
    - Transform back to the original domain
  - Trade off the extra effort of transforming/inverse-transforming for simplification of the solution procedure

Transform Example – Slide Rules

- Slide rules make use of a logarithmic transform



- Multiplication/division of large numbers is difficult
  - Transform the numbers to the logarithmic domain
  - Add/subtract (easy) in the log domain to multiply/divide (difficult) in the linear domain
  - Apply the inverse transform to get back to the original domain
- Extra effort is required, but the problem is simplified

Laplace Transforms

Laplace Transforms

- An ***integral transform*** mapping functions from the ***time domain*** to the ***Laplace domain*** or ***s-domain***

$$g(t) \overset{\mathcal{L}}{\leftrightarrow} G(s)$$

  - Time-domain functions are functions of time, t

$$g(t)$$

  - Laplace-domain functions are functions of $\boldsymbol{s}$

$$G(s)$$

  - s is a complex variable

$$s = \sigma + j\omega$$

Laplace Transforms – Motivation

- We'll use Laplace transforms to ***solve differential equations***
  - ***Differential equations*** in the ***time domain***
    - difficult to solve
  - Apply the Laplace transform
    - Transform to ***the s-domain***
  - ***Differential equations*** become ***algebraic equations***
    - easy to solve
  - Transform the s-domain solution back to the time domain
- Transforming back and forth requires extra effort, but the solution is greatly simplified

Laplace Transform

- **Laplace Transform**:

$$\mathcal{L}\{g(t)\} = G(s) = \int_0^\infty g(t)e^{-st}dt \qquad (1)$$

- ***Unilateral*** or ***one-sided*** transform
  - Lower limit of integration is $t = 0$
  - Assumed that the time domain function is zero for all negative time, i.e.

$$g(t) = 0, \qquad t < 0$$

Laplace Transform Properties

In the following section of notes, we'll derive a few important properties of the Laplace transform.

Laplace Transform – Linearity

- Say we have two time-domain functions:

$$g_1(t) \text{ and } g_2(t)$$

- Applying the transform definition, (1)

$$\mathcal{L}\{\alpha g_1(t) + \beta g_2(t)\} = \int_0^\infty \left(\alpha g_1(t) + \beta g_2(t)\right) e^{-st} dt$$

$$= \int_0^\infty \alpha g_1(t) e^{-st} dt + \int_0^\infty \beta g_2(t) e^{-st} dt$$

$$= \alpha \int_0^\infty g_1(t) e^{-st} dt + \beta \int_0^\infty g_2(t) e^{-st} dt$$

$$= \alpha \cdot \mathcal{L}\{g_1(t)\} + \beta \cdot \mathcal{L}\{g_2(t)\}$$

$$\boxed{\mathcal{L}\{\alpha g_1(t) + \beta g_2(t)\} = \alpha G_1(s) + \beta G_2(s)} \qquad (2)$$

- The Laplace transform is a ***linear operation***

Laplace Transform of a Derivative

- Of particular interest, given that we want to use Laplace transform to solve differential equations

$$\mathcal{L}\{\dot{g}(t)\} = \int_0^\infty \dot{g}(t) e^{-st} dt$$

- Use ***integration by parts*** to evaluate

$$\int u dv = uv - \int v du$$

- Let

$$u = e^{-st} \quad \text{and} \quad dv = \dot{g}(t) dt$$

then

$$du = -se^{-st} dt \quad \text{and} \quad v = g(t)$$

Laplace Transform of a Derivative

$$\mathcal{L}\{\dot{g}(t)\} = e^{-st}g(t)\Big|_0^\infty - \int_0^\infty g(t)(-se^{-st})dt$$

$$= 0 - g(0) + s\int_0^\infty g(t)e^{-st}dt = -g(0) + s\mathcal{L}\{g(t)\}$$

- The Laplace transform of the derivative of a function is the Laplace transform of that function multiplied by s minus the initial value of that function

$$\boxed{\mathcal{L}\{\dot{g}(t)\} = sG(s) - g(0)} \qquad (3)$$

Higher-Order Derivatives

- The Laplace transform of a ***second derivative*** is

$$\mathcal{L}\{\ddot{g}(t)\} = s^2G(s) - sg(0) - \dot{g}(0) \qquad (4)$$

- In general, the Laplace transform of the $\boldsymbol{n^{th}}$ ***derivative*** of a function is given by

$$\mathcal{L}\{g^{(n)}\} = s^nG(s) - s^{n-1}g(0) - s^{n-2}\dot{g}(0) - \cdots - g^{(n-1)}(0) \qquad (5)$$

Laplace Transform of an Integral

- The Laplace Transform of a ***definite integral*** of a function is given by

$$\mathcal{L}\left\{\int_0^t g(\tau)d\tau\right\} = \frac{1}{s}G(s) \qquad (6)$$

- ***Differentiation*** in the time domain corresponds to ***multiplication by s*** in the Laplace domain
- ***Integration*** in the time domain corresponds to ***division by s*** in the Laplace domain

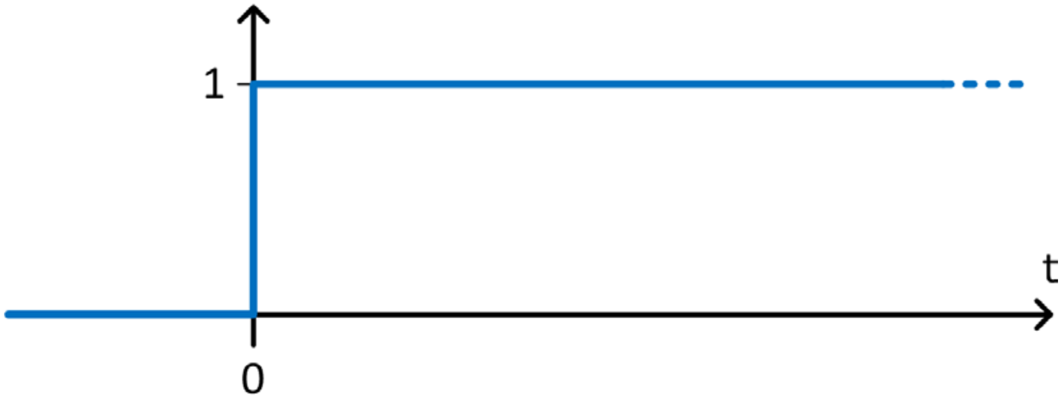
Laplace Transforms of Common Functions

Next, we'll derive the Laplace transform of some common mathematical functions

Unit Step Function

- A useful and common way of characterizing a linear system is with its ***step response***
  - The system's response (output) to a unit step input
- The ***unit step function*** or ***Heaviside step function***:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Unit Step Function – Laplace Transform

- Using the definition of the Laplace transform

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

- The Laplace transform of the unit step

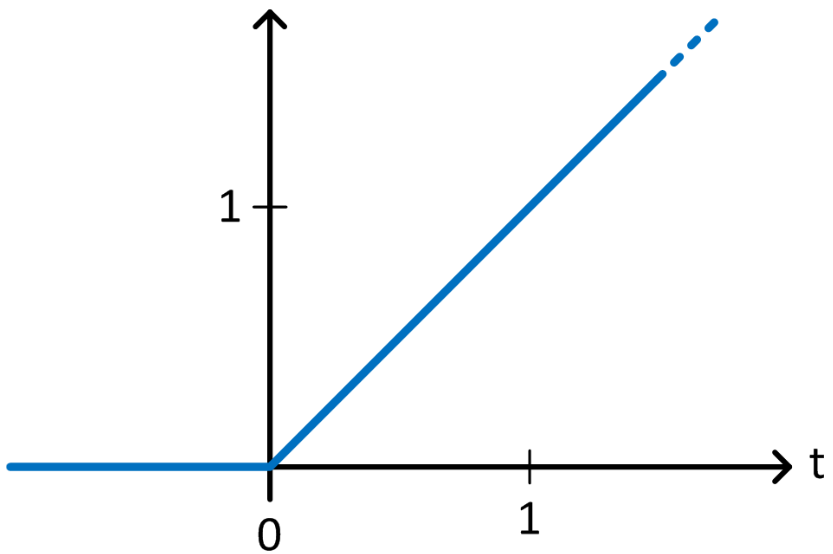
$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}} \quad (7)$$

- Note that the unilateral Laplace transform assumes that the signal being transformed is zero for $t < 0$
  - Equivalent to multiplying any signal by a unit step

Unit Ramp Function

- The unit ramp function is a useful input signal for evaluating how well a system tracks a constantly-increasing input
- The ***unit ramp function***:

$$g(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$



Unit Ramp Function – Laplace Transform

- Could easily evaluate the transform integral
  - Requires integration by parts
- Alternatively, recognize the relationship between the unit ramp and the unit step
  - ***Unit ramp is the integral of the unit step***
- Apply the integration property, (6)

$$\mathcal{L}\{t\} = \mathcal{L}\left\{\int_0^t u(\tau)d\tau\right\} = \frac{1}{s} \cdot \frac{1}{s}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}} \qquad (8)$$

Exponential – Laplace Transform

$$g(t) = e^{-at}$$

- Exponentials are common components of the responses of dynamic systems

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt$$

$$= -\frac{e^{-(s+a)t}}{s+a}\Big|_0^\infty = 0 - \left(-\frac{1}{s+a}\right)$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}} \qquad (9)$$

Sinusoidal functions

- Another class of commonly occurring signals, when dealing with dynamic systems, is ***sinusoidal signals*** – both $\sin(\omega t)$ and $\cos(\omega t)$

$$g(t) = \sin(\omega t)$$

- Recall ***Euler's formula***

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- From which it follows that

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Sinusoidal functions

$$\mathcal{L}\{\sin(\omega t)\} = \frac{1}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt$$

$$= \frac{1}{2j}\int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right)dt$$

$$= \frac{1}{2j}\int_0^\infty e^{-(s-j\omega)t}dt - \frac{1}{2j}\int_0^\infty e^{-(s+j\omega)t}dt$$

$$= \frac{1}{2j}\frac{\left(e^{-(s-j\omega)t}\right)}{-(s-j\omega)}\Bigg|_0^\infty - \frac{1}{2j}\frac{\left(e^{-(s+j\omega)t}\right)}{-(s+j\omega)}\Bigg|_0^\infty$$

$$= \frac{1}{2j}\left[0 + \frac{1}{s-j\omega}\right] - \frac{1}{2j}\left[0 + \frac{1}{s+j\omega}\right] = \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2}$$

$$\boxed{\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}} \qquad (10)$$

Sinusoidal functions

- It can similarly be shown that

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \tag{11}$$

- Note that for neither $\sin(\omega t)$ nor $\cos(\omega t)$ is the function equal to zero for $t < 0$ as the Laplace transform assumes
- Really, what we've derived is

$$\mathcal{L}\{u(t) \cdot \sin(\omega t)\} \quad \text{and} \quad \mathcal{L}\{u(t) \cdot \cos(\omega t)\}$$

More Properties and Theorems

Multiplication by an Exponential, e^{-at}

- We've seen that $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$
- What if another function is multiplied by the decaying exponential term?

$$\mathcal{L}\{g(t)e^{-at}\} = \int_0^\infty g(t)e^{-at}e^{-st}dt = \int_0^\infty g(t)e^{-(s+a)t}dt$$

- This is just the Laplace transform of $g(t)$ with s replaced by $(s+a)$

$$\boxed{\mathcal{L}\{g(t)e^{-at}\} = G(s+a)} \qquad (12)$$

Decaying Sinusoids

- The Laplace transform of a sinusoid is

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

- And, multiplication by an decaying exponential, e^{-at}, results in a substitution of $(s + a)$ for s, so

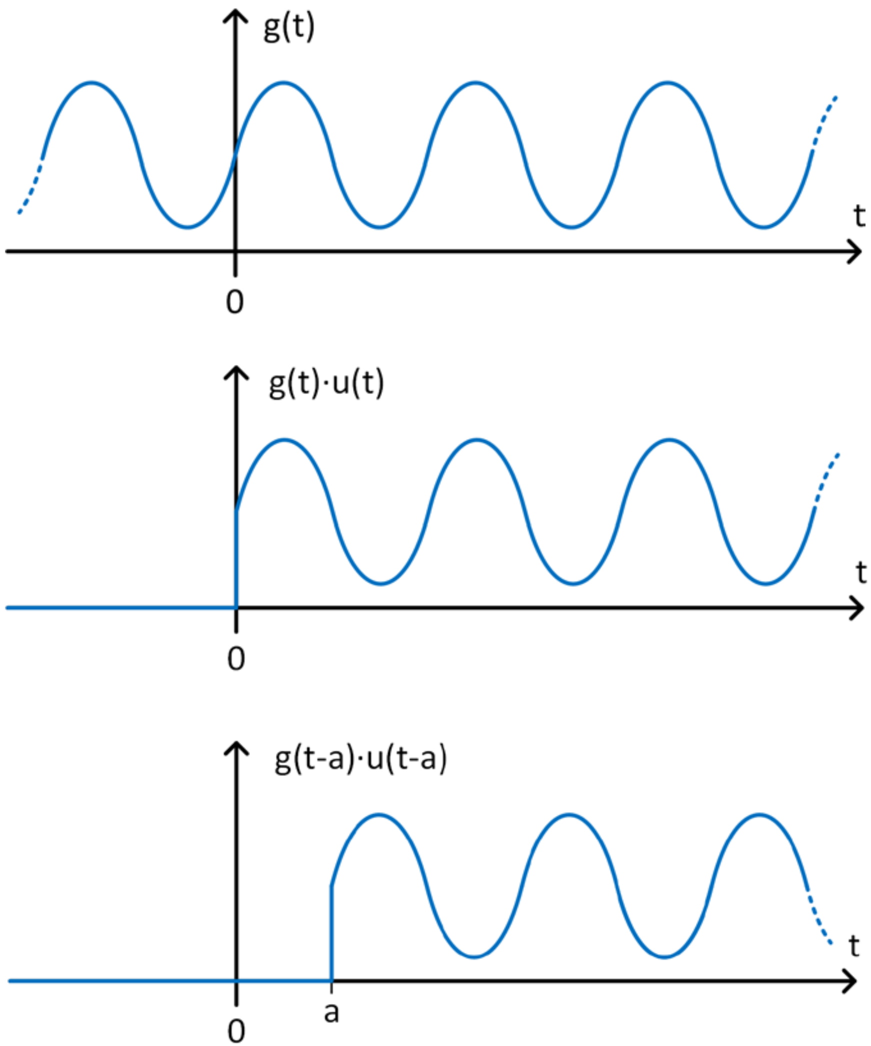
$$\mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s + a)^2 + \omega^2}$$

and

$$\mathcal{L}\{e^{-at}\cos(\omega t)\} = \frac{s + a}{(s + a)^2 + \omega^2}$$

Time Shifting

- Consider a time-domain function, $g(t)$
- To Laplace transform $g(t)$ we've assumed $g(t) = 0$ for $t < 0$, or equivalently multiplied by $u(t)$
- To shift $g(t)$ by an amount, a, in time, we must also multiply by a shifted step function, $u(t-a)$



Time Shifting – Laplace Transform

- The transform of the shifted function is given by

$$\mathcal{L}\{g(t-a)\cdot u(t-a)\} = \int_a^\infty g(t-a)e^{-st}dt$$

- Performing a change of variables, let

$$\tau = (t-a) \text{ and } d\tau = dt$$

- The transform becomes

$$\mathcal{L}\{g(\tau)\cdot u(\tau)\} = \int_0^\infty g(\tau)e^{-s(\tau+a)}d\tau = \int_0^\infty g(\tau)e^{-as}e^{-s\tau}d\tau = e^{-as}\int_0^\infty g(\tau)e^{-s\tau}d\tau$$

- A shift by a in the time domain corresponds to multiplication by e^{-as} in the Laplace domain

$$\boxed{\mathcal{L}\{g(t-a)\cdot u(t-a)\} = e^{-as}G(s)} \qquad (13)$$

Multiplication by time, t

- The Laplace transform of a function multiplied by time:

$$\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds}F(s) \tag{14}$$

- Consider a unit ramp function:

$$\mathcal{L}\{t\} = \mathcal{L}\{t \cdot u(t)\} = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

- Or a parabola:

$$\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

- In general

$$\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}}$$

Initial and Final Value Theorems

- **Initial Value Theorem**
  - Can determine the initial value of a time-domain signal or function from its Laplace transform

$$g(0) = \lim_{s \to \infty} sG(s) \qquad (15)$$

- **Final Value Theorem**
  - Can determine the steady-state value of a time-domain signal or function from its Laplace transform

$$g(\infty) = \lim_{s \to 0} sG(s) \qquad (16)$$

Convolution

- ***Convolution*** of two functions or signals is given by

$$g(t) * x(t) = \int_0^t g(\tau)x(t-\tau)d\tau$$

- Result is a function of time
  - $x(\tau)$ is ***flipped*** in time and ***shifted*** by t
  - Multiply the flipped/shifted signal and the other signal
  - Integrate the result from $\tau = 0 \ldots t$
- May seem like an odd, arbitrary function now, but we'll later see why it is very important
- ***Convolution in the time domain corresponds to multiplication in the Laplace domain***

$$\boxed{\mathcal{L}\{g(t) * x(t)\} = G(s)X(s)} \qquad (17)$$

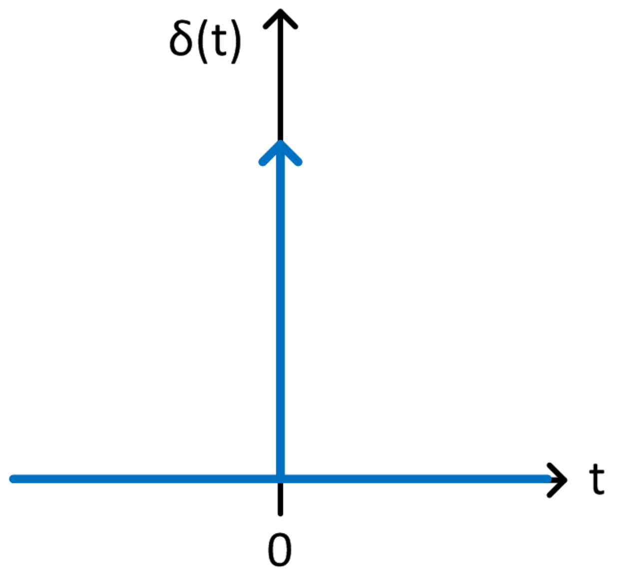
Impulse Function

- Another common way to describe a dynamic system is with its ***impulse response***
  - System output in response to an impulse function input
- ***Impulse function*** defined by

$$\delta(t) = 0, \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

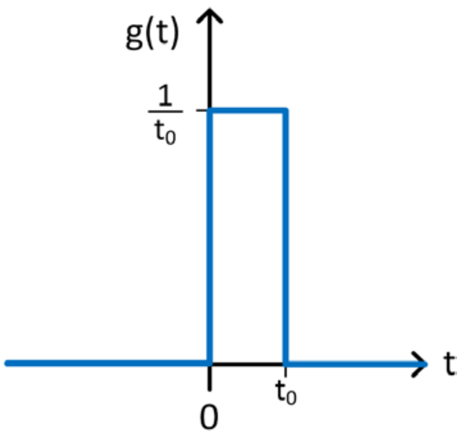
  - An infinitely tall, infinitely narrow pulse



Impulse Function – Laplace Transform

- To derive $\mathcal{L}\{\delta(t)\}$, consider the following function

$$g(t) = \begin{cases} \dfrac{1}{t_0}, & 0 \leq t \leq t_0 \\ 0, & t < 0 \text{ or } t > t_0 \end{cases}$$



- Can think of $g(t)$ as the sum of two step functions:

$$g(t) = \frac{1}{t_0}u(t) - \frac{1}{t_0}u(t-t_0)$$

- The transform of the first term is

$$\mathcal{L}\left\{\frac{1}{t_0}u(t)\right\} = \frac{1}{t_0 s}$$

- Using the time-shifting property, the second term transforms to

$$\mathcal{L}\left\{-\frac{1}{t_0}u(t-t_0)\right\} = -\frac{e^{-t_0 s}}{t_0 s}$$

Impulse Function – Laplace Transform

- In the limit, as $t_0 \rightarrow 0$, $g(t) \rightarrow \delta(t)$, so

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \rightarrow 0} \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \rightarrow 0} \frac{1 - e^{-t_0 s}}{t_0 s}$$

- Apply l'Hôpital's rule

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \rightarrow 0} \frac{\frac{d}{dt_0}(1 - e^{-t_0 s})}{\frac{d}{dt_0}(t_0 s)} = \lim_{t_0 \rightarrow 0} \frac{s e^{-t_0 s}}{s} = \frac{s}{s}$$

- The Laplace transform of an impulse function is one

$$\boxed{\mathcal{L}\{\delta(t)\} = 1} \qquad (18)$$

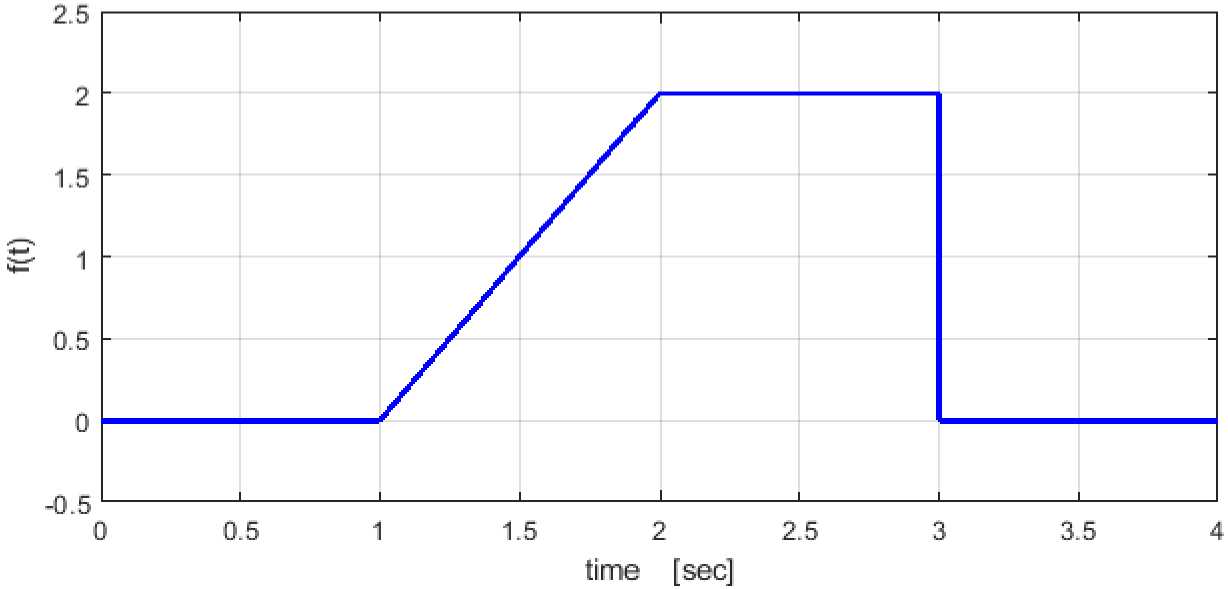
Common Laplace Transforms

$g(t)$	$G(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^m	$\frac{m!}{s^{m+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

$g(t)$	$G(s)$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\dot{g}(t)$	$sG(s)-g(0)$
$\ddot{g}(t)$	$s^2G(s)-sg(0)-\dot{g}(0)$
$\int_0^t g(\tau)d\tau$	$\frac{1}{s}G(s)$
$e^{-at}g(t)$	$G(s+a)$
$g(t-a)\cdot u(t-a)$	$e^{-as}G(s)$
$t\cdot g(t)$	$-\frac{d}{ds}G(s)$

Example – Piecewise Function Laplace Transform

- Determine the Laplace transform of a piecewise function:



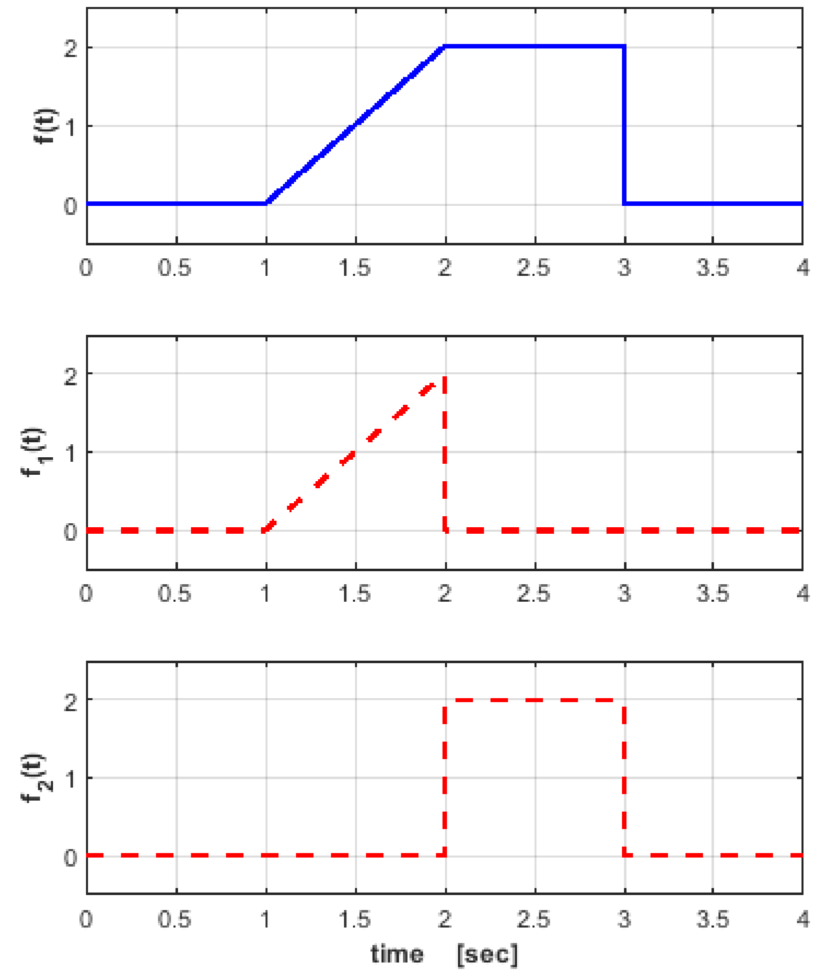
- A summation of functions with known transforms:
  - Ramp
  - Pulse – sum of positive and negative steps
- Transform is the sum of the individual, known transforms

Example – Piecewise Function Laplace Transform

- Treat the piecewise function as a sum of individual functions

$$f(t) = f_1(t) + f_2(t)$$

- $f_1(t)$
  - Time-shifted, gated ramp
- $f_2(t)$
  - Time-shifted pulse
    - Sum of staggered positive and negative steps



Example – Piecewise Function Laplace Transform

- $f_1(t)$: time-shifted, gated ramp
- Ramp w/ slope of 2:

$$r(t) = 2 \cdot t$$

- Time-shifted ramp:

$$r_s(t) = 2 \cdot (t - 1)$$

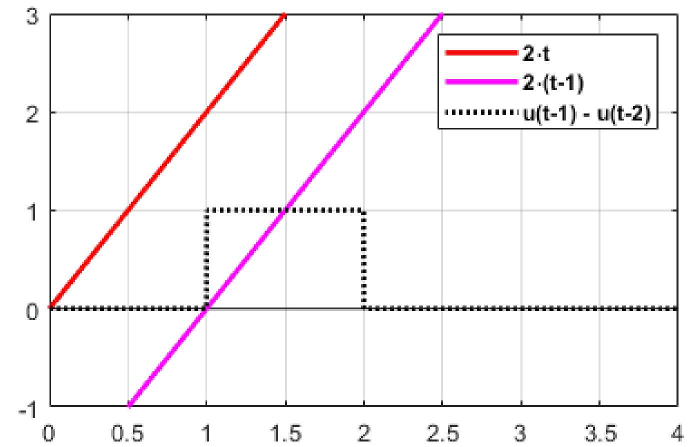
- Gating function
  - Unity-amplitude pulse:

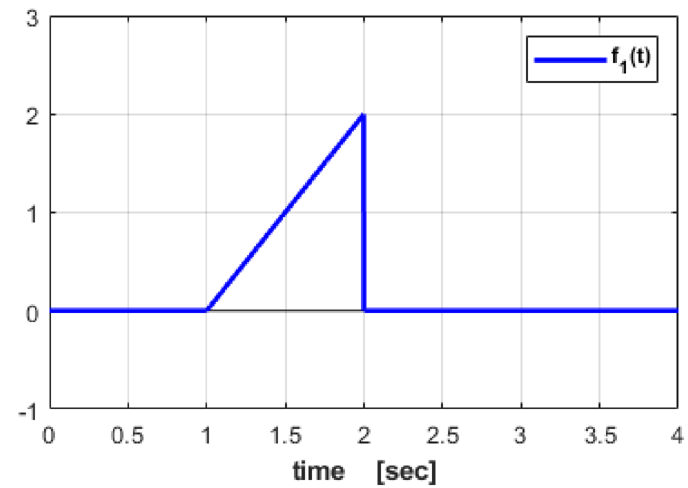
$$g(t) = u(t - 1) - u(t - 2)$$

- Gate the shifted ramp:

$$f_1(t) = r_s(t) \cdot g(t)$$

$$f_1(t) = 2 \cdot (t - 1) \cdot [u(t - 1) - u(t - 2)]$$





Example – Piecewise Function Laplace Transform

- $f_2(t)$: time-shifted pulse
  - Sum of staggered positive and negative steps
- Positive step delayed by 2 sec:

$$s_2(t) = 2 \cdot u(t-2)$$

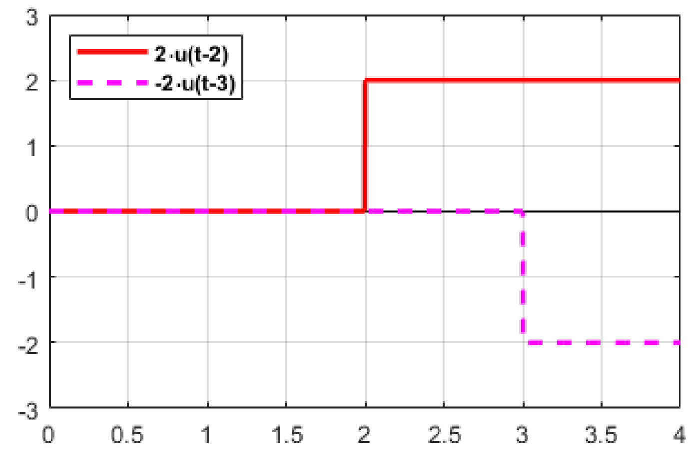
- Negative step delayed by 3 sec:

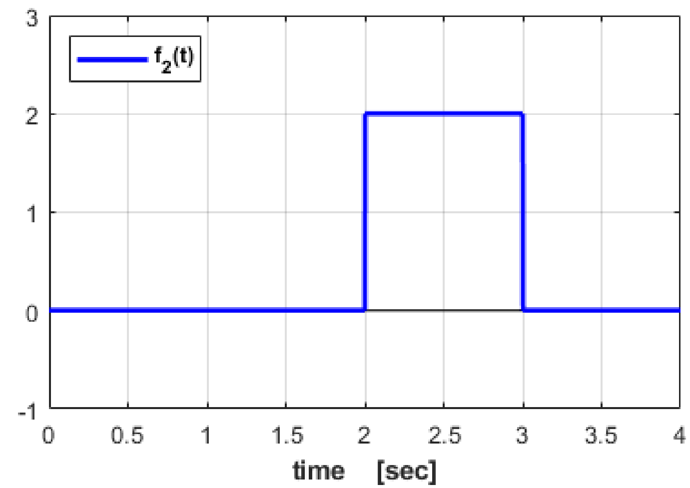
$$s_3(t) = -2 \cdot u(t-3)$$

- Time-shifted pulse

$$f_2(t) = s_2(t) + s_3(t)$$

$$f_2(t) = 2 \cdot u(t-2) - 2 \cdot u(t-3)$$





Example – Piecewise Function Laplace Transform

- Sum the two individual time-domain functions

$$f(t) = f_1(t) + f_2(t)$$

$$f(t) = 2 \cdot (t-1) \cdot [u(t-1) - u(t-2)] + 2 \cdot u(t-2) - 2 \cdot u(t-3)$$

$$\begin{aligned} f(t) = 2[&(t-1) \cdot u(t-1)] \\ &-2[t \cdot u(t-2)] \\ &+4[u(t-2)] \\ &-2[u(t-3)] \end{aligned}$$

- Transform the individual terms in $f(t)$

$$\begin{aligned} F(s) = \mathcal{L}\{&2[(t-1) \cdot u(t-1)]\} \\ &+\mathcal{L}\{-2[t \cdot u(t-2)]\} \\ &+\mathcal{L}\{+4[u(t-2)]\} \\ &+\mathcal{L}\{-2[u(t-3)]\} \end{aligned}$$

Example – Piecewise Function Laplace Transform

- First term is a time-shifted ramp function

$$\mathcal{L}\{2[(t-1)\cdot u(t-1)]\} = \frac{2e^{-s}}{s^2}$$

- The next term is a time-shifted step function multiplied by time

$$\mathcal{L}\{-2[t\cdot u(t-2)]\} = 2\frac{d}{ds}\left[\frac{e^{-2s}}{s}\right]$$

$$= -2\left[\frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}\right]$$

Example – Piecewise Function Laplace Transform

- The last two terms are time-shifted step functions

$$\mathcal{L}\{4 \cdot u(t-2) - 2 \cdot u(t-3)\} = \frac{4e^{-2s}}{s} - \frac{2e^{-3s}}{s}$$

- The piecewise function in the Laplace domain:

$$F(s) = \frac{2e^{-s}}{s^2} - 2\left[\frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}\right] + \frac{4e^{-2s}}{s} - \frac{2e^{-3s}}{s}$$

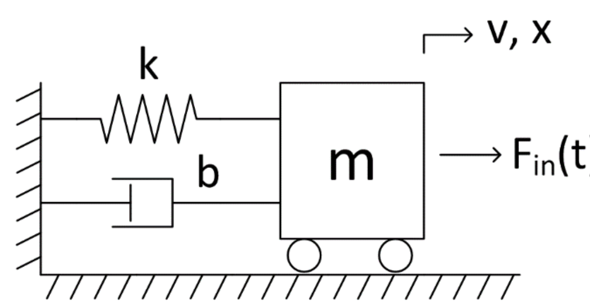
$$\boxed{F(s) = \frac{2e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{2e^{-3s}}{s}}$$

Inverse Laplace Transform

We've just seen how time-domain functions can be transformed to the Laplace domain. Next, we'll look at how we can solve differential equations in the Laplace domain and transform back to the time domain.

Laplace Transforms – Differential Equations

- Consider the simple spring/mass/damper system from the previous section of notes
- State equations are:



$$\dot{p} = -\frac{b}{m}p - kx + F_{in}(t) \quad (1)$$

$$\dot{x} = \frac{1}{m}p \quad (2)$$

- Taking the displacement of the mass as the output

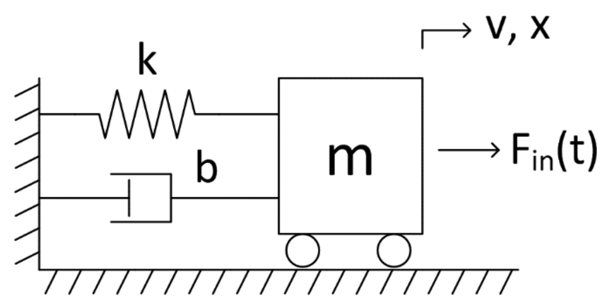
$$y = x \quad (3)$$

- Using (2) and (3) in (1) we get a single second-order differential equation

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{1}{m}F_{in}(t) \quad (4)$$

Laplace Transforms – Differential Equations

- We'll now use Laplace transforms to determine the ***step response*** of the system
- 1N step force input



$$F_{in}(t) = 1N \cdot u(t) = \begin{cases} 0N, & t < 0 \\ 1N, & t \geq 0 \end{cases} \quad (5)$$

- For the step response, we assume ***zero initial conditions***

$$y(0) = 0 \text{ and } \dot{y}(0) = 0 \quad (6)$$

- Using the derivative property of the Laplace transform, (4) becomes

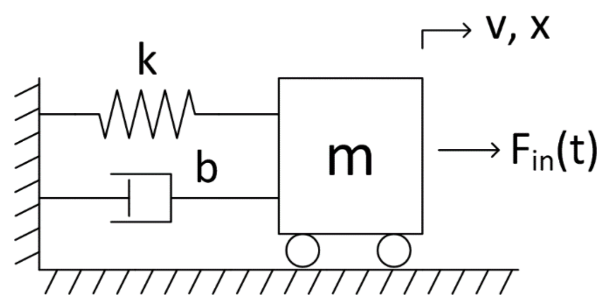
$$s^2Y(s) - sy(0) - \dot{y}(0) + \frac{b}{m}sY(s) - \frac{b}{m}y(0) + \frac{k}{m}Y(s) = \frac{1}{m}F_{in}(s)$$

$$s^2Y(s) + \frac{b}{m}sY(s) + \frac{k}{m}Y(s) = \frac{1}{m}F_{in}(s) \quad (7)$$

Laplace Transforms – Differential Equations

- The input is a step, so (7) becomes

$$s^2Y(s) + \frac{b}{m}sY(s) + \frac{k}{m}Y(s) = \frac{1}{m}1N\frac{1}{s} \qquad (8)$$



- Solving (8) for $Y(s)$

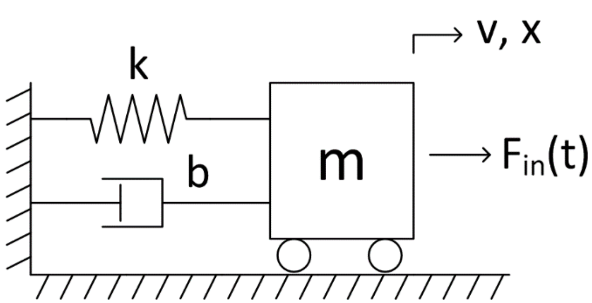
$$Y(s)\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right) = \frac{1}{m}\frac{1}{s}$$

$$Y(s) = \frac{1/m}{s\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)} \qquad (9)$$

- Equation (9) is the solution to the differential equation of (4), given the step input and I.C.'s
  - The system step response in the Laplace domain
  - Next, we need to transform back to the time domain

Laplace Transforms – Differential Equations

$$Y(s) = \frac{1/m}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \qquad (9)$$



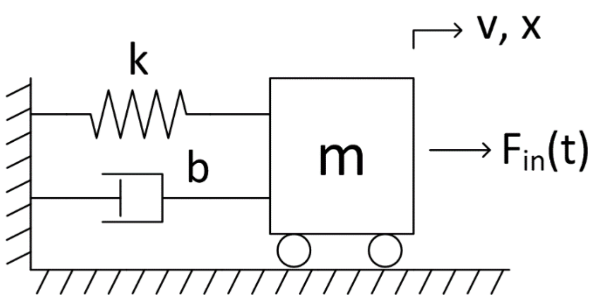
- The form of (9) is typical of Laplace transforms when dealing with linear systems
  - A ***rational polynomial*** in s
  - Here, the numerator is 0th-order

$$Y(s) = \frac{B(s)}{A(s)}$$

- Roots of the numerator polynomial, $B(s)$, are called the ***zeros*** of the function
- Roots of the denominator polynomial, $A(s)$, are called the ***poles*** of the function

Inverse Laplace Transforms

$$Y(s) = \frac{1/m}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \qquad (9)$$



- To get (9) back into the time domain, we need to perform an ***inverse Laplace transform***
  - An integral inverse transform exists, but we don't use it
  - Instead, we use ***partial fraction expansion***

- **Partial fraction expansion**
  - Idea is to express the Laplace transform solution, (9), as a sum of Laplace transform terms that appear in the table
  - Procedure depends on the type of roots of the denominator polynomial
    - Real and distinct
    - Repeated
    - Complex

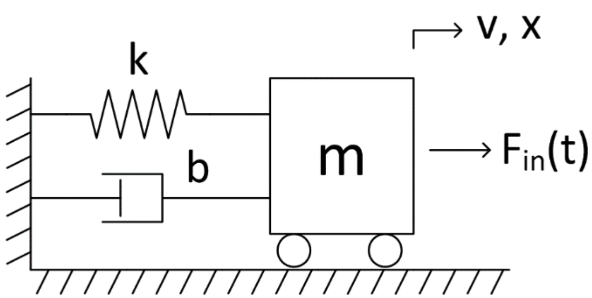
Inverse Laplace Transforms – Example 1

- Consider the following system parameters

$$m = 1kg$$

$$k = \frac{16N}{m}$$

$$b = 10\frac{N \cdot s}{m}$$



- Laplace transform of the step response becomes

$$Y(s) = \frac{1}{s(s^2+10s+16)} \tag{10}$$

- Factoring the denominator

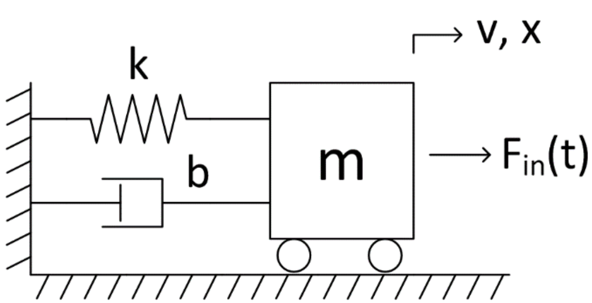
$$Y(s) = \frac{1}{s(s+2)(s+8)} \tag{11}$$

- In this case, the denominator polynomial has three ***real, distinct roots***

$$s_1 = 0, \qquad s_2 = -2, \qquad s_3 = -8$$

Inverse Laplace Transforms – Example 1

- Partial fraction expansion of (11) has the form



$$Y(s) = \frac{1}{s(s+2)(s+8)} = \frac{r_1}{s} + \frac{r_2}{s+2} + \frac{r_3}{s+8} \qquad (12)$$

  - The numerator coefficients, r_1, r_2, and r_3, are called ***residues***

- Can already see the form of the time-domain function
  - Sum of a ***constant*** and ***two decaying exponentials***

- To determine the residues, multiply both sides of (12) by the denominator of the left-hand side

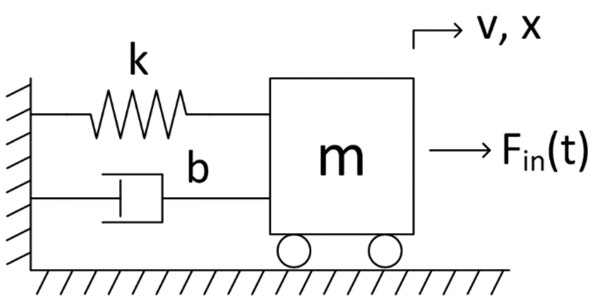
$$1 = r_1(s+2)(s+8) + r_2 s(s+8) + r_3 s(s+2)$$

$$1 = r_1 s^2 + 10 r_1 s + 16 r_1 + r_2 s^2 + 8 r_2 s + r_3 s^2 + 2 r_3 s$$

- Collecting terms, we have

$$1 = s^2(r_1 + r_2 + r_3) + s(10 r_1 + 8 r_2 + 2 r_3) + 16 r_1 \qquad (13)$$

Inverse Laplace Transforms – Example 1



- Equating coefficients of powers of s on both sides of (13) gives a system of three equations in three unknowns

$$s^2: \quad r_1 + r_2 + r_3 = 0$$
$$s^1: \quad 10r_1 + 8r_2 + 2r_3 = 0$$
$$s^0: \quad 16r_1 = 1$$

- Solving for the residues gives

$$r_1 = 0.0625$$
$$r_2 = -0.0833$$
$$r_3 = 0.0208$$

- The Laplace transform of the step response is

$$Y(s) = \frac{0.0625}{s} - \frac{0.0833}{s+2} + \frac{0.0208}{s+8} \tag{14}$$

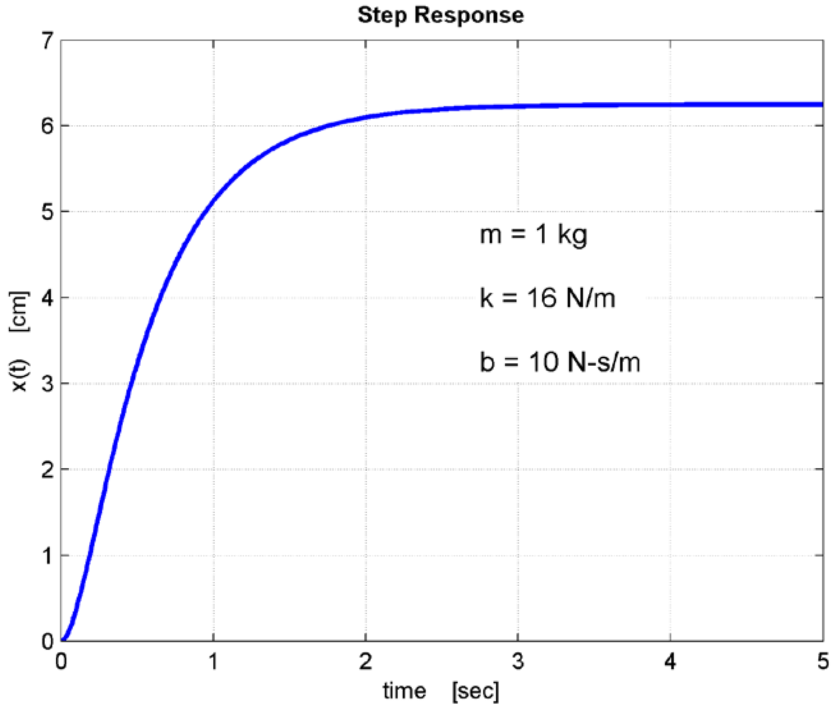
- Equation (14) can now be transformed back to the time domain using the Laplace transform table

Inverse Laplace Transforms – Example 1

- The time-domain step response of the system is the ***sum of a constant term and two decaying exponentials***:

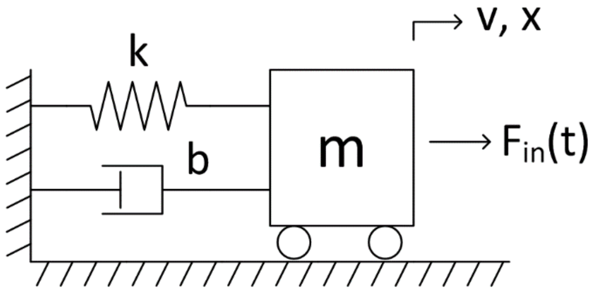
$$y(t) = 0.0625 - 0.0833e^{-2t} + 0.0208e^{-8t} \tag{15}$$

- Step response plotted in MATLAB
- Characteristic of a signal having ***only real poles***
  - No overshoot/ringing
- Steady-state displacement agrees with intuition
  - $1N$ force applied to a $16N/m$ spring



Inverse Laplace Transforms – Example 1

- Go back to (10) and apply the ***initial value theorem***



$$y(0) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{1}{(s^2 + 10s + 16)} = 0cm$$

- Which is, in fact our assumed initial condition

- Next, apply the ***final value theorem*** to the Laplace transform step response, (10)

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{1}{(s^2 + 10s + 16)}$$

$$y(\infty) = \frac{1}{16} = 0.0625m = 6.25cm$$

- This final value agrees with both intuition and our numerical analysis

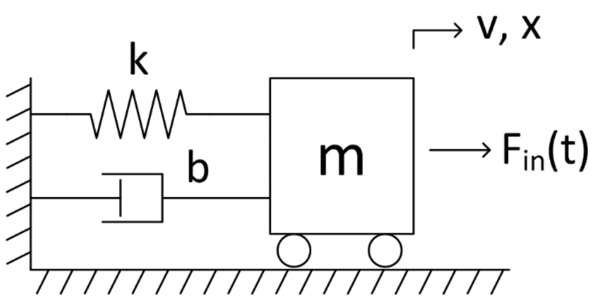
Inverse Laplace Transforms – Example 2

- ***Reduce the damping*** and re-calculate the step response

$$m = 1kg$$

$$k = \frac{16N}{m}$$

$$b = 8\frac{N \cdot s}{m}$$



- Laplace transform of the step response becomes

$$Y(s) = \frac{1}{s(s^2+8s+16)} \tag{16}$$

- Factoring the denominator

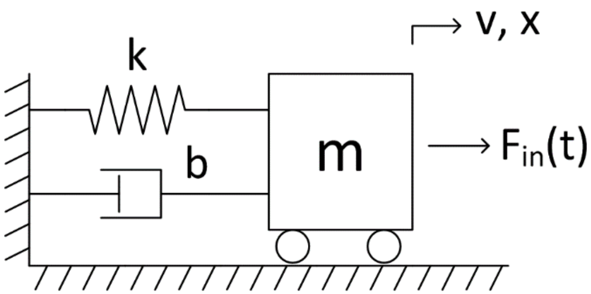
$$Y(s) = \frac{1}{s(s+4)^2} \tag{17}$$

- In this case, the denominator polynomial has three ***real roots***, two of which are ***identical***

$$s_1 = 0, \qquad s_2 = -4, \qquad s_3 = -4$$

Inverse Laplace Transforms – Example 2

- Partial fraction expansion of (17) has the form



$$Y(s) = \frac{1}{s(s+4)^2} = \frac{r_1}{s} + \frac{r_2}{s+4} + \frac{r_3}{(s+4)^2} \qquad (18)$$

- Again, find residues by multiplying both sides of (18) by the left-hand side denominator

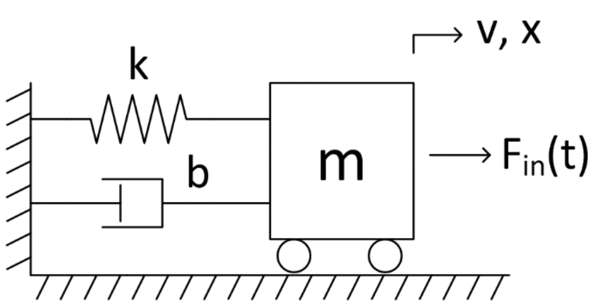
$$1 = r_1(s+4)^2 + r_2 s(s+4) + r_3 s$$

$$1 = r_1 s^2 + 8r_1 s + 16r_1 + r_2 s^2 + 4r_2 s + r_3 s$$

- Collecting terms, we have

$$1 = s^2(r_1 + r_2) + s(8r_1 + 4r_2 + r_3) + 16r_1 \qquad (19)$$

Inverse Laplace Transforms – Example 2



- Equating coefficients of powers of s on both sides of (19) gives a system of three equations in three unknowns

$$s^2: \quad r_1 + r_2 = 0$$
$$s^1: \quad 8r_1 + 4r_2 + r_3 = 0$$
$$s^0: \quad 16r_1 = 1$$

- Solving for the residues gives

$$r_1 = 0.0625$$
$$r_2 = -0.0625$$
$$r_3 = -0.2500$$

- The Laplace transform of the step response is

$$Y(s) = \frac{0.0625}{s} - \frac{0.0625}{s+4} - \frac{0.25}{(s+4)^2} \qquad (20)$$

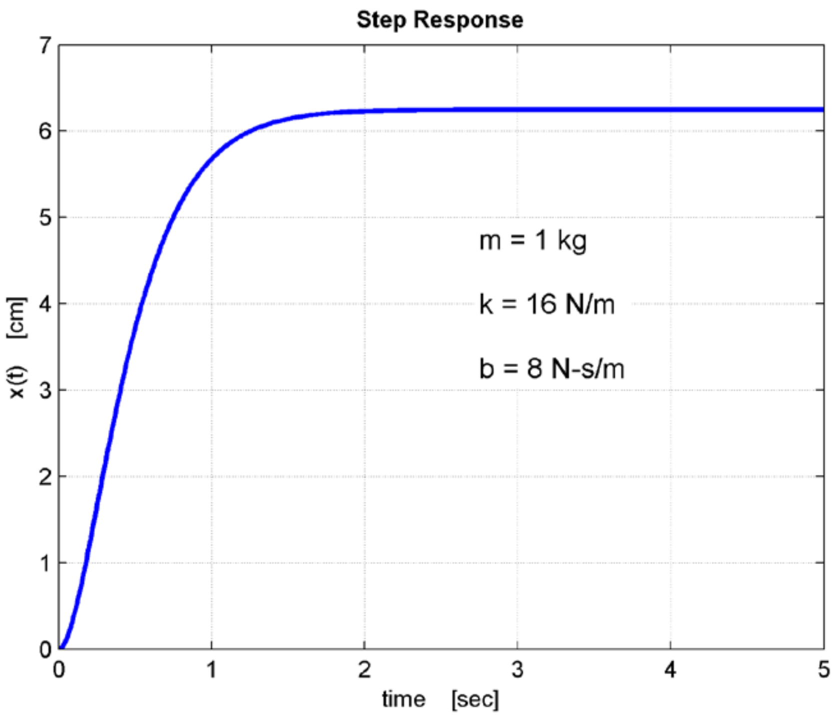
- Equation (20) can now be transformed back to the time domain using the Laplace transform table

Inverse Laplace Transforms – Example 2

- The time-domain step response of the system is the ***sum of a constant, a decaying exponential, and a decaying exponential scaled by time***:

$$y(t) = 0.0625 - 0.0625e^{-4t} - 0.25te^{-4t} \quad (21)$$

- Step response plotted in MATLAB
- Again, characteristic of a signal having ***only real poles***
  - Similar to the last case
  - A bit faster – slow pole at $s = -2$ was eliminated



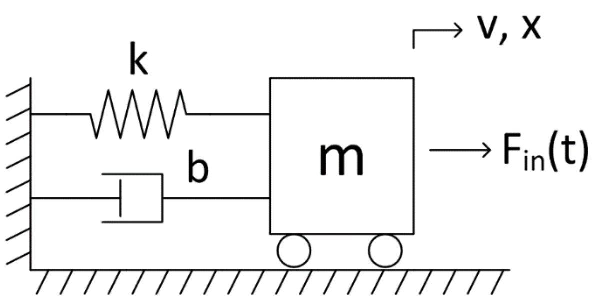
Inverse Laplace Transforms – Example 3

- ***Reduce the damping even further*** and go through the process once again

$$m = 1kg$$

$$k = \frac{16N}{m}$$

$$b = 4\frac{N \cdot s}{m}$$



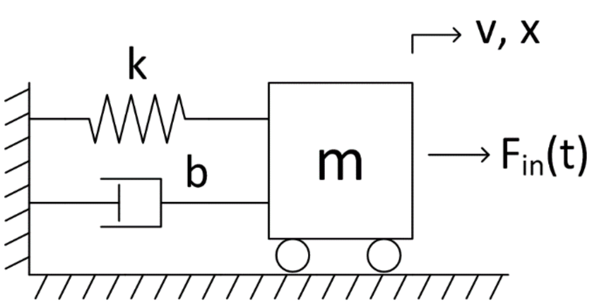
- Laplace transform of the step response becomes

$$Y(s) = \frac{1}{s(s^2+4s+16)} \quad (22)$$

- The second-order term in the denominator now has ***complex roots***, so we won't factor any further
- The denominator polynomial still has a root at zero and now has two roots which are a ***complex-conjugate pair***

$$s_1 = 0, \qquad s_2 = -2 + j3.464, \qquad s_3 = -2 - j3.464$$

Inverse Laplace Transforms – Example 3



- Want to cast the partial fraction terms into forms that appear in the Laplace transform table
- Second-order terms should be of the form

$$\frac{r_i(s+\sigma)+r_{i+1}\omega}{(s+\sigma)^2+\omega^2} \qquad (23)$$

- This will transform into the sum of ***damped sine*** and ***cosine*** terms

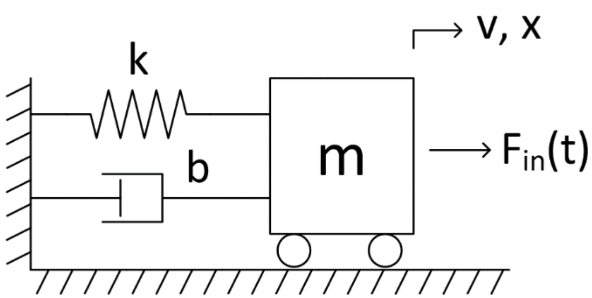
$$\mathcal{L}^{-1}\left\{r_i\frac{(s+\sigma)}{(s+\sigma)^2+\omega^2}+r_{i+1}\frac{\omega}{(s+\sigma)^2+\omega^2}\right\}=r_ie^{-\sigma t}\cos(\omega t)+r_{i+1}e^{-\sigma t}\sin(\omega t)$$

- To get the second-order term in the denominator of (22) into the form of (23), ***complete the square***, to give the following partial fraction expansion

$$Y(s)=\frac{1}{s(s^2+4s+16)}=\frac{r_1}{s}+\frac{r_2(s+2)+r_3(3.464)}{(s+2)^2+(3.464)^2} \qquad (24)$$

Inverse Laplace Transforms – Example 3

- Note that the σ and ω terms in (23) and (24) are the ***real*** and ***imaginary parts*** of the complex-conjugate denominator roots



$$s_{2,3} = -\sigma \pm j\omega = -2 \pm j3.464$$

- Multiplying both sides of (24) by the left-hand-side denominator, equate coefficients and solve for residues as before:

$$r_1 = 0.0625$$
$$r_2 = -0.0625$$
$$r_3 = -0.0361$$

- Laplace transform of the step response is

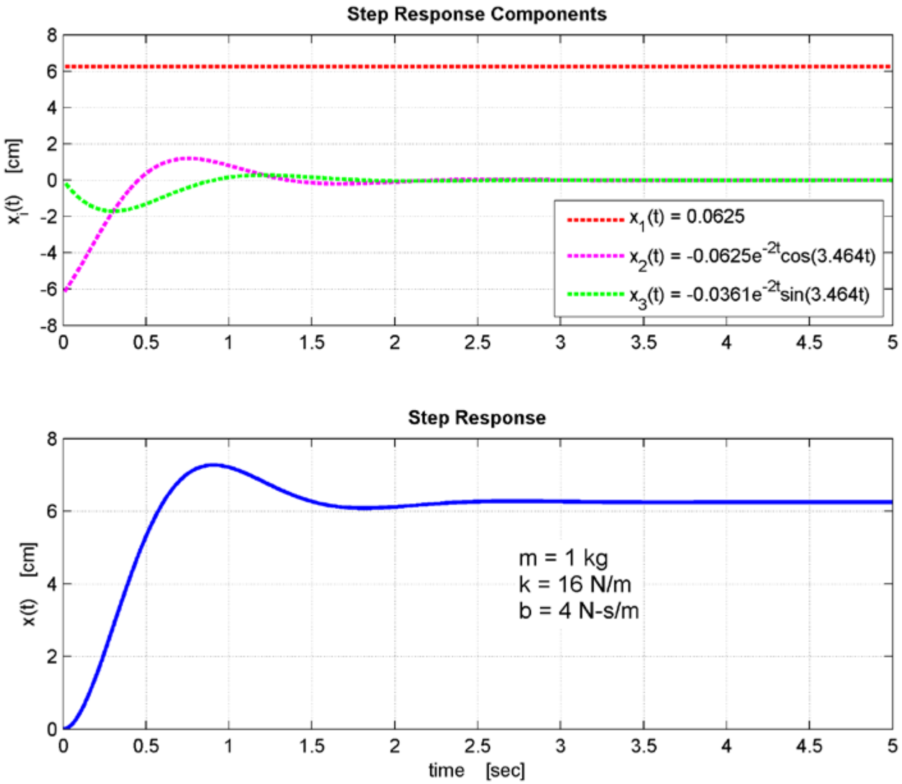
$$Y(s) = \frac{0.0625}{s} - \frac{0.0625(s+2)}{(s+2)^2+(3.464)^2} - \frac{0.0361(3.464)}{(s+2)^2+(3.464)^2} \qquad (25)$$

Inverse Laplace Transforms – Example 3

- The time-domain step response of the system is the ***sum of a constant and two decaying sinusoids***:

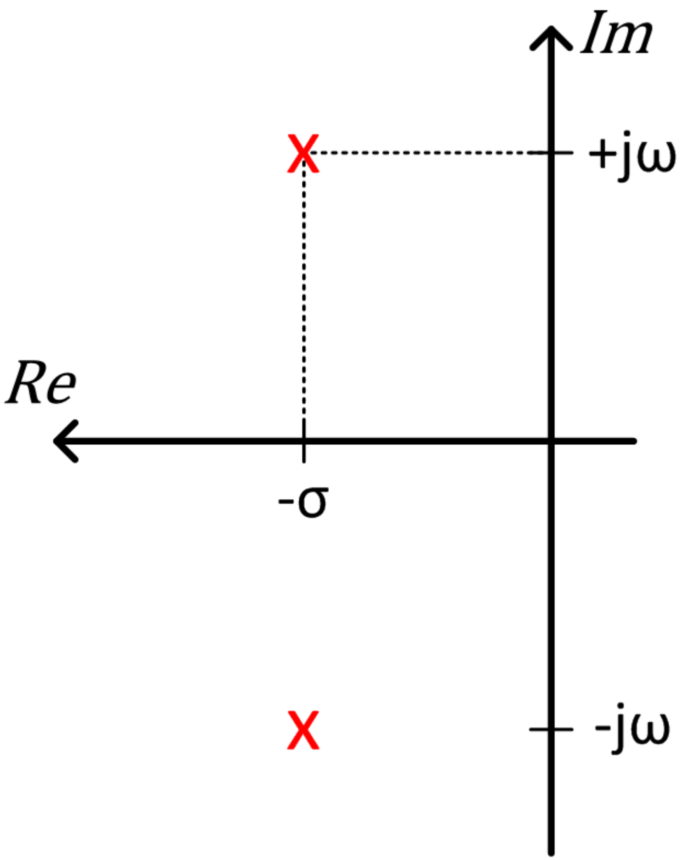
$$y(t) = 0.0625 - 0.0625e^{-2t}\cos(3.464t) - 0.0361e^{-2t}\sin(3.464t) \quad (26)$$

- Step response and individual components plotted in MATLAB
- Characteristic of a signal having ***complex poles***
  - Sinusoidal terms result in overshoot and (possibly) ringing



Laplace-Domain Signals with Complex Poles

- The Laplace transform of the step response in the last example had ***complex poles***
  - A ***complex-conjugate pair***: $s = -\sigma \pm j\omega$



- Results in sine and cosine terms in the time domain

$$Ae^{-\sigma t}\cos(\omega t) + Be^{-\sigma t}\sin(\omega t)$$

- ***Imaginary part*** of the roots, ω
  - ***Frequency of oscillation*** of sinusoidal components of the signal
- ***Real part*** of the roots, σ,
  - ***Rate of decay*** of the sinusoidal components
- Much more on this later