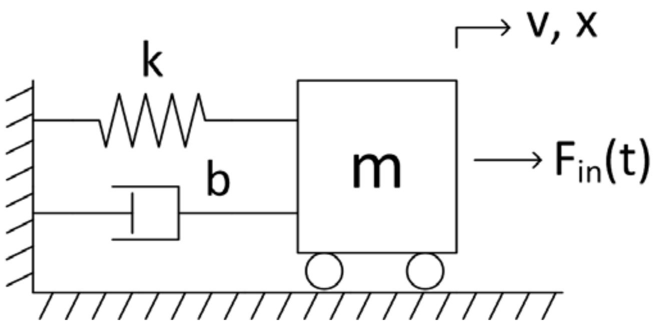
SECTION 6: TIME-DOMAIN ANALYSIS

ESE 330 – Modeling & Analysis of Dynamic Systems

Natural and Forced Responses

This first sub-section of notes continues where the previous section left off, and will explore the difference between the forced and natural responses of a dynamic system.

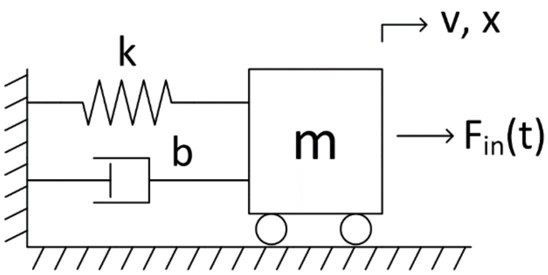
Natural and Forced Responses



- In the previous section we used Laplace transforms to determine the response of a system to a step input, given zero initial conditions
  - The ***driven response***
- Now, consider the response of the same system to non-zero initial conditions only
  - The ***natural response***

Natural Response

- Same spring/mass/damper system
- Set the input to zero
- Second-order ODE for displacement of the mass:



$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = 0 \qquad (1)$$

- Use the derivative property to Laplace transform (1)
  - Allow for non-zero initial-conditions

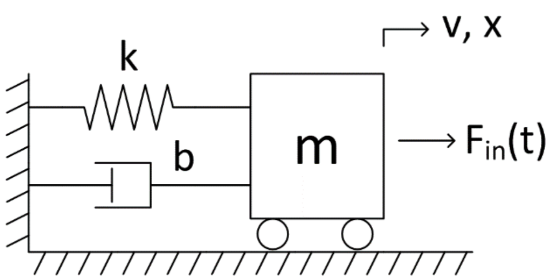
$$s^2Y(s) - sy(0) - \dot{y}(0) + \frac{b}{m}sY(s) - \frac{b}{m}y(0) + \frac{k}{m}Y(s) = 0 \qquad (2)$$

Natural Response

- Solving (2) for $Y(s)$ gives the Laplace transform of the output due solely to ***initial conditions***
  - Laplace transform of the ***natural response***

$$Y(s) = \frac{s\,y(0) + \dot{y}(0) + \frac{b}{m}y(0)}{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \tag{3}$$

- Consider the under-damped system with the following initial conditions
  - $y(0) = 0.15\ m$
  - $\dot{y}(0) = 0.1\ \frac{m}{s}$



- $m = 1\ kg$
- $k = 16\ \frac{N}{m}$
- $b = 4\ \frac{N \cdot s}{m}$

Natural Response

- Substituting component parameters and initial conditions into (3)

$$Y(s) = \frac{0.15s + 0.7}{(s^2+4s+16)} \tag{4}$$

- Remember, it's the ***roots of the denominator polynomial*** that dictate the form of the response
  - **Real roots** – decaying exponentials
  - **Complex roots** – decaying sinusoids
- For the under-damped case, roots are complex
  - Complete the square
  - Partial fraction expansion has the form

$$Y(s) = \frac{0.15s + 0.7}{(s^2+4s+16)} = \frac{r_1(s+2)+r_2(3.464)}{(s+2)^2+(3.464)^2} \tag{5}$$

Natural Response

$$Y(s) = \frac{0.15s + 0.7}{(s^2+4s+16)} = \frac{r_1(s+2)+r_2(3.464)}{(s+2)^2+(3.464)^2} \qquad (5)$$

- Multiply both sides of (5) by the denominator of the left-hand side

$$0.15s + 0.7 = r_1 s + 2r_1 + 3.464r_2$$

- Equating coefficients and solving for r_1 and r_2 gives

$$r_1 = 0.15, \ r_2 = 0.115$$

- The Laplace transform of the natural response:

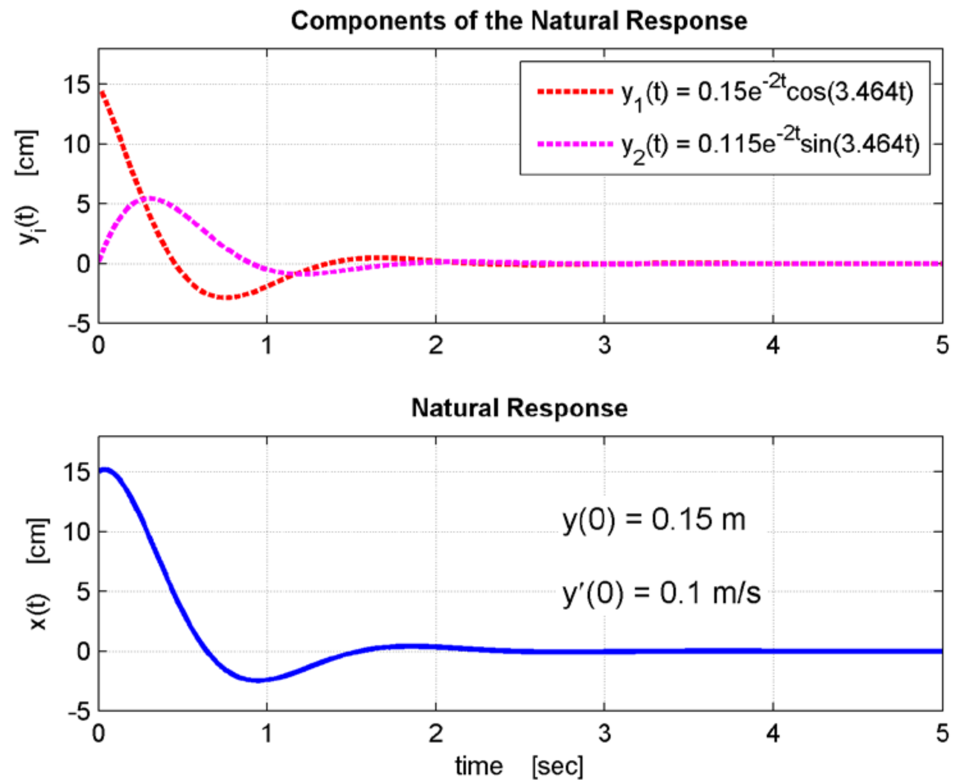
$$Y(s) = \frac{0.15(s+2)}{(s+2)^2+(3.464)^2} + \frac{0.115(3.464)}{(s+2)^2+(3.464)^2} \qquad (6)$$

Natural Response

- Inverse Laplace transform is the ***natural response***

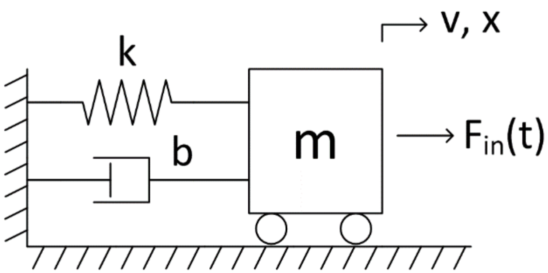
$$y(t) = 0.15e^{-2t}\cos(3.464 \cdot t) + 0.115e^{-2t}\sin(3.464 \cdot t) \qquad (7)$$

- Under-damped response is the sum of ***decaying sine and cosine*** terms



Driven Response with Non-Zero I.C.'s

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{1}{m}F_{in}(t)$$



- Now, Laplace transform, allowing for both ***non-zero input and initial conditions***

$$s^2Y(s) - sy(0) - \dot{y}(0) + \frac{b}{m}Y(s) - \frac{b}{m}y(0) + \frac{k}{m}Y(s) = \frac{1}{m}F_{in}(s)$$

- Solving for $Y(s)$, gives the Laplace transform of the response to both the input and the initial conditions

$$Y(s) = \frac{s\,y(0) + \dot{y}(0) + \frac{b}{m}y(0) + \frac{1}{m}F_{in}(s)}{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \qquad (8)$$

Driven Response with Non-Zero I.C.'s

- Laplace transform of the response has two components

$$Y(s) = \underbrace{\frac{s\,y(0) + \dot{y}(0) + \frac{b}{m}y(0)}{\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)}} + \underbrace{\frac{\frac{1}{m}F_{in}(s)}{\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)}}$$

Natural response - initial conditions ***Driven response*** - input

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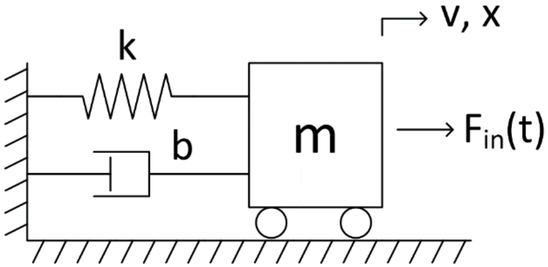
- Total response is a superposition of the initial condition response and the driven response
- Both have the same denominator polynomial
  - Same roots, same type of response
    - Over-, under-, critically-damped

Driven Response with Non-Zero I.C.'s

- $y(0) = 0.15\,m$
- $\dot{y}(0) = 0.1\,\frac{m}{s}$
- $F_{in}(t) = 1N \cdot u(t)$



- $m = 1\,kg$
- $k = 16\,\frac{N}{m}$
- $b = 4\,\frac{N\cdot s}{m}$

- Laplace transform of the ***total*** response

$$Y(s) = \frac{0.15s + 0.7 + \frac{1}{s}}{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} = \frac{0.15s^2 + 0.7s + 1}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)}$$

- Transform back to time domain via partial fraction expansion

$$Y(s) = \frac{r_1}{s} + \frac{r_2(s+2)}{(s+2)^2 + (3.464)^2} + \frac{r_3(3.464)}{(s+2)^2 + (3.464)^2}$$

- Solving for the residues gives

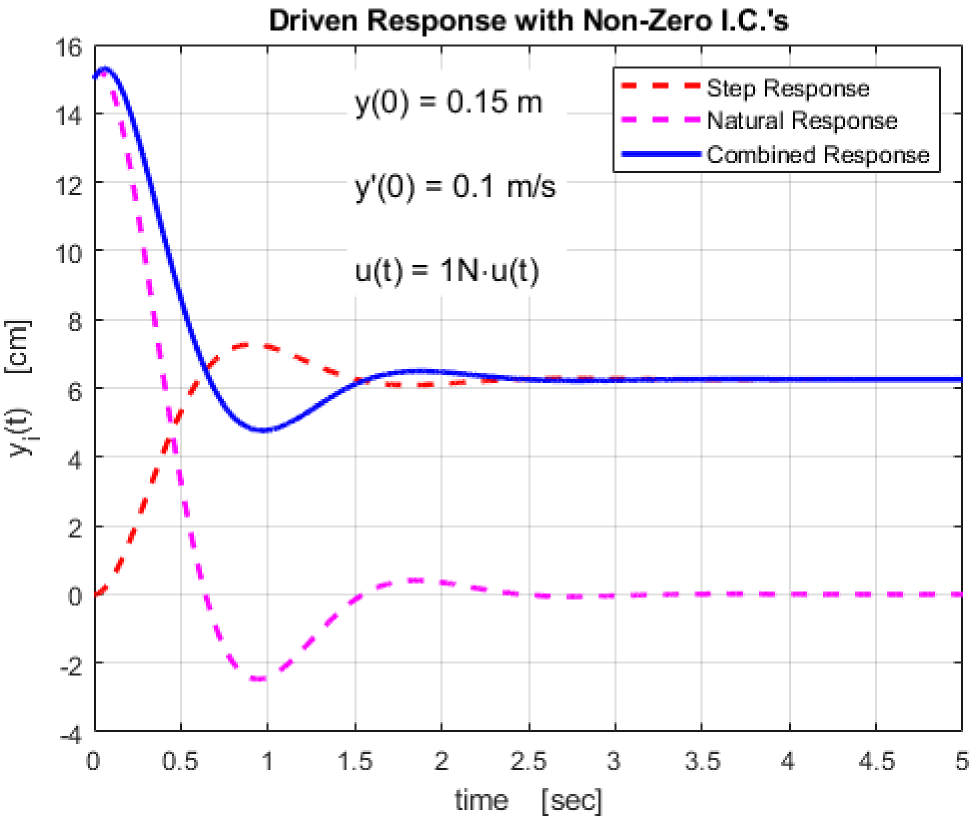
$$r_1 = 0.0625, \quad r_2 = 0.0875, \quad r_3 = 0.0794$$

Driven Response with Non-Zero I.C.'s

- Total response:

$$y(t) = 0.0625 + 0.0875e^{-2t}\cos(3.464 \cdot t) + 0.0794e^{-2t}\sin(3.464 \cdot t)$$

- Superposition of two components
  - ***Natural response*** due to initial conditions
  - ***Driven response*** due to the input



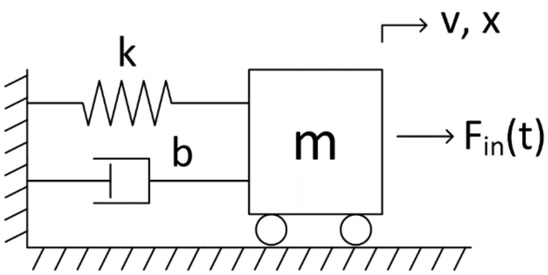
Solving the State-Space Model

Next, we'll apply the Laplace transform to the entire state-space model in matrix form, just as we did for single differential equations.

Solving the State-Space Model

- We’ve seen how to use the Laplace transform to solve individual differential equations
- Now, we’ll ***apply the Laplace transform to the full state-space system model***

- First, we’ll look at the same simple example
  - Later, we’ll take a more generalized approach
  - State-space model is



$$\begin{bmatrix} \dot{p} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} p \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ x \end{bmatrix} \tag{1}$$

- Note that, because this model was derived from a bond graph model, the state variables are now ***momentum*** and ***displacement***

Laplace Transform of the State-Space Model

- For now, focus on the ***state equation***
  - Output is a linear combination of states and inputs
  - Determining the state trajectory is the important thing
- Use the derivative property to ***Laplace transform the state equation***

$$s\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} - \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} P(s) \\ X(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s)$$

- Rearranging to put all transformed state vectors on the left-hand side

$$s\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} - \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} P(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s) \qquad (2)$$

Laplace Transform of the State-Space Model

$$s\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} - \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix}\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s) \qquad (2)$$

- Can factor out the transformed state vector from the left-hand side
  - Must multiply s by a 2×2 identity matrix

$$\left(sI - \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix}\right)\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s)$$

$$\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{b}{m} & -k \\ \frac{1}{m} & 0 \end{bmatrix}\right)\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s)$$

$$\begin{bmatrix} s + \frac{b}{m} & k \\ -\frac{1}{m} & s \end{bmatrix}\begin{bmatrix} P(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s) \qquad (3)$$

Laplace Transform of the State-Space Model

$$\begin{bmatrix} s+\frac{b}{m} & k \\ -\frac{1}{m} & s \end{bmatrix} \begin{bmatrix} P(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} p(0) \\ x(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{in}(s) \tag{3}$$

- Note the form of (3)
  - The LHS is $(sI - A)\mathbf{X}(s)$, where A is the system matrix
  - Everything on the RHS reduces to a 2×1 vector
  - A known matrix times a vector of unknowns equals a known vector
- If we can solve for $P(s)$ and/or $X(s)$, we can inverse transform to get $p(t)$ and/or $x(t)$
  - Use ***Cramer's Rule***

Cramer's Rule

- Given a matrix equation

$$\mathbf{Ax} = \mathbf{y}$$

- We can solve for elements of $\mathbf{x}$ as follows

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})} = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}$$

- The matrix $\mathbf{A}_i$ is formed by replacing the i^{th} column of $\mathbf{A}$ with the vector $\mathbf{y}$

Laplace Transform of the State-Space Model

- According to Cramer's Rule

$$X(s) = \frac{\begin{vmatrix} s+\frac{b}{m} & p(0)+F_{in}(s) \\ -\frac{1}{m} & x(0) \end{vmatrix}}{\begin{vmatrix} s+\frac{b}{m} & k \\ -\frac{1}{m} & s \end{vmatrix}}$$

$$X(s) = \frac{\left(s\,x(0)+\frac{b}{m}x(0)\right)-\left(-\frac{1}{m}p(0)-\frac{1}{m}F_{in}(s)\right)}{s^2+\frac{b}{m}s+\frac{k}{m}} \qquad (4)$$

- According to the output equation from (1)

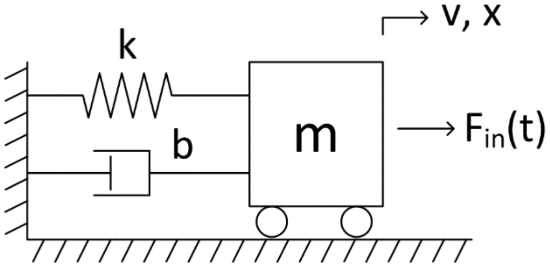
$$Y(s) = X(s)$$

- Equation (4) is identical to (8) from the previous subsection of notes, which we arrived at differently

Laplace Transform of the State-Space Model

$$Y(s)=\frac{\left(s\,y(0)+\frac{b}{m}y(0)\right)-\left(-\frac{1}{m}p(0)-\frac{1}{m}F_{in}(s)\right)}{s^2+\frac{b}{m}s+\frac{k}{m}} \qquad (5)$$

- Next, sub in parameter values, I.C.'s and an input
- Use PFE to ***inverse transform*** to $y(t)$
- Again, consider the under-damped system:
  - $x(0)=0.15\,m$
  - $p(0)=0.1\,N\cdot s$



  - $m=1\,kg$
  - $k=16\,\frac{N}{m}$
  - $b=4\,\frac{N\cdot s}{m}$
- Let the input be a $1N$ step: $F_{in}(t)=1N\cdot u(t)$

Laplace Transform of the State-Space Model

- The Laplace transform of the output becomes

$$Y(s) = \frac{(0.15s + 0.6) - \left(-0.1 - \frac{1}{s}\right)}{s^2 + 4s + 16}$$

$$Y(s) = \frac{0.15s^2 + 0.7s + 1}{s(s^2 + 4s + 16)} \qquad (6)$$

- Inverse transform via ***partial fraction expansion***

$$Y(s) = \frac{0.15s^2 + 0.7s + 1}{s(s^2 + 4s + 16)} = \frac{r_1}{s} + \frac{r_2(s+2) + r_3(3.464)}{(s+2)^2 + (3.464)^2} \qquad (7)$$

- Multiply both sides by left-hand-side denominator

$$0.15s^2 + 0.7s + 1 = r_1 s^2 + 4r_1 s + 16r_1 + r_2 s^2 + 2r_2 s + 3.464 r_3 s$$

- Equating coefficients and solving yields

$$r_1 = 0.0625, \;\; r_2 = 0.0875, \;\; r_3 = 0.0794$$

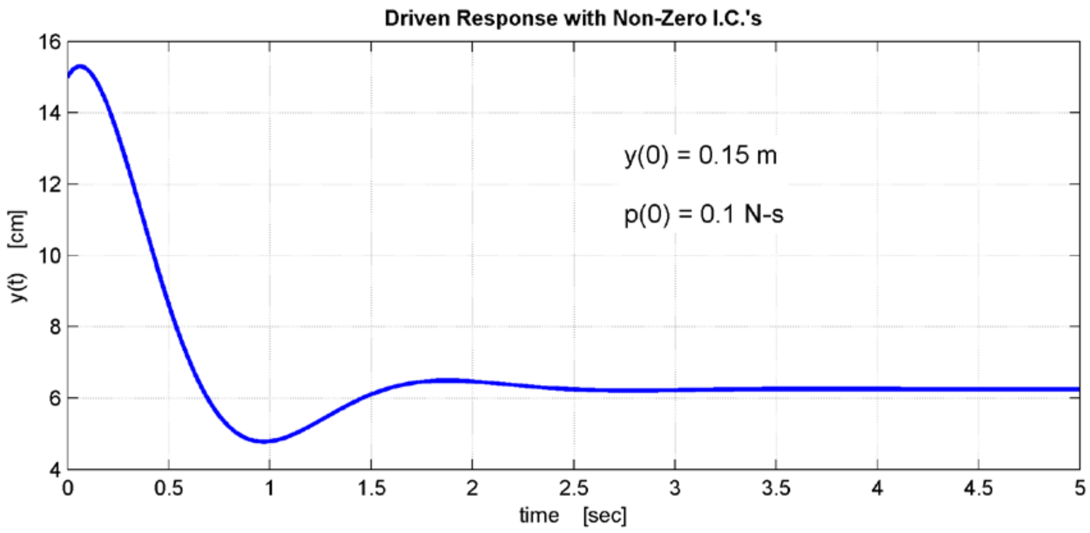
Laplace Transform of the State-Space Model

- The Laplace transform of the system response is

$$Y(s) = \frac{0.0625}{s} + \frac{0.0875(s+2)}{(s+2)^2+(3.464)^2} + \frac{0.0794(3.464)}{(s+2)^2+(3.464)^2} \quad (8)$$

- The time-domain response is

$$y(t) = 0.0625 + 0.0875e^{-2t}\cos(3.464t) + 0.0794e^{-2t}\sin(3.464t) \quad (9)$$



- ***Transient*** portion
  - Due to initial conditions and input step
  - Decays to zero
- ***Steady-State*** portion
  - Due to constant input
  - Does not decay

Laplace Transform of the State-Space Model

- Now, we'll apply the Laplace transform to the solution of the state-space model in general form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- For now, focus on the state equation only
  - Output is derived from states and inputs
- Laplace transform of the state equation

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

- Rearranging

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0)$$

- Factoring out the transformed state from the left-hand side

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0) \qquad (10)$$

Laplace Transform of the State-Space Model

$$(s\mathbf{I}-\mathbf{A})\mathbf{X}(s)=\mathbf{B}\mathbf{U}(s)+\mathbf{x}(0) \qquad (10)$$

- Remember the dimensions of each term in (10)
  - $(s\mathbf{I}-\mathbf{A})$: $n\times n$
  - $\mathbf{X}(s)$: $n\times 1$
  - $(s\mathbf{I}-\mathbf{A})\mathbf{X}(s)$: $n\times 1$
  - $\mathbf{B}\mathbf{U}(s)$: $n\times 1$
  - $\mathbf{x}(0)$: $n\times 1$

- Apply ***Cramer's rule*** to solve for the Laplace transform of the i^{th} state variable

$$X_i(s)=\frac{|(s\mathbf{I}-\mathbf{A})_i|}{|s\mathbf{I}-\mathbf{A}|} \qquad (11)$$

- The matrix $(s\mathbf{I}-\mathbf{A})_i$ is formed by replacing the i^{th} column of $(s\mathbf{I}-\mathbf{A})$ with $\mathbf{B}\mathbf{U}(s)+\mathbf{x}(0)$
  - An $n\times 1$ vector of known values

Laplace Transform of the State-Space Model

$$X_i(s) = \frac{|(s\mathbf{I}-\mathbf{A})_i|}{|s\mathbf{I}-\mathbf{A}|} \tag{11}$$

- Denominator of (11) is the determinant of $(s\mathbf{I} - \mathbf{A})$
  - $(s\mathbf{I} - \mathbf{A})$ is an $n \times n$ matrix
    - Each diagonal term is a first-order polynomial in s
  - One term in the determinant is the ***trace*** of the matrix, the product terms along the diagonal
  - $|s\mathbf{I} - \mathbf{A}|$ is an $\boldsymbol{n^{th}}$***-order polynomial in s***
- The ***characteristic polynomial***:

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| \tag{12}$$

- Roots of $\Delta(s)$ are values of s that satisfy the ***characteristic equation***

$$\Delta(s) = 0 \tag{13}$$

  - ***Poles*** of (11)
  - ***Eigenvalues*** of system matrix, $\mathbf{A}$

Laplace Transform of the State-Space Model

- ***Denominator of every state variable's Laplace transform contains the characteristic polynomial***

$$X_i(s) = \frac{|(s\mathbf{I}-\mathbf{A})_i|}{\Delta(s)} \tag{14}$$

  - A *characteristic* of the system
- Remember, denominator roots (i.e. poles) determine the nature of the response
  - **Real roots** – decaying exponentials
  - **Complex roots** – decaying sinusoids
- Responses of all state variables have same components
  - Numerators of transforms determine the differences
- ***Output transform has the same denominator***, $\Delta(s)$
  - Linear combination of states and input
  - Response includes the same sinusoidal and/or exponential components

Laplace Transform of the State-Space Model

- Assume:
  - zero initial conditions: $\mathbf{x}(0) = \mathbf{0}$
  - SISO system: single input – $U(s)$ is a scalar transform
- Can factor out the input from the numerator of (14)

$$|(s\mathbf{I} - \mathbf{A})_i| = U(s)|(s\mathbf{I} - \mathbf{A})_{i^*}|$$

where $(s\mathbf{I} - \mathbf{A})_{i^*}$ is the $n \times n$ matrix formed by replacing the i^{th} column of $(s\mathbf{I} - \mathbf{A})$ with the $n \times 1$ vector $\mathbf{B}$

  - $U(s)$ appears in every term of one column of $(s\mathbf{I} - \mathbf{A})_i$
  - $U(s)$ appears in every term of the determinant

Laplace Transform of the State-Space Model

- Can now write the Laplace transform of the state variable response as

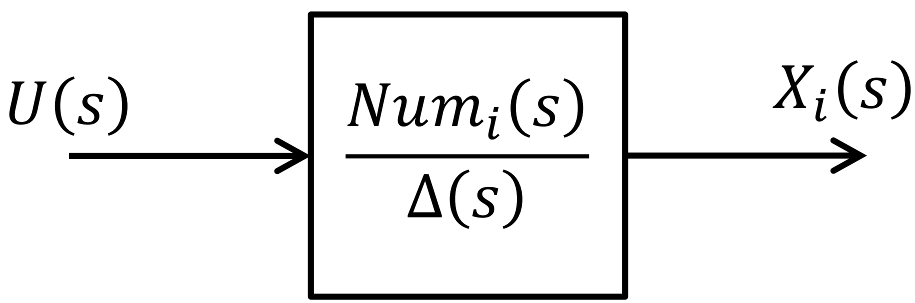
$$X_i(s) = U(s)\frac{\left|(s\mathbf{I}-\mathbf{A})_{i^*}\right|}{|sI-A|} = U(s)\frac{Num_i(s)}{\Delta(s)} \qquad (15)$$

- $Num_i(s)$ is, in general, different for each state variable
  - At most, an $(n-1)^{st}$ order polynomial in s
- Components of ***every*** state variable (and output) response determined by
  - The ***characteristic polynomial***, $\Delta(s)$
  - The ***input***, $U(s)$
- Numerator, $Num_i(s)$, determines exact response
  - Weighting of each sinusoidal and/or exponential component

Laplace Transform of the State-Space Model

$$X_i(s) = U(s)\,\frac{Num_i(s)}{\Delta(s)} \tag{15}$$

- Laplace transform of each state variable response, $X_i(s)$, is the Laplace transform of the input scaled by $\dfrac{Num_i(s)}{\Delta(s)}$



- In the next sub-section, we'll explore a related concept – ***transfer functions***

Transfer Functions

Transfer Functions

- Now, come back to the full state-space model, including the output equation – (SISO case assumed here - u and y are scalars)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathrm{D}u$$

- Assume zero initial conditions and Laplace transform the whole model

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{1}$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + \mathrm{D}U(s) \tag{2}$$

- Simplify the state equation as before

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

- Solving for the state vector

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) \tag{3}$$

Transfer Functions

- Substituting (3) into (2) gives the Laplace transform of the output

$$Y(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + \mathrm{D}U(s)$$

- Factoring out the input

$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathrm{D}]U(s) \tag{4}$$

- Transform of the output is the input scaled by the stuff in the square brackets
- Dividing through by the input gives the ***transfer function***

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathrm{D} \tag{5}$$

  - ***Ratio of system's output to input in the Laplace domain, assuming zero initial conditions***
  - An alternative to the state-space (time-domain) model for mathematically representing a system

Transfer Matrix – MIMO Systems

- For MIMO systems

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

  - m inputs: $\mathbf{u}$ is $m \times 1$, $\mathbf{B}$ is $n \times m$
  - p outputs: $\mathbf{y}$ is $p \times 1$, $\mathbf{C}$ is $p \times n$
- ***Transfer function becomes a $p \times m$ matrix***

$$\mathbf{G}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- Transfer function $G_{ij}(s)$ relates the i^{th} output to the j^{th} input

$$G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}$$

- We'll continue to assume SISO systems in this course

Transfer Functions

- System output in the Laplace domain is the input multiplied by the transfer function

$$Y(s) = U(s) \cdot G(s)$$

- We saw earlier that state variables are given by

$$X_i(s) = U(s)\frac{Num_i(s)}{\Delta(s)}$$

where $\Delta(s) = |s\mathbf{I} - \mathbf{A}|$ is the ***characteristic polynomial***

- Output is linear combination of states and input, so we'd expect the denominator of $G(s)$ to be $\Delta(s)$ as well
  - Is it? What is the denominator of $G(s)$?

Transfer Functions

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathrm{D} \tag{5}$$

- The matrix inverse term in (5) is given by

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{adj(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|}$$

  where the numerator is the ***adjoint*** of $(s\mathbf{I} - \mathbf{A})$
- Equation (5) can be rewritten as

$$G(s) = \frac{\mathbf{C}\, adj(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathrm{D}|s\mathbf{I} - \mathbf{A}|}{|s\mathbf{I} - \mathbf{A}|} \tag{6}$$

- ***Transfer function denominator is the characteristic polynomial***
- Poles of the transfer function are roots of $\Delta(s)$
  - ***System poles*** or ***eigenvalues***
  - ***Eigenvalues of the system matrix,*** $\mathbf{A}$
  - Along with the input, system poles determine the nature of the time-domain response

Eigenvalues

This sub-section of notes takes a bit of a tangent to explain the use of the term ***eigenvalues*** when referring to system poles.

Eigenvalues

- We've been using the term ***eigenvalue*** when referring to system poles – why?
- Recall from linear algebra, the ***eigenvalue problem***

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

where: $\mathbf{A}$ is an $n \times n$ matrix

$\mathbf{v}$ is an $n \times 1$ vector – an ***eigenvector***

λ is a scalar – an ***eigenvalue***

- Eigenvalue problem involves finding both the ***eigenvalues*** and the ***eigenvectors*** that satisfy (1)
- Eigenvalues and eigenvectors are specific to (characteristics of) the matrix $\mathbf{A}$
- An $n \times n$ matrix will have, at most, n eigenvalues and n corresponding eigenvectors
- Equation (1) says:
  - An $n \times 1$ eigenvector, $\mathbf{v}$, left-multiplied by an $n \times n$ matrix, $\mathbf{A}$, results in an $n \times 1$ vector
  - The resulting vector is the eigenvector ***scaled*** by the eigenvalue, λ
  - Result is ***in the same direction*** as $\mathbf{v}$ – i.e., ***not rotated***

Eigenvalues and Eigenvectors

- Geometrically, multiplication of a vector by a matrix results in two things
  - ***Scaling*** and ***rotation***
- Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

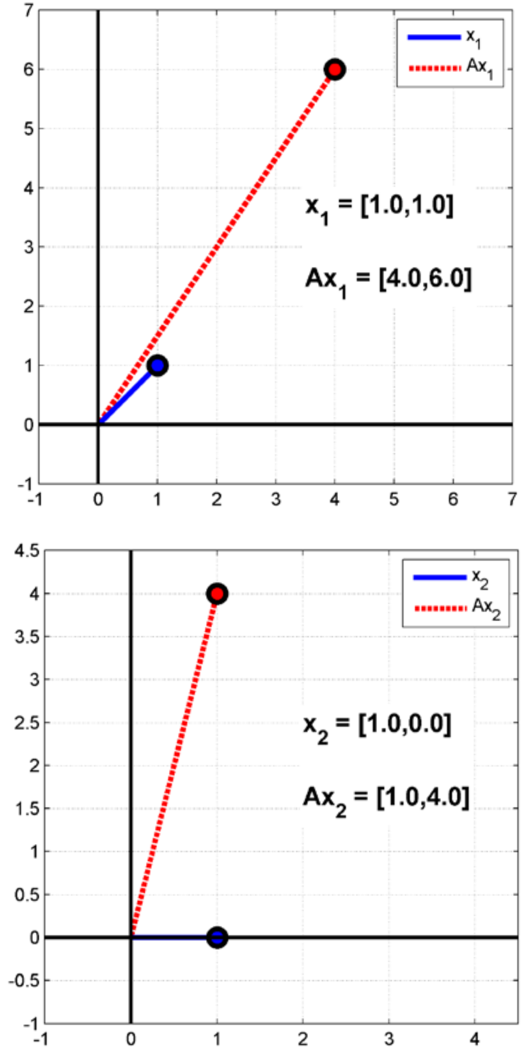
- And the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Compute the product

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

- In both cases, results have ***different magnitudes*** and ***different directions***



Eigenvalues and Eigenvectors

- Multiplication of a matrix and one of its ***eigenvectors*** results in ***scaling only***
  - ***No rotation***
- The 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

has two eigenvectors (normalized)

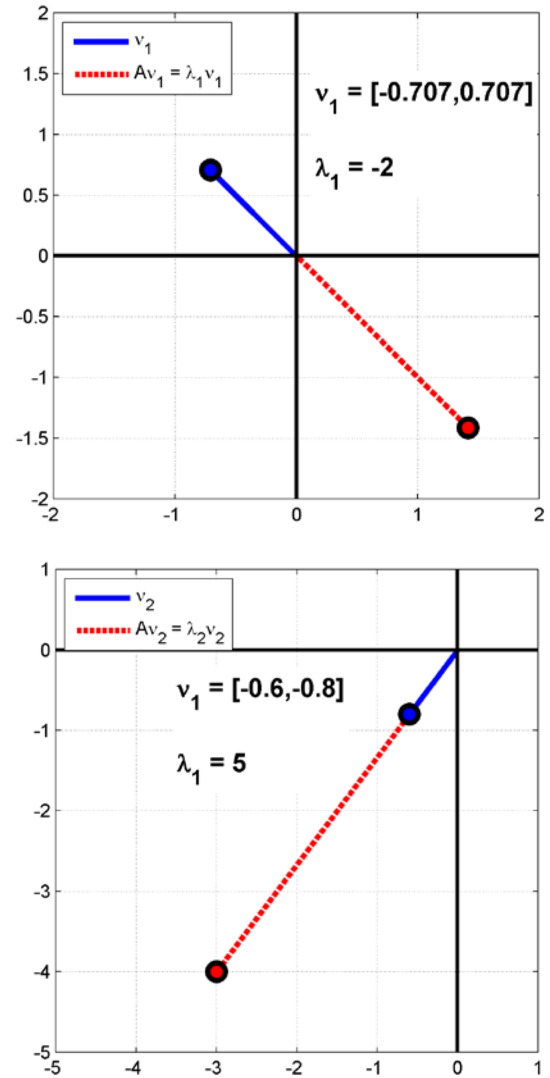
$$\mathbf{v}_1 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix}$$

and two corresponding eigenvalues

$$\lambda_1 = -2 \text{ and } \lambda_2 = 5$$

such that

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \text{ and } \mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$$



Eigenvalues and Eigenvectors

- A ***full-rank***, $n \times n$ matrix will have n pairs of eigenvalues and eigenvectors
- To find all eigenvalues and eigenvectors that satisfy (1)

$$\mathbf{Av} = \lambda\mathbf{v} \tag{1}$$

rearrange

$$\lambda\mathbf{v} - \mathbf{Av} = \mathbf{0}$$

and factor out the eigenvector term

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = 0 \tag{2}$$

- If $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ exists, then $\mathbf{v} = \mathbf{0}$, which is the trivial solution and of no interest
- We're interested in values of λ and $\mathbf{v}$ that satisfy (2) when $(\lambda\mathbf{I} - \mathbf{A})$ is not invertible – when it is ***singular***

Eigenvalues and Eigenvectors

- Want to find values of λ for which $(\lambda\mathbf{I} - \mathbf{A})$ is singular
  - A matrix is singular if its determinant is zero

$$|\lambda\mathbf{I} - \mathbf{A}| = 0 \tag{3}$$

- Equation (3) is the ***characteristic equation*** for $\mathbf{A}$
  - $|\lambda\mathbf{I} - \mathbf{A}|$ is the ***characteristic polynomial***, $\Delta(\lambda)$
  - An n^{th}-order polynomial in λ
- ***Eigenvalues*** of matrix $\mathbf{A}$ are all n values of λ that satisfy (3)
  - Roots of the characteristic polynomial
  - Find the corresponding ***eigenvectors*** by substituting λ into (2) and solving for $\mathbf{v}$
- Letting $\lambda = s$, (3) becomes the ***denominator of the system transfer function***, $G(s)$

Using the Transfer Function to Determine System Response

Using $G(s)$ to determine System Response

- System output in the Laplace domain can be expressed in terms of the transfer function as

$$Y(s) = U(s)G(s) \tag{1}$$

  - Laplace-domain output is the product of the Laplace-domain input and the transfer function
- Response to two specific types of inputs often used to characterize dynamic systems
  - ***Impulse response***
  - ***Step response***
- We'll use the approach of (1) to determine these responses

Impulse response

- ***Impulse function***

$$\delta(t) = 0,\ t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

- Laplace transform of the impulse function is

$$\mathcal{L}\{\delta(t)\} = 1$$

- Impulse response in the Laplace domain is

$$Y(s) = 1 \cdot G(s) = G(s)$$

- The ***transfer function is the Laplace transform of the impulse response***
- Impulse response in the time domain is the inverse transform of the transfer function

$$y(t) = g(t) = \mathcal{L}^{-1}\{G(s)\}$$

Step Response

- ***Step function***: $$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$
- Laplace transform of the step function

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

- Laplace-domain step response

$$Y(s) = \frac{1}{s} \cdot G(s)$$

- Time-domain step response

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot G(s)\right\}$$

- Recall the integral property of the Laplace transform

$$\mathcal{L}\left\{\int_0^t g(\tau)d\tau\right\} = \frac{1}{s} \cdot G(s), \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot G(s)\right\} = \int_0^t g(\tau)d\tau$$

- ***The step response is the integral of the impulse response***

First- and Second-Order Systems

- All transfer functions can be decomposed into 1st- and 2nd-order terms by factoring $\Delta(s)$
  - Real poles – ***1st-order terms***
  - Complex-conjugate poles – ***2nd-order terms***

- These terms and, therefore, the poles determine the nature of the time-domain response
  - Real poles – ***decaying exponentials***
  - Complex-conjugate poles - ***decaying sinusoids***

- All time-domain responses will be a superposition of decaying exponentials and decaying sinusoids
  - These are the ***natural modes*** or ***eigenmodes*** of the system

- Instructive to examine the responses of 1st- and 2nd-order systems
  - Gain insight into relationships between pole location and response

Response of First-Order Systems

First-Order System – Impulse Response

- First-order transfer function:

$$G(s) = \frac{A}{s+\sigma}$$

- Single real pole at

$$s = -\sigma = -\frac{1}{\tau}$$

where τ is the system ***time constant***

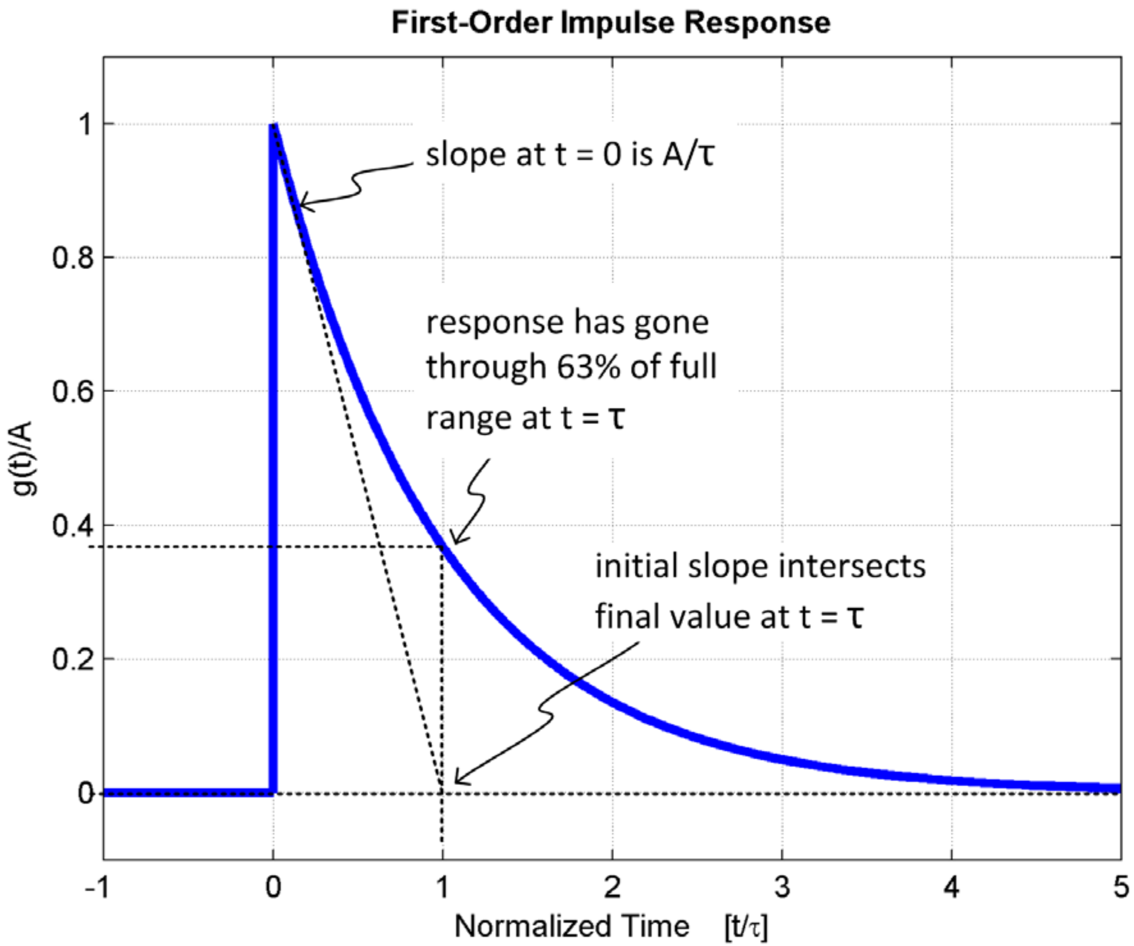
- Impulse response:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = Ae^{-\sigma t} = Ae^{-\frac{t}{\tau}}$$

$$\boxed{g(t) = Ae^{-\frac{t}{\tau}}}$$

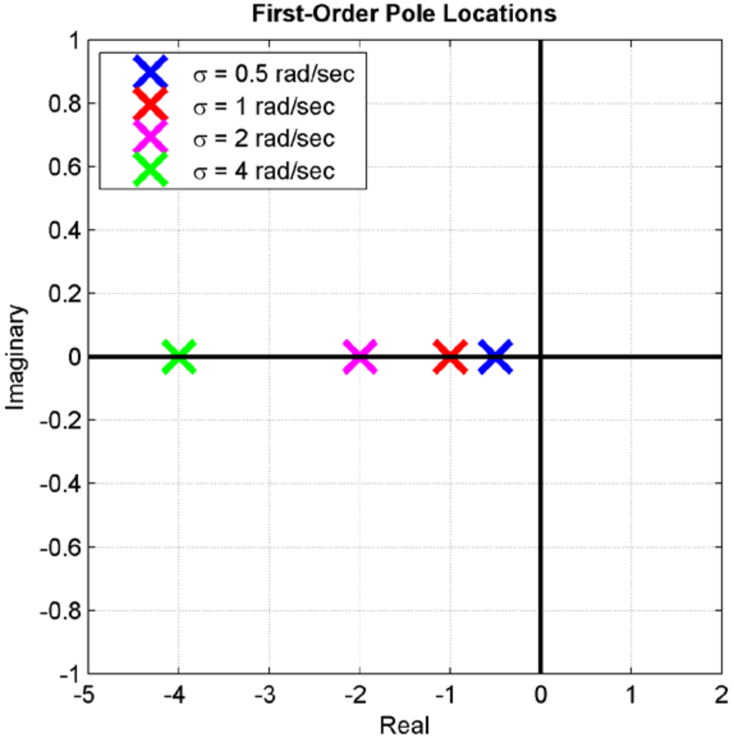
First-Order System – Impulse Response

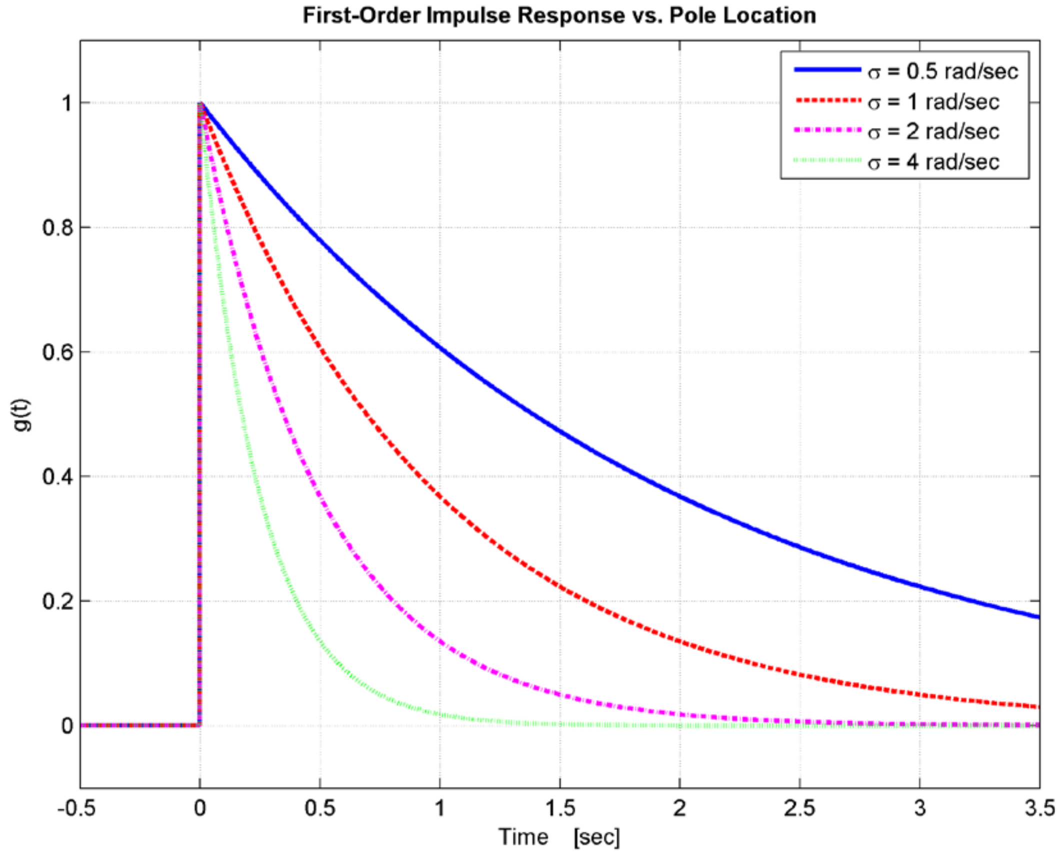
- Initial slope is inversely proportional to time constant
- Response completes 63% of transition after one time constant
- Decays to zero as long as the pole is negative



Impulse Response vs. Pole Location

- Increasing σ corresponds to decreasing τ and a faster response





First-Order System – Step Response

- Step response in the Laplace domain

$$Y(s) = \frac{1}{s} \cdot G(s) = \frac{A}{s(s+\sigma)}$$

- Inverse transform back to time domain via partial fraction expansion

$$Y(s) = \frac{A}{s(s+\sigma)} = \frac{r_1}{s} + \frac{r_2}{s+\sigma}$$

$$A = (r_1 + r_2)s + \sigma r_1$$

$$s^0: \ \sigma r_1 = A \ \rightarrow \ r_1 = \frac{A}{\sigma}$$

$$s^1: \ r_1 + r_2 = 0 \ \rightarrow \ r_2 = -\frac{A}{\sigma}$$

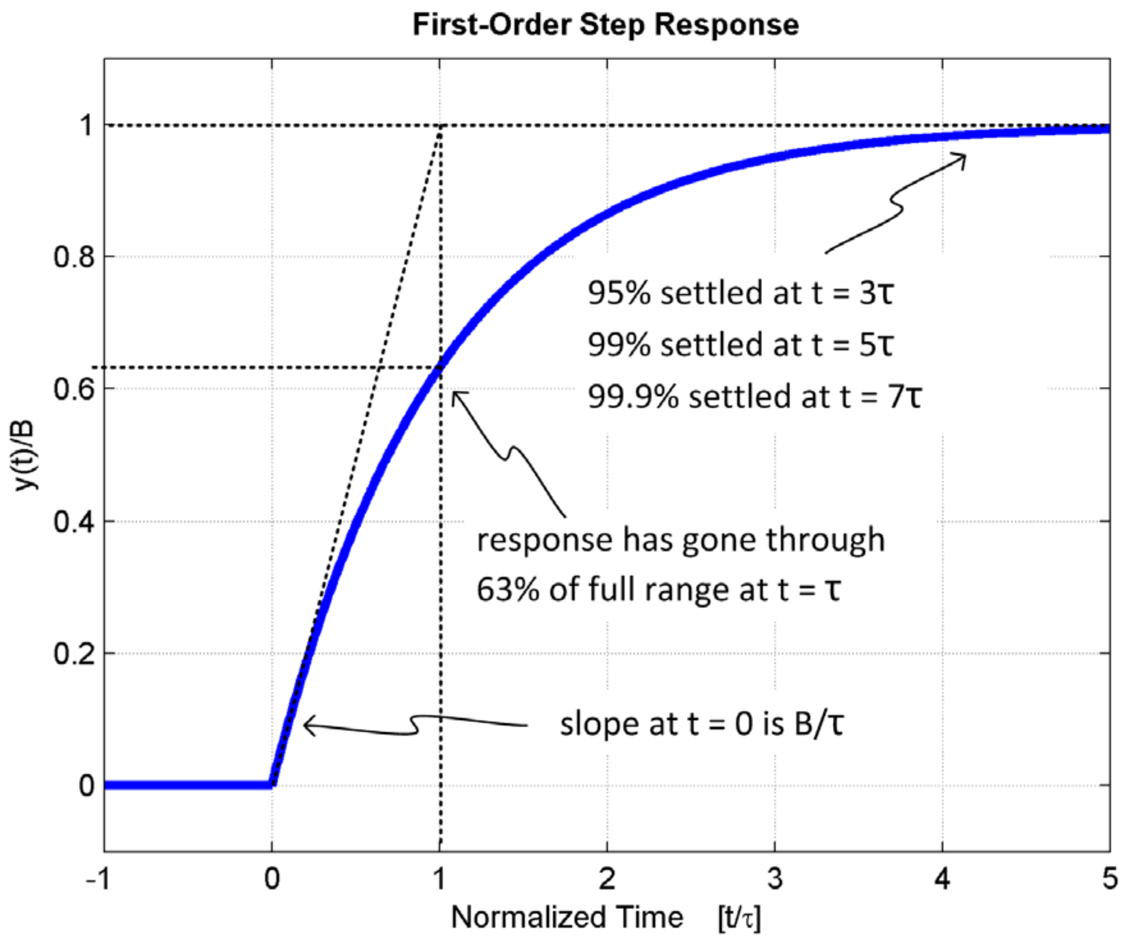
$$Y(s) = \frac{A/\sigma}{s} - \frac{A/\sigma}{s+\sigma}$$

- Time-domain step response

$$y(t) = \frac{A}{\sigma} - \frac{A}{\sigma} e^{-\sigma t} = B - Be^{-\frac{t}{\tau}}$$

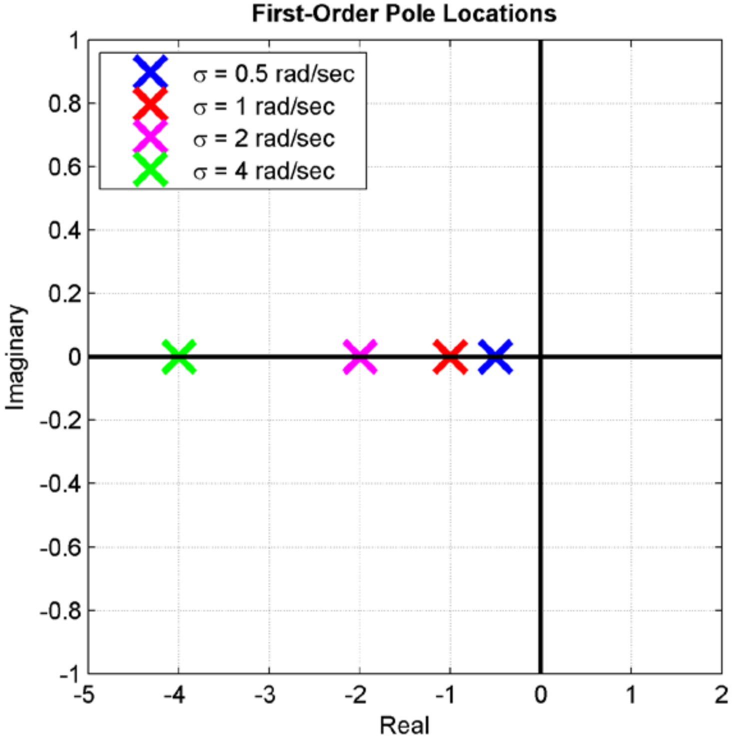
First-Order System – Step Response

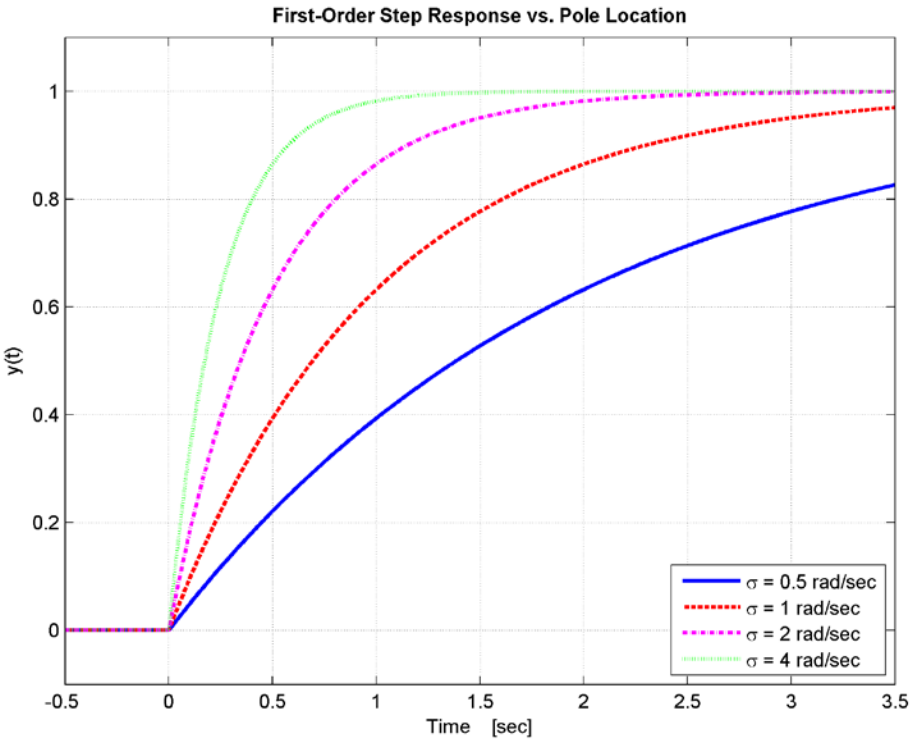
- Initial slope is inversely proportional to time constant
- Response completes 63% of transition after one time constant
- Almost completely settled after 7τ



Step Response vs. Pole Location

- Increasing σ corresponds to decreasing τ and a faster response





Pole Location and Stability

- First-order transfer function

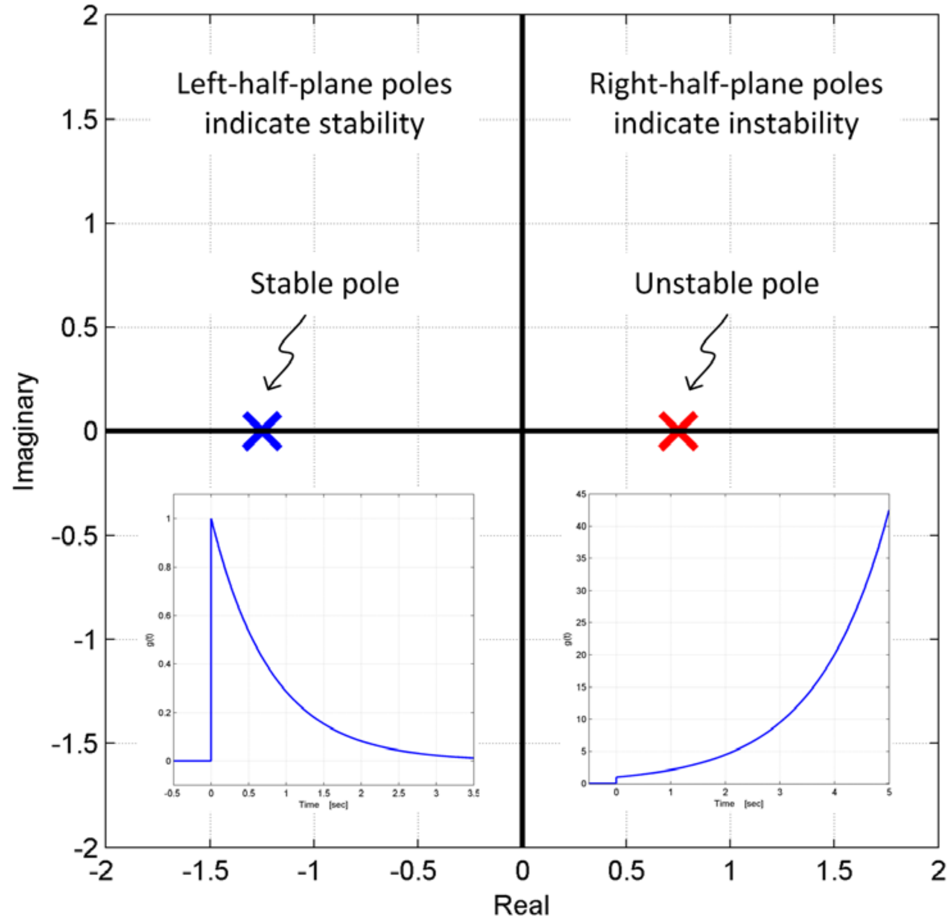
$$G(s) = \frac{A}{s-p}$$

  where p is the system pole

- Impulse response is

$$g(t) = Ae^{pt}$$

- If $p < 0$, $g(t)$ decays to zero
  - Pole in the ***left half-plane***
  - System is ***stable***
- If $p > 0$, $g(t)$ grows without bound
  - Pole in the ***right half-plane***
  - System is ***unstable***



Response of Second-Order Systems

Second-Order Systems

- Second-order transfer function

$$G(s) = \frac{Num(s)}{s^2 + a_1 s + a_0} = \frac{Num(s)}{(s+\sigma)^2 + \omega_d^2} \quad (1)$$

where ω_d is the ***damped natural frequency***

- Can also express the 2nd-order transfer function as

$$G(s) = \frac{Num(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (2)$$

where ω_n is the ***un-damped natural frequency***, and ζ is the ***damping ratio***

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\zeta = \frac{\sigma}{\omega_n}$$

- Two poles at

$$s_{1,2} = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

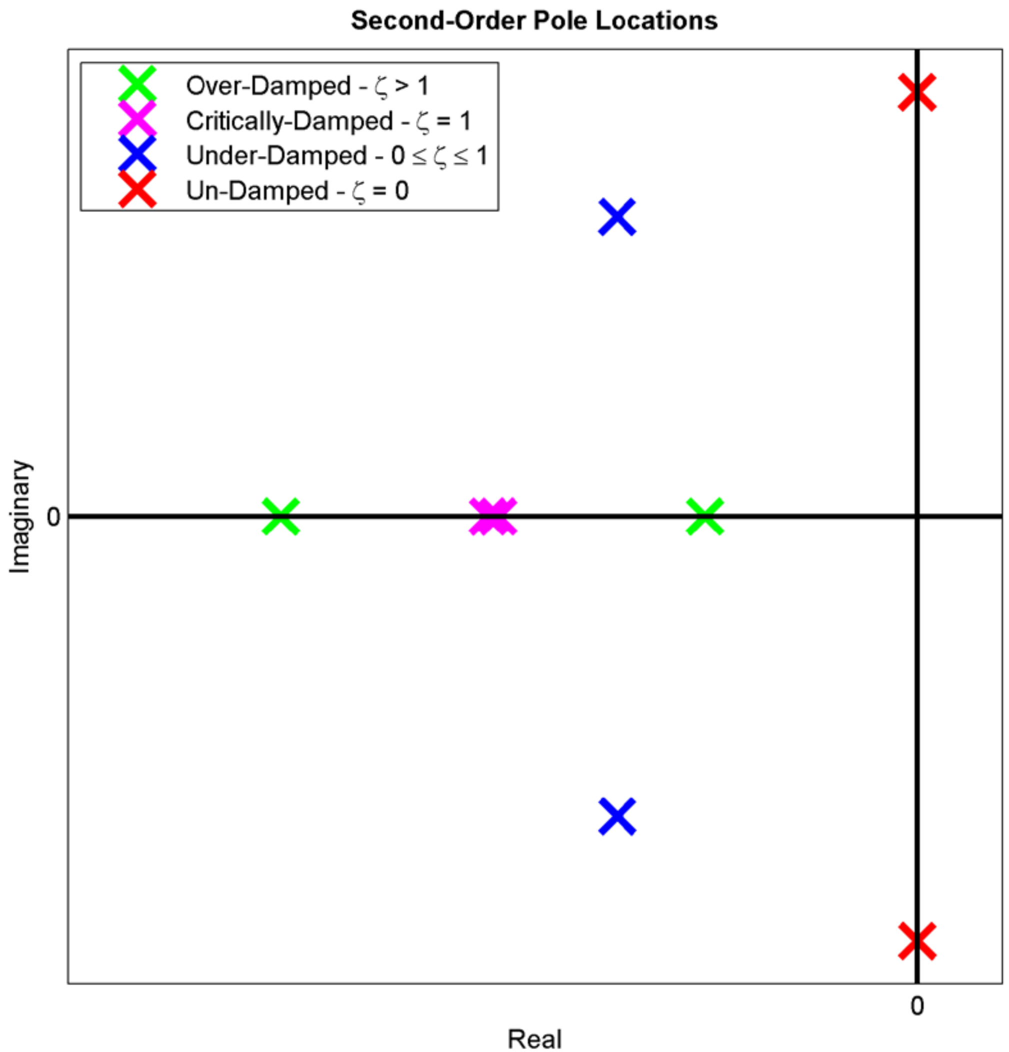
Categories of Second-Order Systems

- The 2nd-order system poles are

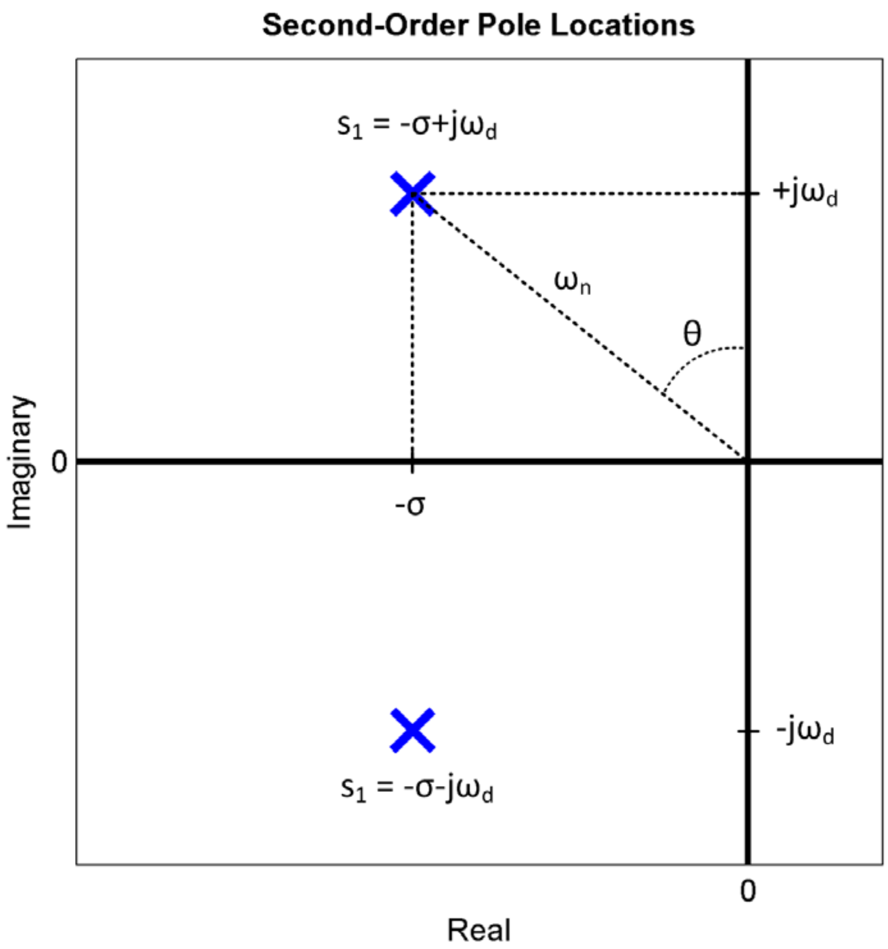
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- Value of ζ determines the nature of the poles and, therefore, the response
- $\boldsymbol{\zeta > 1}$: ***Over-damped***
  - Two distinct, real poles – sum of decaying exponentials – treat as two first-order terms
  - $s_1 = -\sigma_1,\ s_2 = -\sigma_2$
- $\boldsymbol{\zeta = 1}$: ***Critically-damped***
  - Two identical, real poles – time-scaled decaying exponentials
  - $s_{1,2} = -\sigma = -\zeta\omega_n = -\omega_n$
- $\boldsymbol{0 < \zeta < 1}$: ***Under-damped***
  - Complex-conjugate pair of poles – sum of decaying sinusoids
  - $s_{1,2} = -\sigma \pm j\omega_d = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
- $\boldsymbol{\zeta = 0}$: ***Un-damped***
  - Purely-imaginary, conjugate pair of poles – sum of non-decaying sinusoids
  - $s_{1,2} = \pm j\omega_n$

2nd-Order Pole Locations and Damping



Second-Order Poles - $0 \leq \zeta \leq 1$



- Can relate σ, ω_d, ω_n, and ζ to pole location geometry
- ω_n is the magnitude of the poles
- ζ is a measure of system damping

$$\zeta = \frac{\sigma}{\omega_n} = \sin(\theta)$$

- $\zeta = 0$
  - Two purely imaginary poles
- $\zeta = 1$
  - Two identical real poles

Impulse Response – Critically-Damped

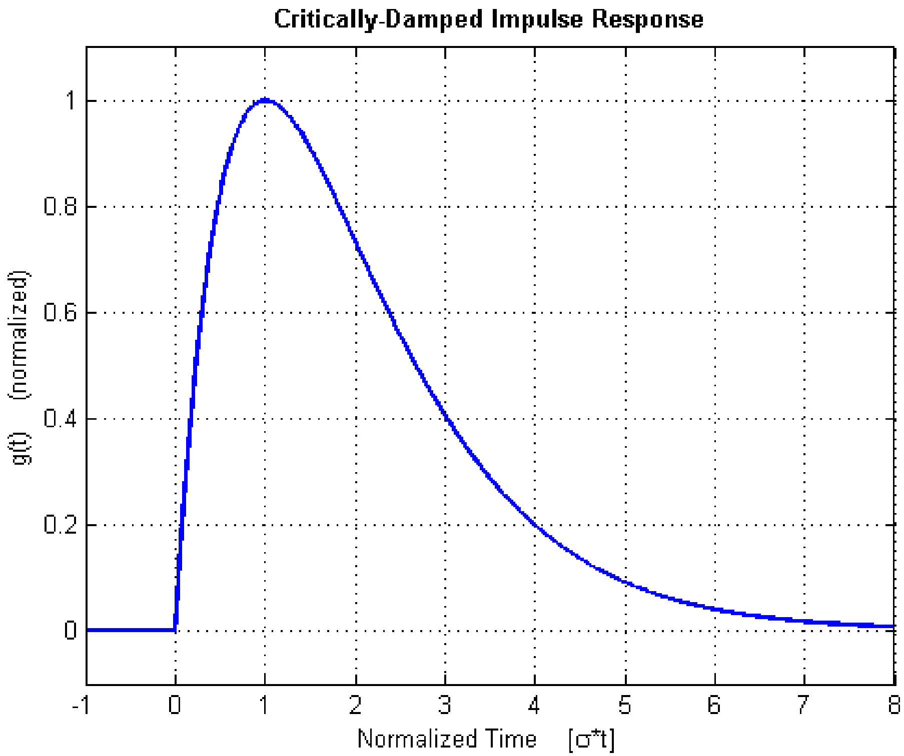
- For $\zeta = 1$, the transfer function reduces to

$$G(s) = \frac{A}{s^2 + 2\omega_n s + \omega_n^2} = \frac{A}{(s+\omega_n)^2} = \frac{A}{(s+\sigma)^2}$$

- Impulse response

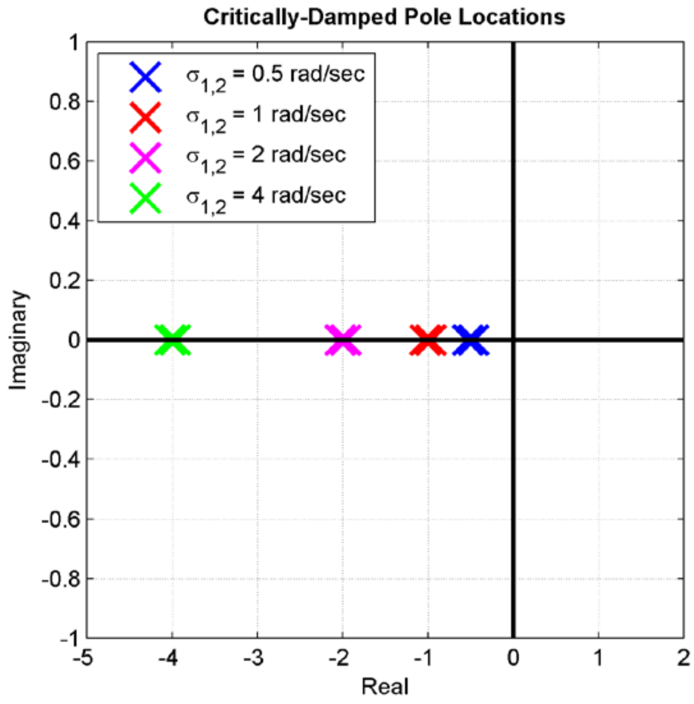
$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

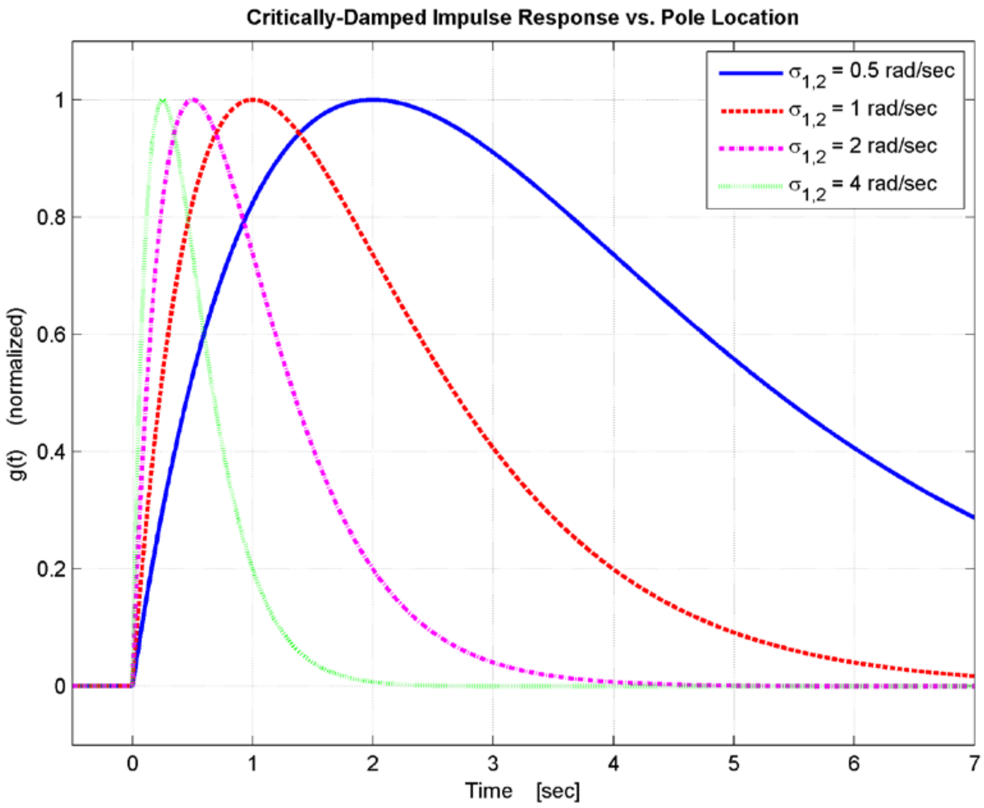
$$g(t) = Ate^{-\sigma t}$$



Impulse Response – Critically-Damped

- Speed of response is proportional to σ





Impulse Response – Under-Damped

- For $0 < \zeta < 1$, the transfer function is

$$G(s) = \frac{A}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Complete the square on the denominator

$$G(s) = \frac{A}{(s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2} = \frac{A}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Rewrite in the form of a damped sinusoid

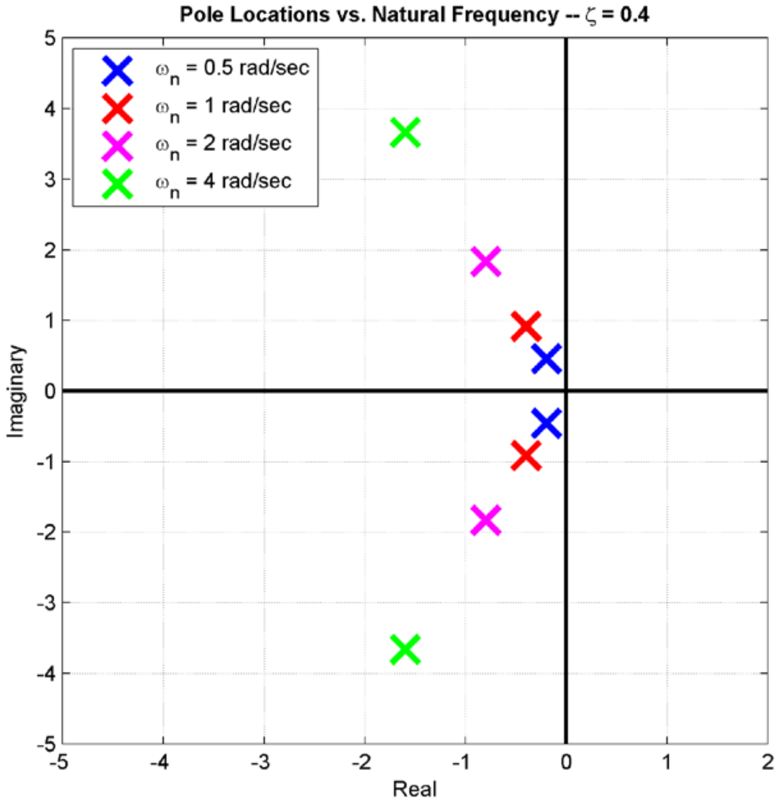
$$G(s) = \frac{A}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2} = \frac{A}{\omega_d}\frac{\omega_d}{(s + \sigma)^2 + \omega_d^2}$$

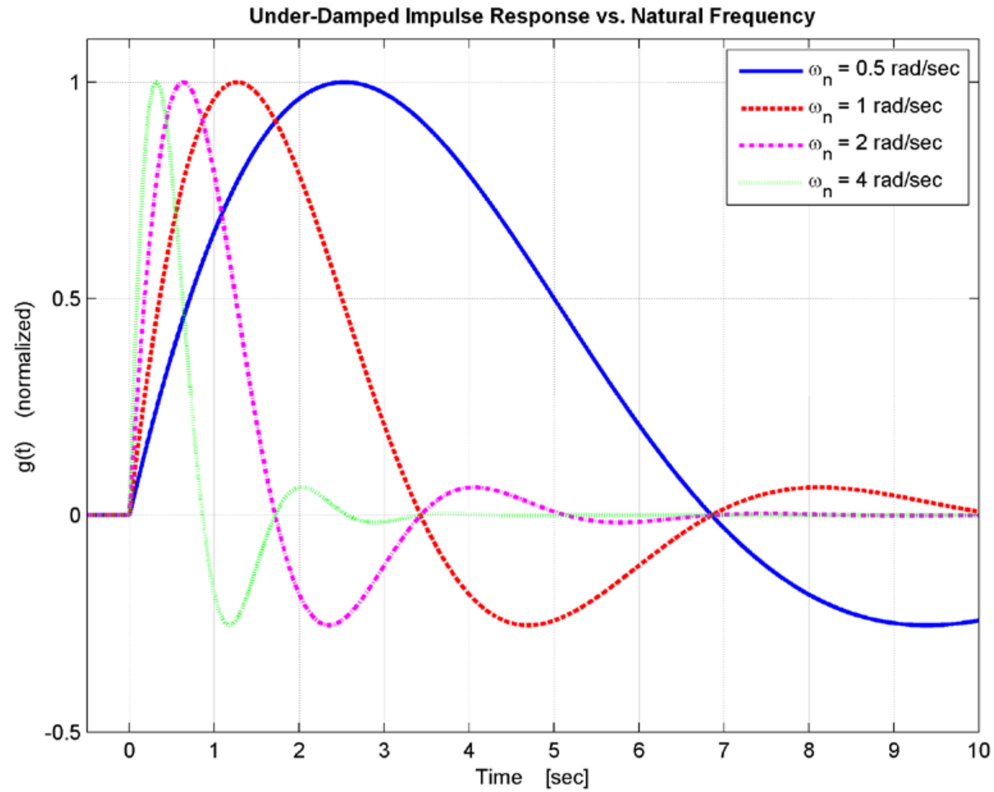
- Inverse Laplace transform for the time-domain impulse response

$$g(t) = \frac{A}{\omega_d} e^{-\sigma t}\sin(\omega_d t)$$

Under-Damped Impulse Response vs. ω_n

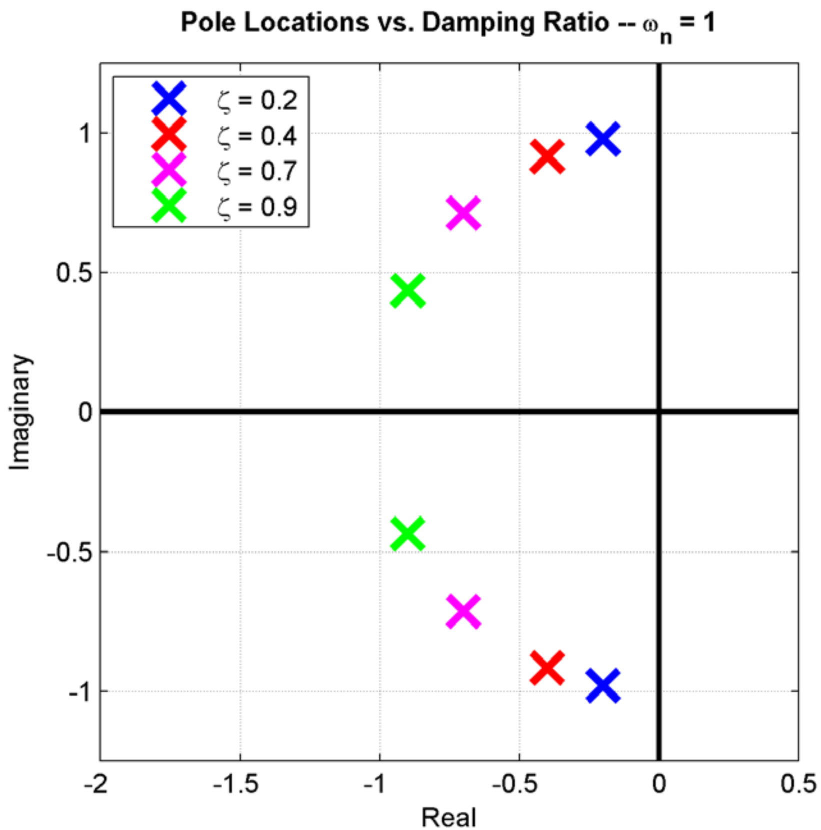
$$g(t) = \frac{A}{\omega_d} e^{-\sigma t} \sin(\omega_d t) = B e^{-\zeta \omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t\right)$$

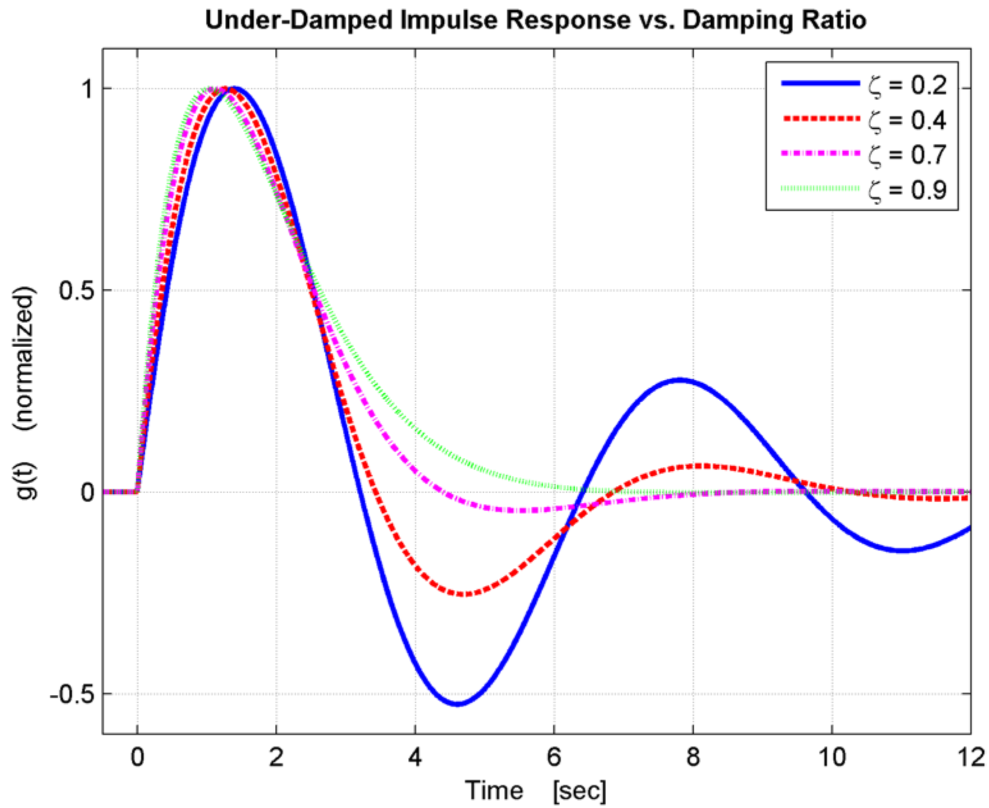




Under-Damped Impulse Response vs. ζ

$$g(t) = \frac{A}{\omega_d} e^{-\sigma t} \sin(\omega_d t) = B e^{-\zeta \omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t\right)$$





Impulse Response – Un-Damped

- For $\zeta = 0$, the transfer function reduces to

$$G(s) = \frac{A}{s^2 + \omega_n^2}$$

- Putting into the form of a sinusoid

$$G(s) = \frac{A}{\omega_n} \frac{\omega_n}{s^2 + \omega_n^2}$$

- Inverse transform to get the time-domain impulse response

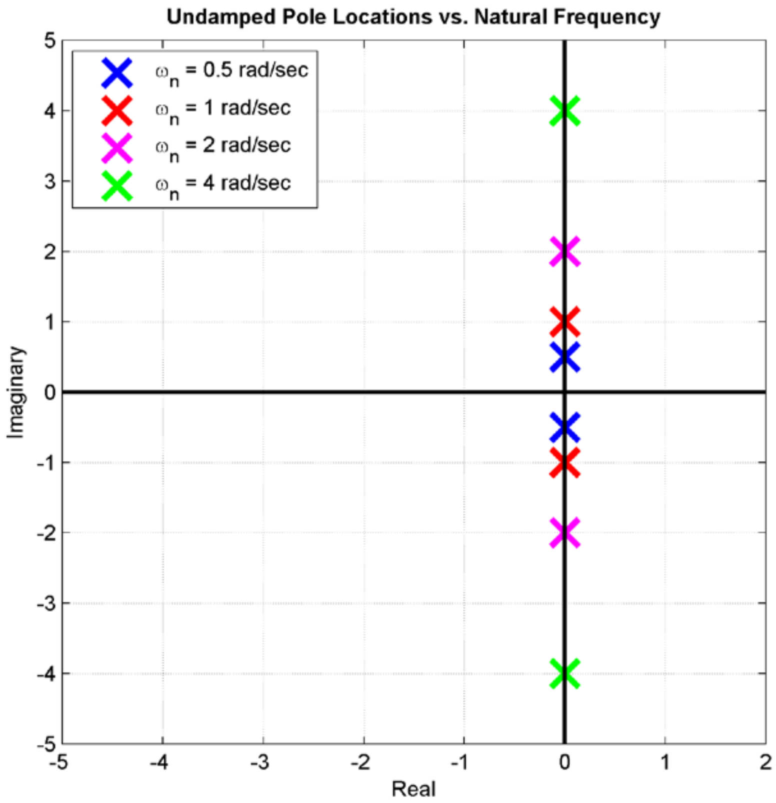
$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

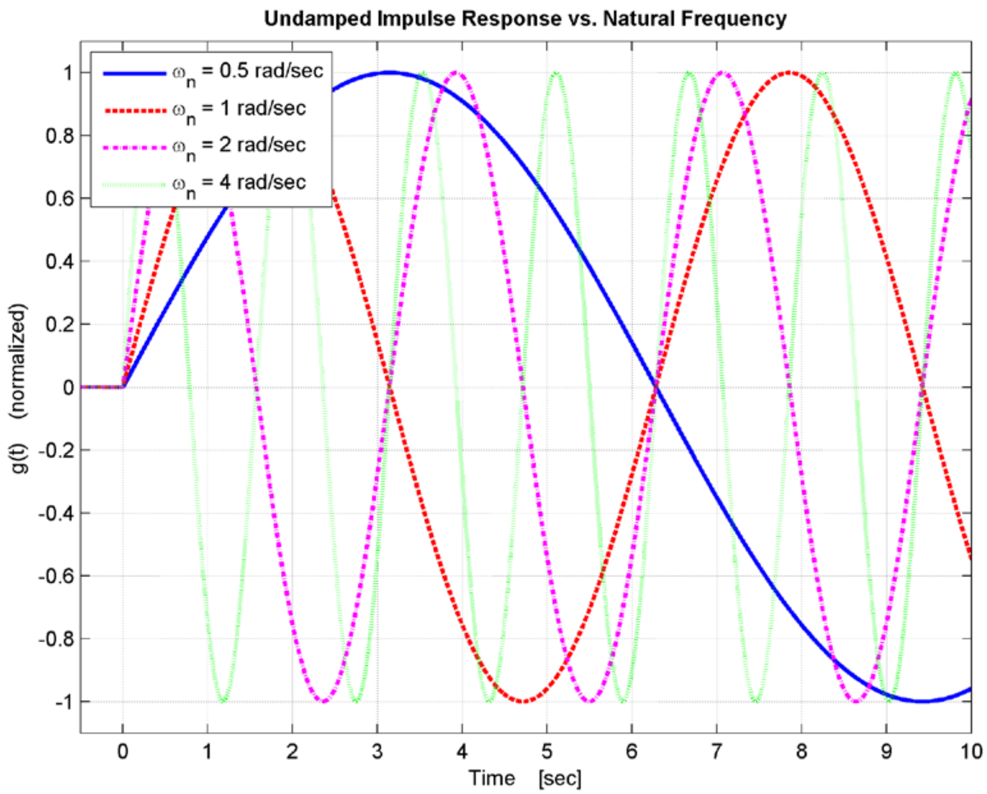
- An un-damped sinusoid

$$g(t) = \frac{A}{\omega_n} \sin(\omega_n t)$$

Un-Damped Impulse Response vs. ω_n

$$g(t) = \frac{A}{\omega_n} \sin(\omega_n t)$$





Second-Order Step Response

- The Laplace transform of the step response is

$$Y(s) = \frac{1}{s} G(s)$$

- The time-domain step response for each damping case can be derived as the the inverse transform of $Y(s)$

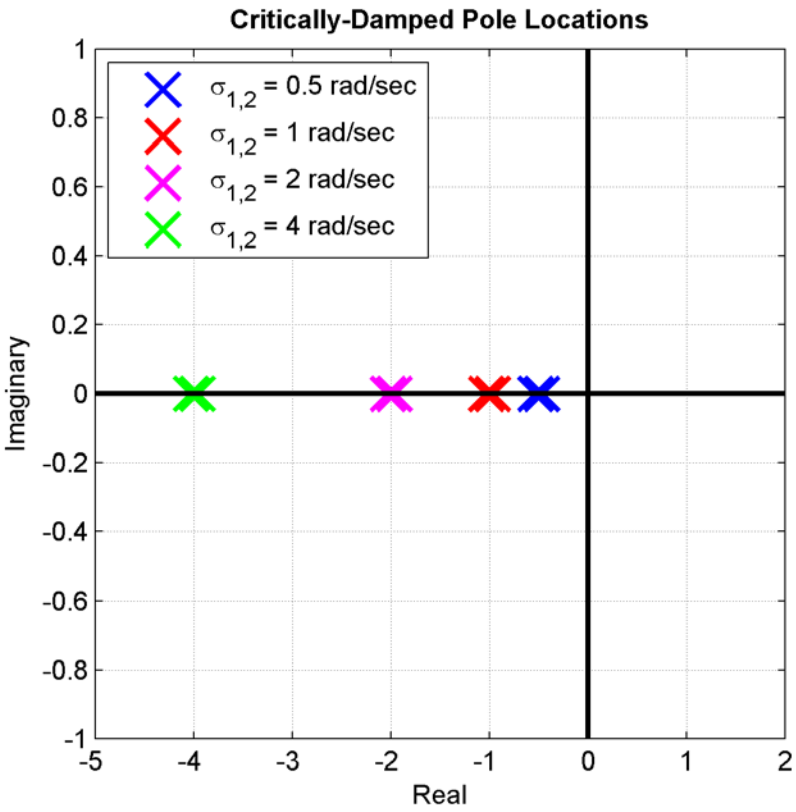
$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

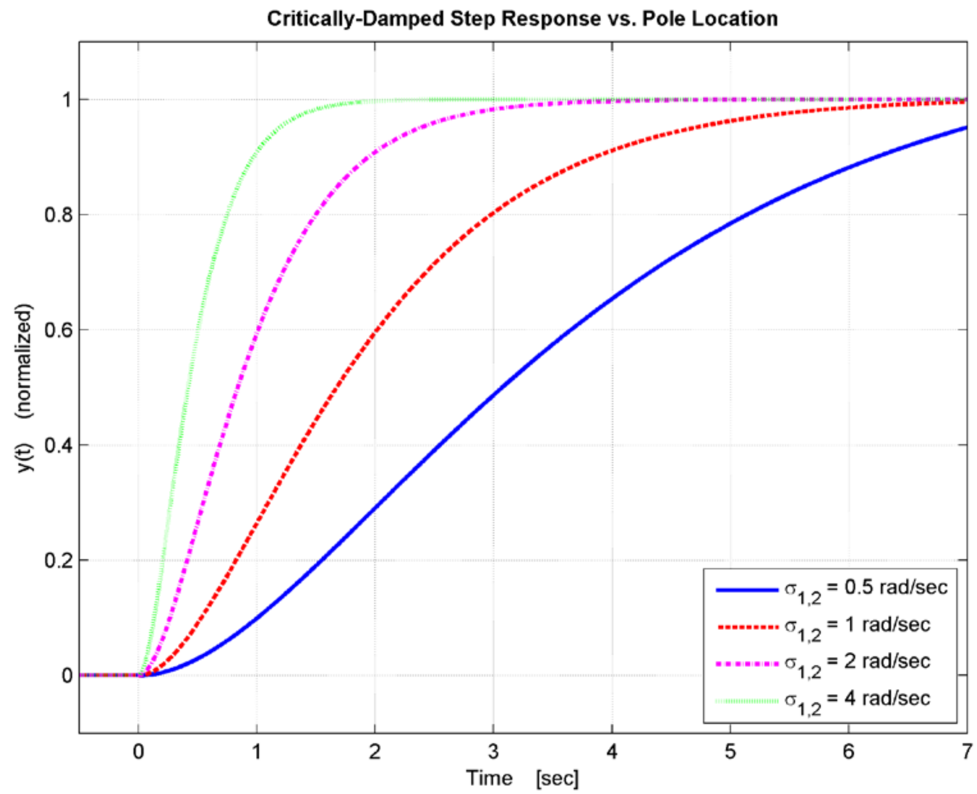
or as the integral of the corresponding impulse response

$$y(t) = \int_0^t g(\tau) d\tau$$

Critically-Damped Step Response vs. σ

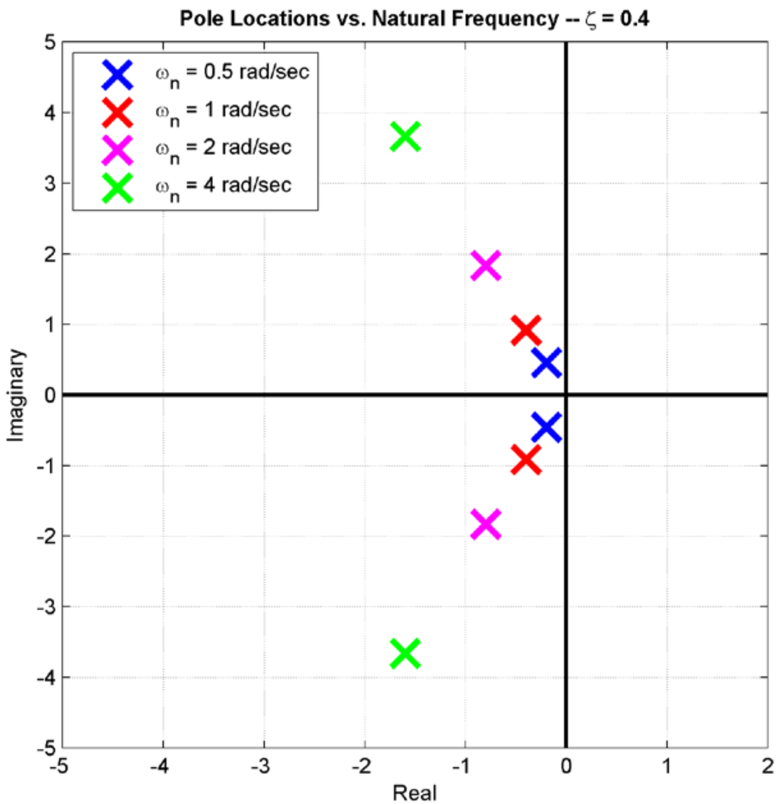
$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \frac{A}{\sigma^2}\left(1 - e^{-\sigma t} - \sigma t e^{-\sigma t}\right)$$

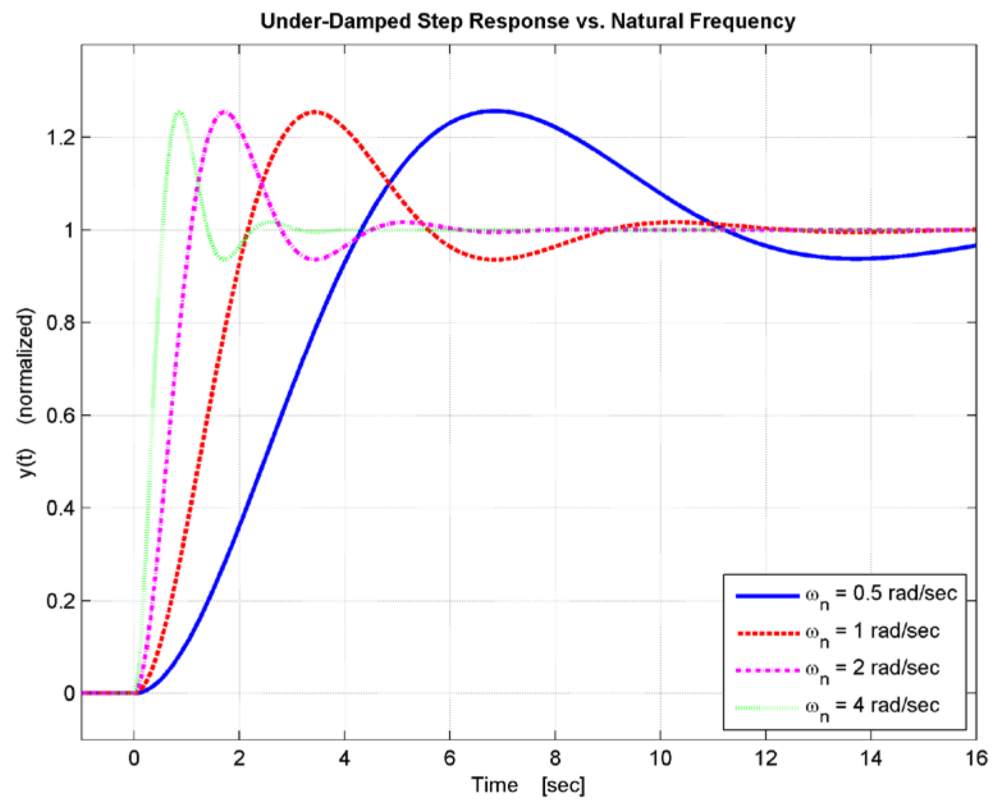




Under-Damped Step Response vs. ω_n

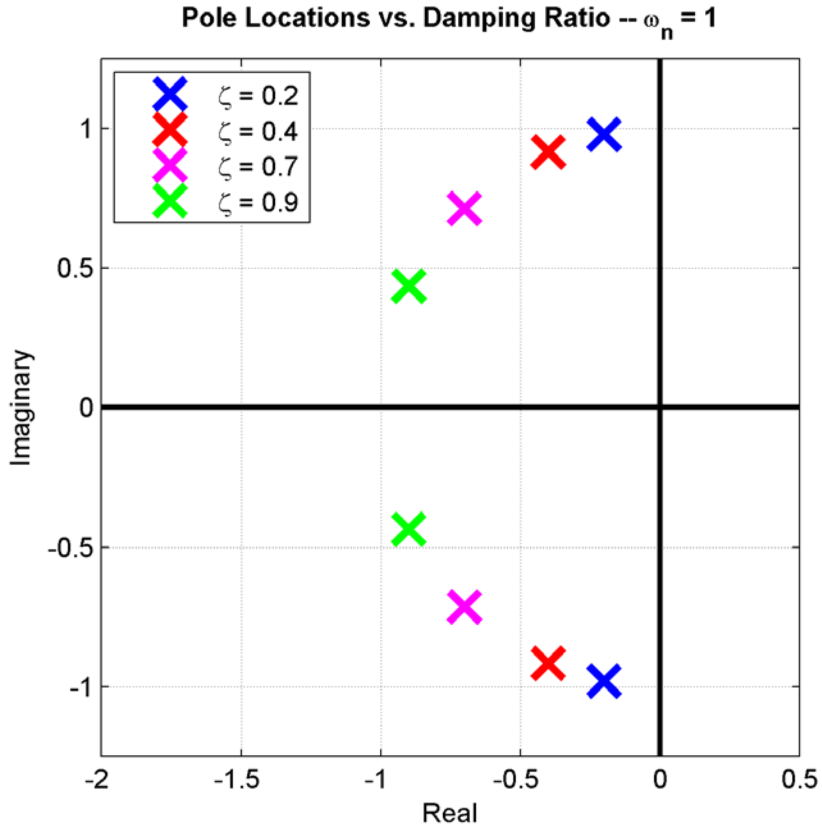
$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \frac{A}{\omega_n^2}\left[1 - e^{-\sigma t}\cos(\omega_d t) - \frac{\sigma}{\omega_d}e^{-\sigma t}\sin(\omega_d t)\right]$$

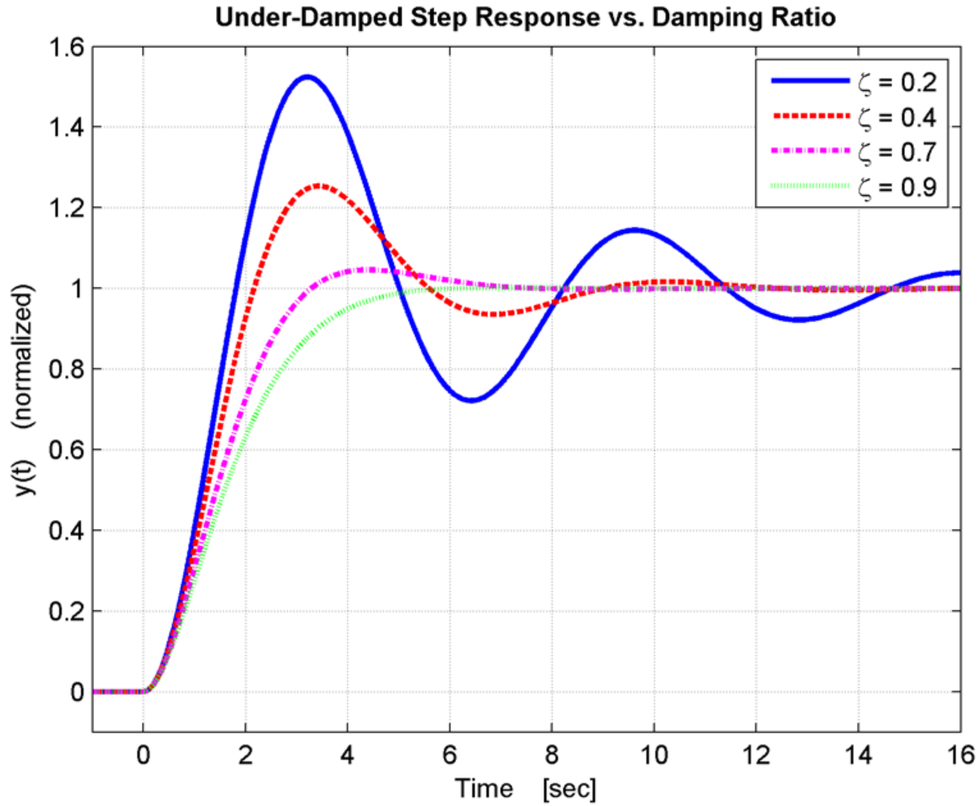




Under-Damped Step Response vs. ζ

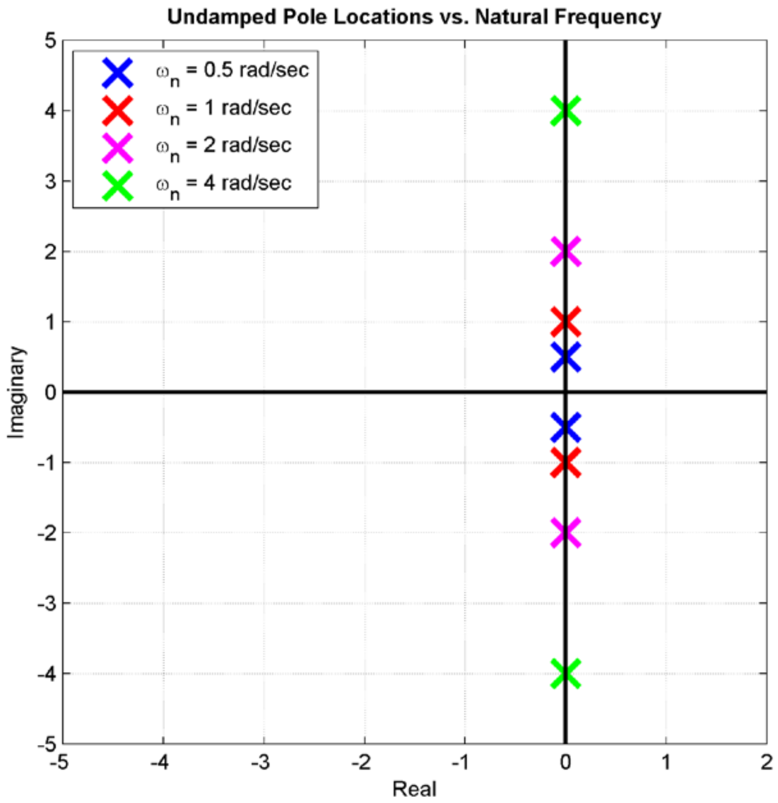
$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \frac{A}{\omega_n^2}\left[1 - e^{-\sigma t}\cos(\omega_d t) - \frac{\sigma}{\omega_d}e^{-\sigma t}\sin(\omega_d t)\right]$$

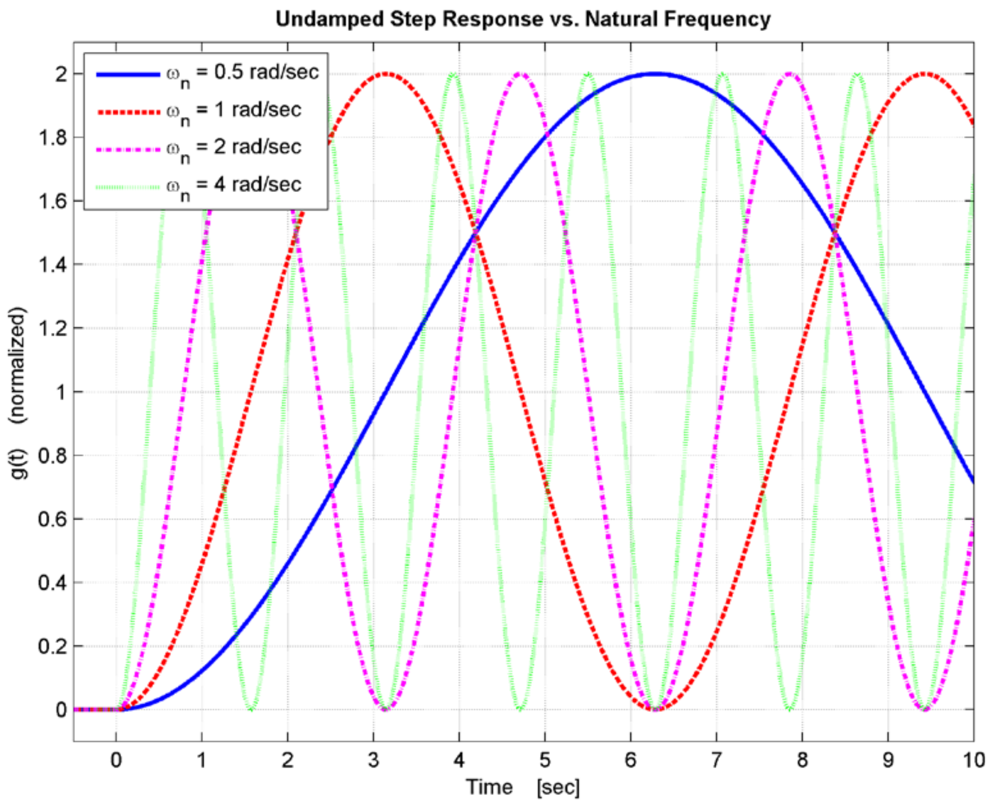




Un-Damped Step Response vs. ω_n

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \frac{A}{\omega_n^2}\left[1 - \cos(\omega_n t)\right]$$

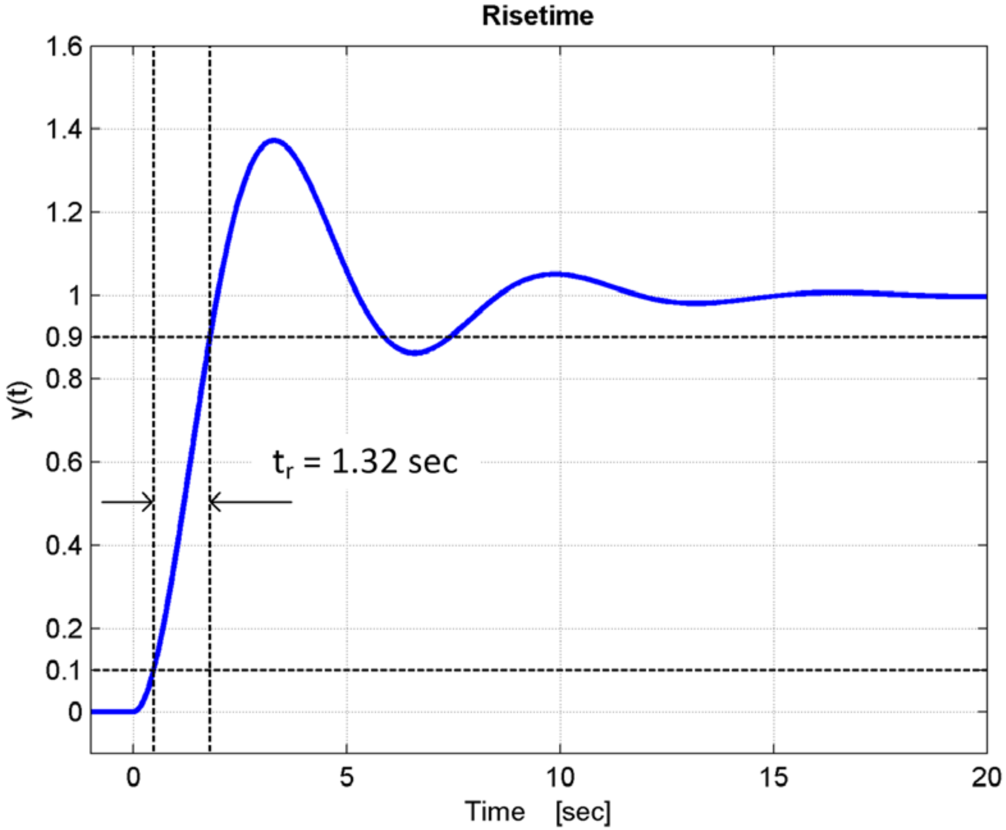




Step Response Characteristics

Step Response – Risetime

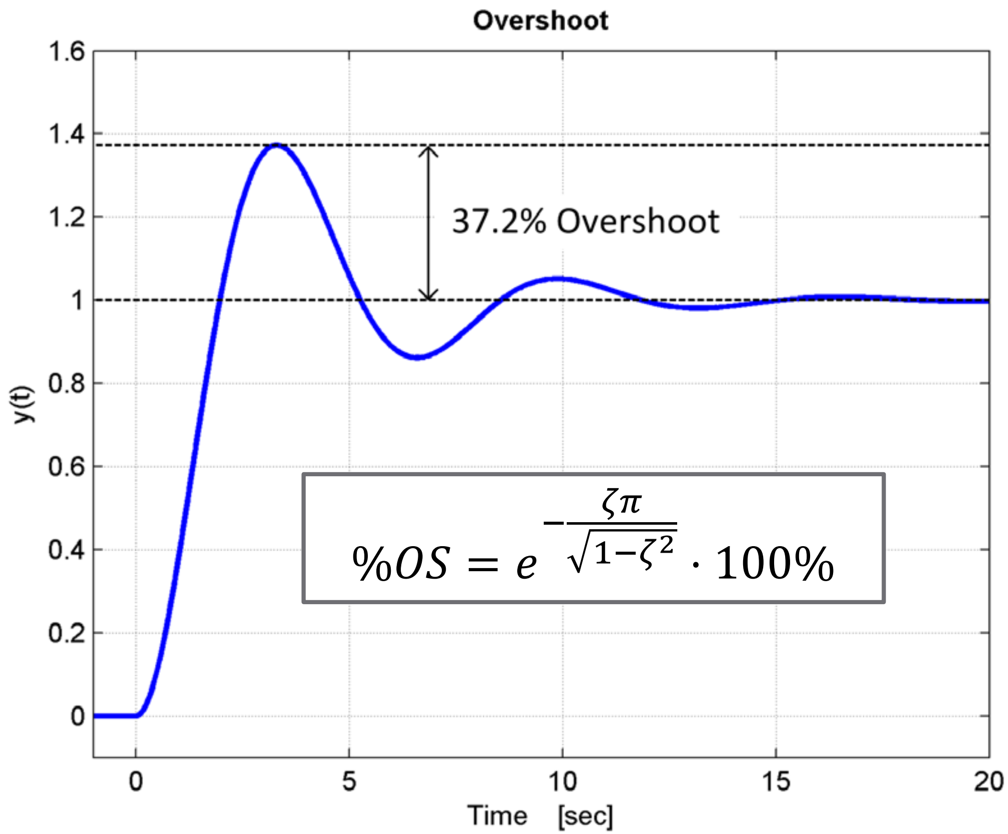
- ***Risetime*** is the time it takes a signal to transition between two set levels
  - Typically 10% to 90% of full transition
  - Sometimes 20% to 80%
- A measure of the speed of a response
- Very rough approximation:
  - $t_r \approx \frac{1.8}{\omega_n}$



Step Response – Overshoot

- ***Overshoot*** is the magnitude of a signal's excursion beyond its final value
  - Expressed as a percentage of full-scale swing
- Overshoot increases as ζ decreases

ζ	%OS
0.45	20
0.5	16
0.6	10
0.7	5

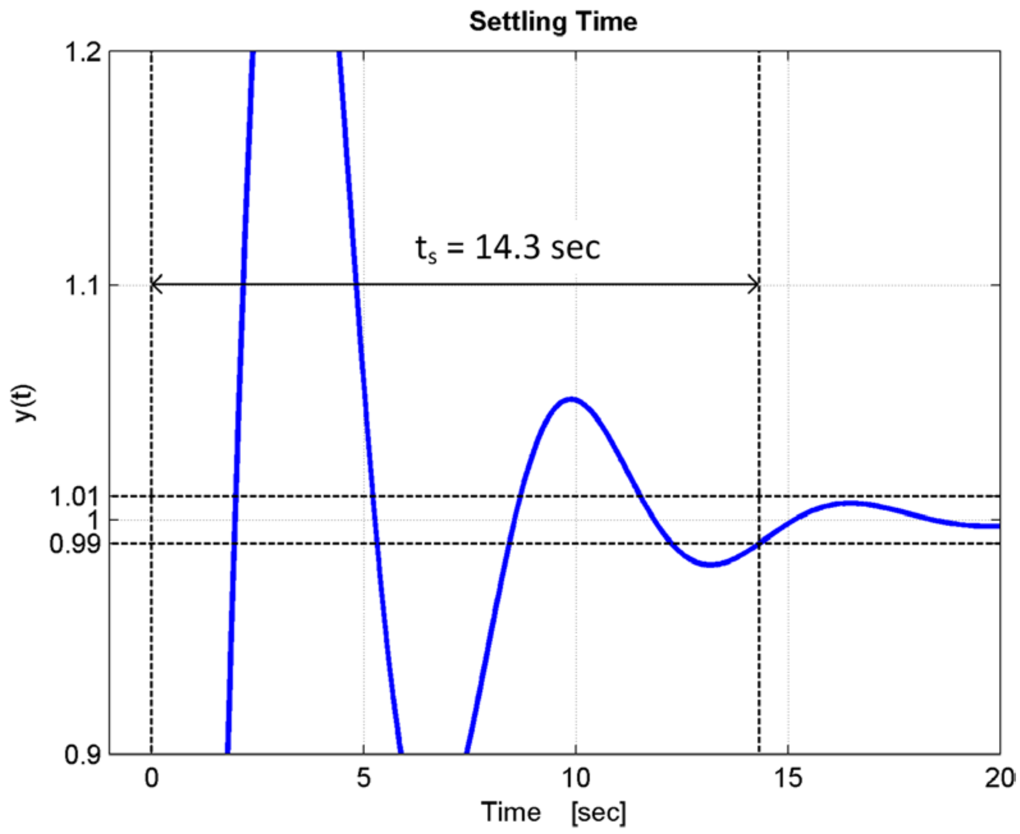


$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \cdot 100\%$$

$$\zeta = \frac{-\ln\left(\%OS/100\right)}{\sqrt{\pi^2 + \ln^2\left(\%OS/100\right)}}$$

Step Response –Settling Time

- ***Settling time*** is the time it takes a signal to settle, finally, to within some percentage of its final value
  - Typically $\pm 1\%$ or $\pm 5\%$
- Inversely proportional to the real part of the poles, σ
- For $\pm 1\%$ settling:
  - $t_s \approx \frac{4.6}{\sigma} = \frac{4.6}{\zeta \omega_n}$



The Convolution Integral

In this sub-section, we'll see that the time-domain output of a system is given by the convolution of its time-domain input and its impulse response.

Convolution Integral

- Laplace transform of a system output is given by the product of the transform of the input signal and the transfer function

$$Y(s) = G(s) \cdot U(s)$$

- Recall that ***multiplication in the Laplace domain corresponds to convolution in the time domain***

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = g(t) * u(t)$$

- ***Time-domain output given by the convolution of the input signal and the impulse response***

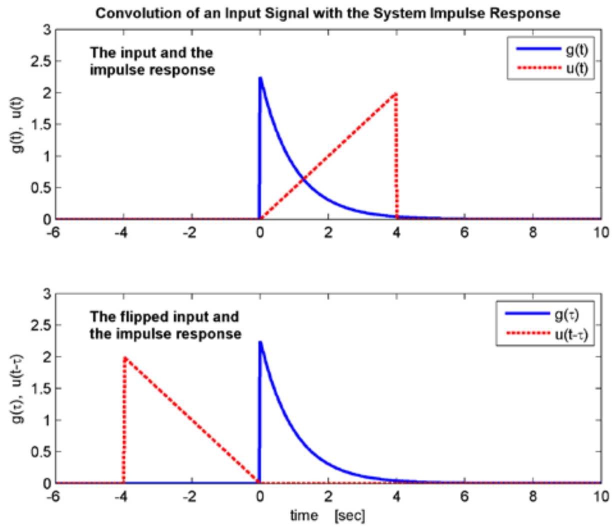
$$y(t) = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

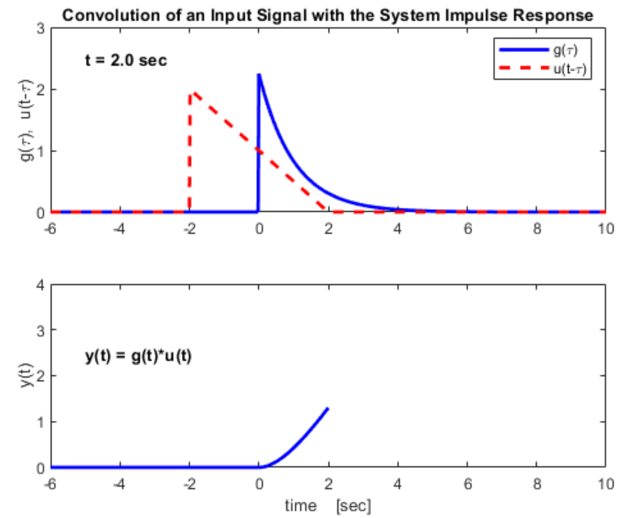
Convolution

- Time-domain output is the input ***convolved*** with the impulse response

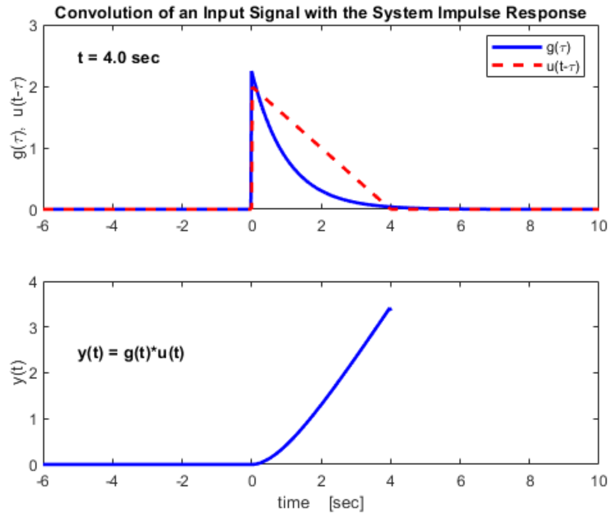
$$y(t) = g(t) * u(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

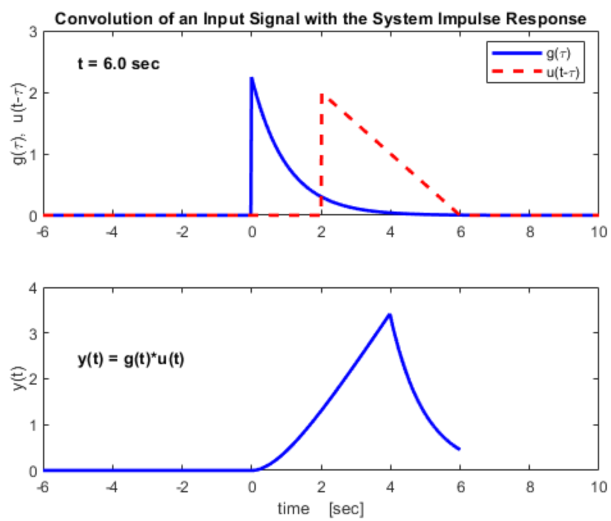
  - Input is flipped in time and shifted by t
  - Multiply impulse response and flipped/shifted input
  - Integrate over $\tau = 0 \ldots t$
- Each time point of $y(t)$ given by result of integral with $u(-\tau)$ shifted by t

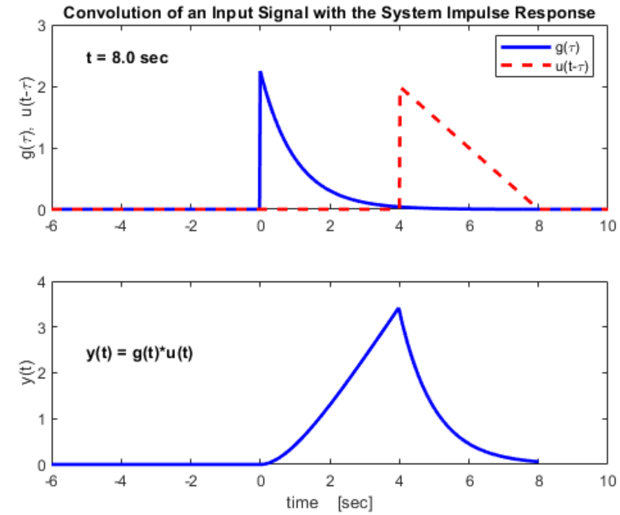


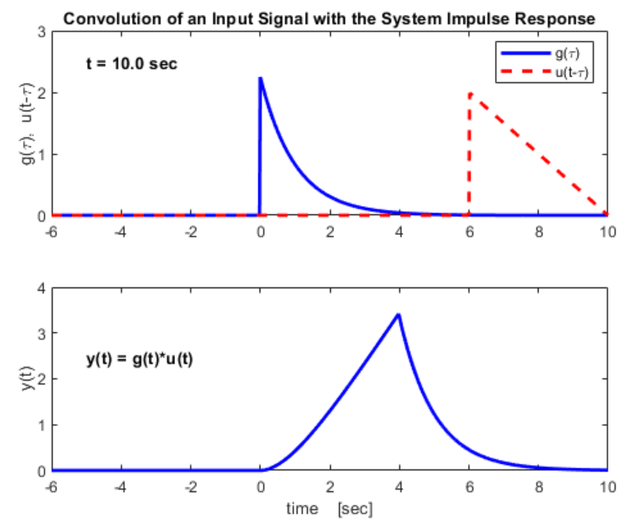


Convolution









Convolution

Time-Domain Analysis in MATLAB

A few of MATLAB's many built-in functions that are useful for simulating linear systems are listed in the following sub-section.

System Objects

- MATLAB has data types dedicated to linear system models
- Two primary system model objects:
  - ***State-space model***
  - ***Transfer function model***
- Objects created by calling MATLAB functions
  - `ss.m` – creates a state-space model
  - `tf.m` – creates a transfer function model

State-Space Model – `ss(…)`


```
sys = ss(A,B,C,D)
```

- A: system matrix - $n \times n$
- B: input matrix - $n \times m$
- C: output matrix - $p \times n$
- D: feed-through matrix - $p \times m$
- `sys`: state-space model object

- State-space model object will be used as an input to other MATLAB functions

Transfer Function Model – `tf(…)`


```
sys = tf(Num,Den)
```

- `Num`: vector of numerator polynomial coefficients
- `Den`: vector of denominator polynomial coefficients
- `sys`: transfer function model object

- Transfer function is assumed to be of the form

$$G(s) = \frac{b_1 s^r + b_2 s^{r-1} + \cdots + b_r s + b_{r+1}}{a_1 s^n + a_2 s^{n-1} + \cdots + a_n s + a_{n+1}}$$

- Inputs to `tf(…)` are
  - `Num = [b1,b2,…,br+1];`
  - `Den = [a1,a2,…,an+1];`

Step Response Simulation – `step(…)`


```
[y,t] = step(sys,t)
```

- `sys`: system model – state-space or transfer function
- `t`: *optional* time vector *or* final time value
- `y`: output step response
- `t`: output time vector

- If no outputs are specified, step response is automatically plotted
- Time vector (or final value) input is optional
  - If not specified, MATLAB will generate automatically

Impulse Response Simulation – `impulse(…)`


```
[y,t] = impulse(sys,t)
```

- `sys`: system model – state-space or transfer function
- `t`: *optional* time vector *or* final time value
- `y`: output impulse response
- `t`: output time vector

- If no outputs are specified, impulse response is automatically plotted
- Time vector (or final value) input is optional
  - If not specified, MATLAB will generate automatically

Natural Response – `initial(…)`


```
[y,t,x] = initial(sys,x0,t)
```

- `sys`: ***state-space*** system model function
- `x0`: initial value of the state - $n \times 1$ vector
- `t`: *optional* time vector *or* final time value
- `y`: response to initial conditions - `length(t)`$\times$ 1 vector
- `t`: output time vector
- `x`: trajectory of all states - `length(t)`$\times n$ matrix

- If no outputs are specified, response to initial conditions is automatically plotted
- Time vector (or final value) input is optional
  - If not specified, MATLAB will generate automatically

Linear System Simulation – `lsim(…)`


```
[y,t,x] = lsim(sys,u,t,x0)
```

- `sys`: system model – state-space or transfer function
- `u`: input signal vector
- `t`: time vector corresponding to the input signal
- `x0`: *optional* initial conditions – (*for ss model only*)
- `y`: output response
- `t`: output time vector
- `x`: *optional* trajectory of all states – (*for ss model only*)

- If no outputs are specified, response is automatically plotted
- Input can be any arbitrary signal

More MATLAB Functions

- A few more useful MATLAB functions
  - Pole/zero analysis:
    - `pzmap(…)`
    - `pole(…)`
    - `zero(…)`
    - `eig(…)`
  - Input signal generation:
    - `gensig(…)`
- Refer to MATLAB help documentation for more information