SECTION 7: FREQUENCY-DOMAIN ANALYSIS

ESE 330 – Modeling & Analysis of Dynamic Systems

Response to Sinusoidal Inputs

Frequency-Domain Analysis – Introduction

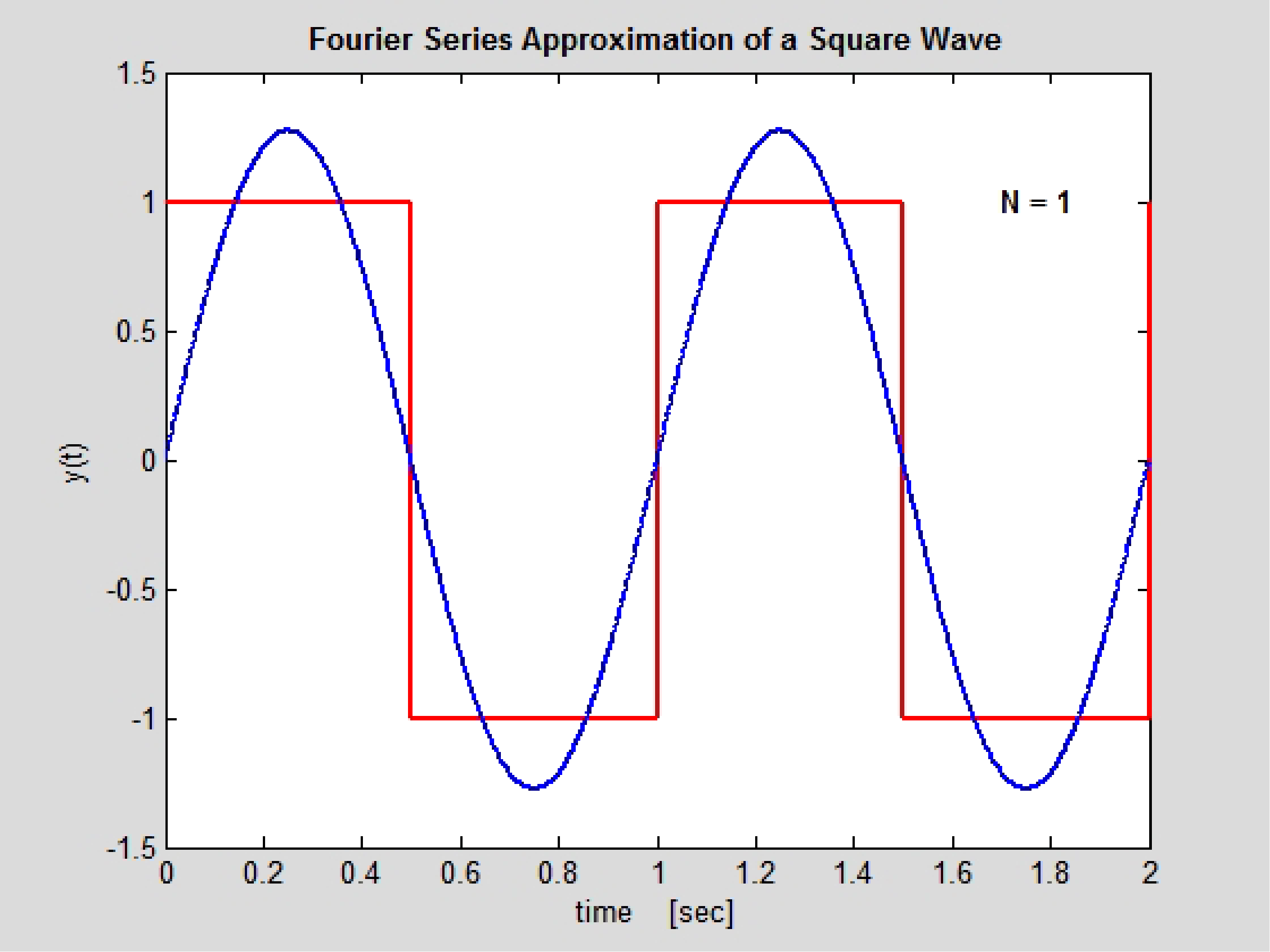
- We've looked at system impulse and step responses
  - Also interested in the response to ***periodic inputs***
- ***Fourier theory*** tells us that any periodic signal can be represented as a sum of harmonically-related sinusoids
- The ***Fourier series***:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n f t) + b_n \sin(2\pi n f t)]$$

where a_n and b_n are given by the Fourier integrals

- Sinusoids are basis signals from which all other periodic signals can be constructed
  - ***Sinusoidal system response*** is of particular interest

Fourier Series



System Response to a Sinusoidal Input

- Consider an n^{th}-order system
  - n poles: $p_1, p_2, \ldots p_n$
    - Real or complex
    - Assume all are distinct
  - Transfer function is:

$$G(s) = \frac{Num(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{1}$$

- Apply a sinusoidal input to the system

$$u(t) = A\sin(\omega t) \quad \xrightarrow{\mathcal{L}} \quad U(s) = A\frac{\omega}{s^2+\omega^2}$$

- Output is given by

$$Y(s) = G(s)U(s) = \frac{Num(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \cdot A\frac{\omega}{s^2+\omega^2} \tag{2}$$

System Response to a Sinusoidal Input

- Partial fraction expansion of (2) gives

$$Y(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} + \frac{r_{n+1}s}{s^2+\omega^2} + \frac{r_{n+2}\omega}{s^2+\omega^2} \tag{3}$$

- Inverse transform of (3) gives the time-domain output

$$y(t) = \underbrace{r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}}_{\textit{transient}} + \underbrace{r_{n+1}\cos(\omega t) + r_{n+2}\sin(\omega t)}_{\textit{steady state}} \tag{4}$$

- Two portions of the response:
  - Transient
    - Decaying exponentials or sinusoids – goes to zero in steady state
    - Natural response to initial conditions
  - Steady state
    - Due to the input – sinusoidal in steady state

Steady-State Sinusoidal Response

- We are interested in the ***steady-state response***

$$y_{ss}(t) = r_{n+1}\cos(\omega t) + r_{n+2}\sin(\omega t) \tag{5}$$

- A trig. identity provides insight into $y_{ss}(t)$:

$$\alpha\cos(\omega t) + \beta\sin(\omega t) = \sqrt{\alpha^2 + \beta^2}\sin(\omega t + \phi)$$

where

$$\phi = \tan^{-1}\left(\frac{\alpha}{\beta}\right)$$

- Steady-state response to a sinusoidal input

$$u(t) = A\sin(\omega t)$$

is a sinusoid of the same frequency, but, in general different amplitude and phase

$$y_{ss}(t) = B\sin(\omega t + \phi)$$

Where (6)

$$B = \sqrt{r_{n+1}^2 + r_{n+2}^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{r_{n+1}}{r_{n+2}}\right)$$

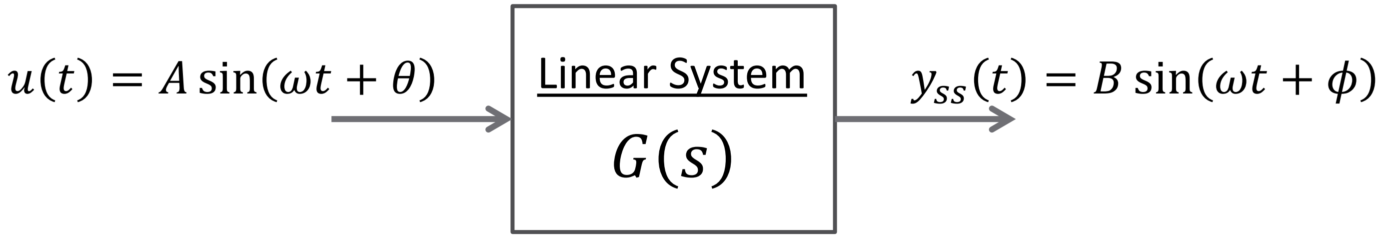
Steady-State Sinusoidal Response

$$u(t) = A\sin(\omega t) \quad \rightarrow \quad y_{ss}(t) = B\sin(\omega t + \phi)$$

- Steady-state sinusoidal response is a ***scaled*** and ***phase-shifted*** sinusoid of the same frequency
  - Equal frequency is a property of linear systems
- Note the ω term in the numerator of (3)
  - ω will affect the residues
  - Residues determine amplitude and phase of the output
  - ***Output amplitude and phase are frequency-dependent***

$$y_{ss}(t) = B(\omega)\sin\big(\omega t + \phi(\omega)\big)$$

Steady-State Sinusoidal Response



- ***Gain*** – the ratio of amplitudes of the output and input of the system

$$Gain = \frac{B}{A}$$

- ***Phase*** – phase difference between system input and output

$$Phase = \phi - \theta$$

- Systems will, in general, exhibit ***frequency-dependent*** gain and phase
- We'd like to be able to determine these functions of frequency
  - The system's ***frequency response***

Frequency Response

A system's frequency response, or sinusoidal transfer function, describes its gain and phase shift for sinusoidal inputs as a function of frequency.

Frequency Response

- System output in the Laplace domain is

$$Y(s) = U(s) \cdot G(s)$$

- Multiplication in the Laplace domain corresponds to ***convolution*** in the time domain

$$y(t) = u(t) * g(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

- Consider an exponential input of the form

$$u(t) = e^{st}$$

where s is the complex Laplace variable: $s = \sigma + j\omega$

- Now the output is

$$y(t) = u(t) * g(t) = \int_0^t g(\tau)e^{s(t-\tau)}d\tau = \int_0^t g(\tau)e^{st}e^{-s\tau}d\tau$$

$$y(t) = \int_0^t g(\tau)e^{-s\tau}d\tau \cdot e^{st} \qquad (1)$$

Frequency Response

$$y(t) = \int_0^t g(\tau)e^{-s\tau}d\tau \cdot e^{st} \qquad (1)$$

- We're interested in the steady-state response, so let the upper limit of integration go to infinity

$$y(t) = \int_0^\infty g(\tau)e^{-s\tau}d\tau \cdot e^{st}$$

$$y(t) = G(s) \cdot e^{st} \qquad (2)$$

- Time-domain response to an exponential input is the time-domain input multiplied by the system transfer function
- What is this input?

$$u(t) = e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} \qquad (3)$$

- If we let $\sigma \rightarrow 0$, i.e. let $s \rightarrow j\omega$, then we have

$$y(t) = G(j\omega) \cdot e^{j\omega t} \qquad (4)$$

Euler's Formula

- Recall ***Euler's formula***:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \tag{5}$$

- From which it follows that

$$\cos(\omega t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2} \tag{6}$$

and

$$\sin(\omega t) = \frac{e^{j\omega t}-e^{-j\omega t}}{2j} \tag{7}$$

Frequency Response

- We're interested in the sinusoidal steady-state system response, so let the input be

$$u(t) = A\cos(\omega t) = A\frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

- A sum of complex exponentials in the form of (3)
  - We've let $s \rightarrow j\omega$ in the first term and $s \rightarrow -j\omega$ in the second

$$u(t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} \tag{8}$$

- According to (4) the output in response to (8) will be

$$y(t) = \frac{A}{2}G(j\omega)\cdot e^{j\omega t} + \frac{A}{2}G(-j\omega)\cdot e^{-j\omega t} \tag{9}$$

Frequency Response

$$y(t) = \frac{A}{2} G(j\omega) \cdot e^{j\omega t} + \frac{A}{2} G(-j\omega) \cdot e^{-j\omega t} \tag{9}$$

- $G(j\omega)$ is a complex function of frequency
  - Evaluates to a complex number at each value of ω
  - Has both ***magnitude*** and ***phase***
  - Can be expressed in ***polar form*** as

$$G(j\omega) = Me^{j\phi} \tag{10}$$

where

$$M = |G(j\omega)| \text{ and } \phi = \angle G(j\omega)$$

- It follows that

$$G(-j\omega) = Me^{-j\phi} \tag{11}$$

Frequency Response

- Using (11), the output given by (9) becomes

$$y(t) = \frac{A}{2} M\left[e^{j\omega t} e^{j\phi} + e^{-j\omega t} e^{-j\phi}\right]$$

$$y(t) = \frac{A}{2} M\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right] \quad (12)$$

$$y(t) = M \cdot A\cos(\omega t + \phi) \quad (13)$$

where, again

$$M = |G(j\omega)| \text{ and } \phi = \angle G(j\omega) \quad (14)$$

Frequency response Function – $G(j\omega)$

- $G(j\omega)$ is the system's ***frequency response function***
  - Transfer function, where $s \rightarrow j\omega$

$$G(j\omega) = G(s)|_{s\rightarrow j\omega} \tag{15}$$

  - A complex-valued function of frequency
- $|G(j\omega)|$ at each ω is the ***gain*** at that frequency
  - Ratio of output amplitude to input amplitude
- $\angle G(j\omega)$ at each ω is the ***phase*** at that frequency
  - Phase shift between input and output sinusoids
- Another representation of system behavior
  - Along with state-space model, impulse/step responses, transfer function, etc.
  - Typically represented graphically

Plotting the Frequency Response Function

- $G(j\omega)$ is a complex-valued function of frequency
  - Has both magnitude and phase
  - Plot gain and phase separately
- Frequency response plots formatted as ***Bode plots***
  - Two sets of axes: gain on top, phase below
  - Identical, logarithmic frequency axes
  - Gain axis is logarithmic – either explicitly or as units of decibels (dB)
  - Phase axis is linear with units of degrees

Bode Plots

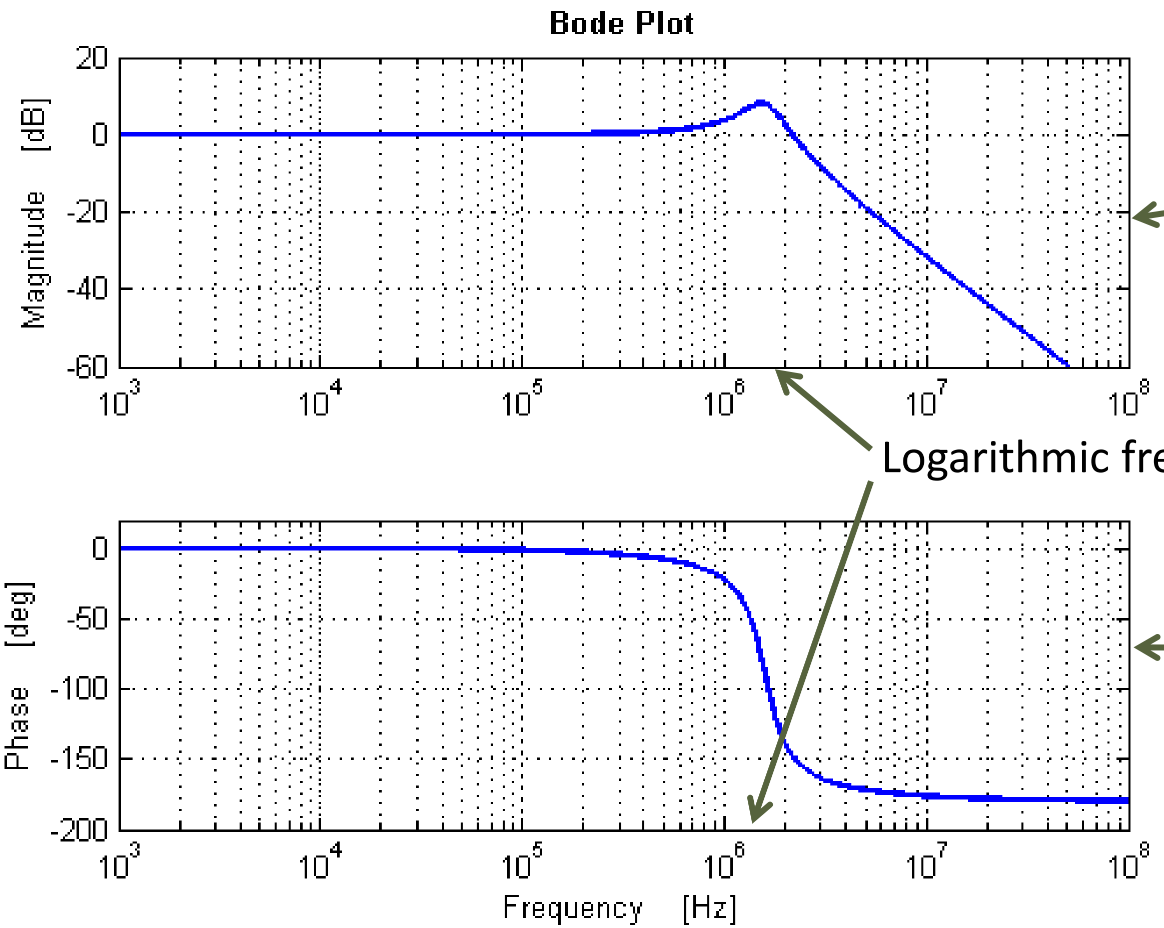
Units of magnitude are dB

Magnitude plot on top

Logarithmic frequency axes

Units of phase are degrees

Phase plot below



Decibels - dB

- Frequency response gain most often expressed and plotted with units of decibels (dB)
  - A logarithmic scale
  - Provides detail of very large and very small values on the same plot
  - Commonly used for ratios of powers or amplitudes
- Conversion from a linear scale to dB:

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}(|G(j\omega)|)$$

- Conversion from dB to a linear scale:

$$|G(j\omega)| = 10^{\frac{|G(j\omega)|_{dB}}{20}}$$

Decibels – dB

- Multiplying two gain values corresponds to adding their values in dB
  - E.g., the overall gain of cascaded systems

$$|G_1(j\omega) \cdot G_2(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB}$$

- Negative dB values corresponds to sub-unity gain
- Positive dB values are gains greater than one

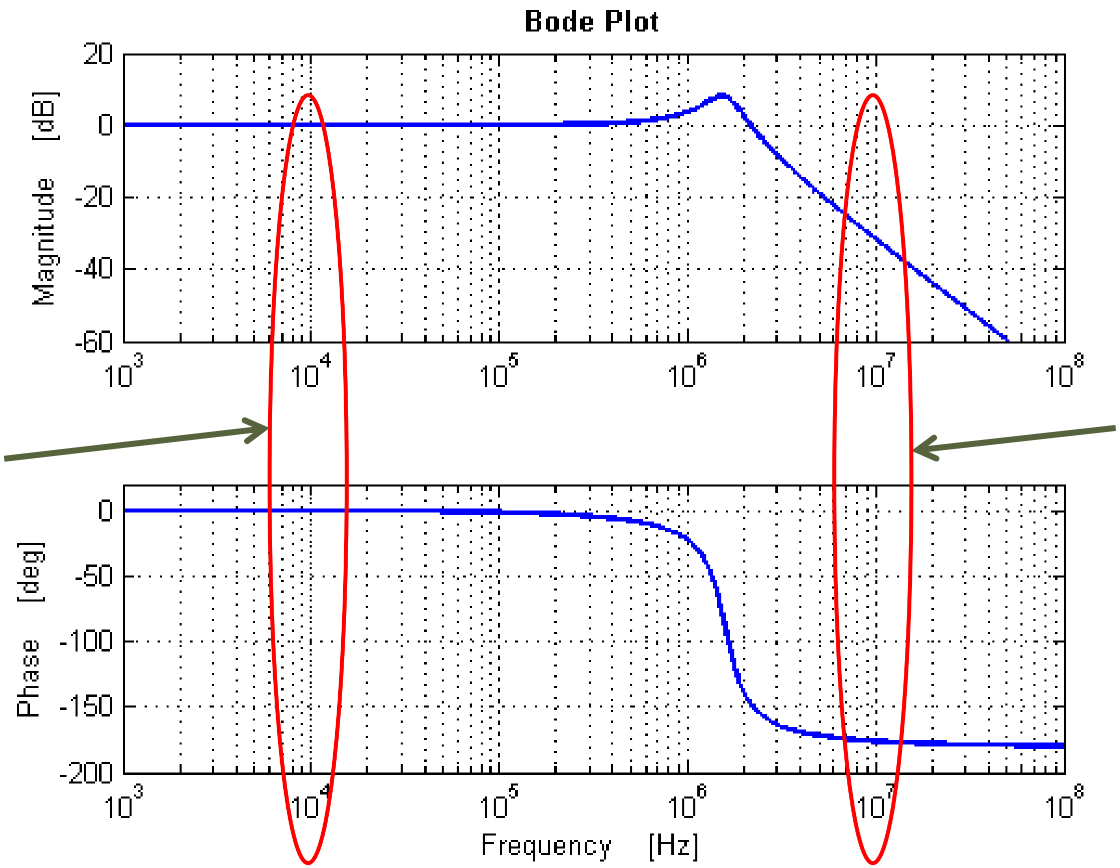
dB	Linear
60	1000
40	100
20	10
0	1

dB	Linear
6	2
-3	$1/\sqrt{2} = 0.707$
-6	0.5
-20	0.1

Interpreting Bode Plots

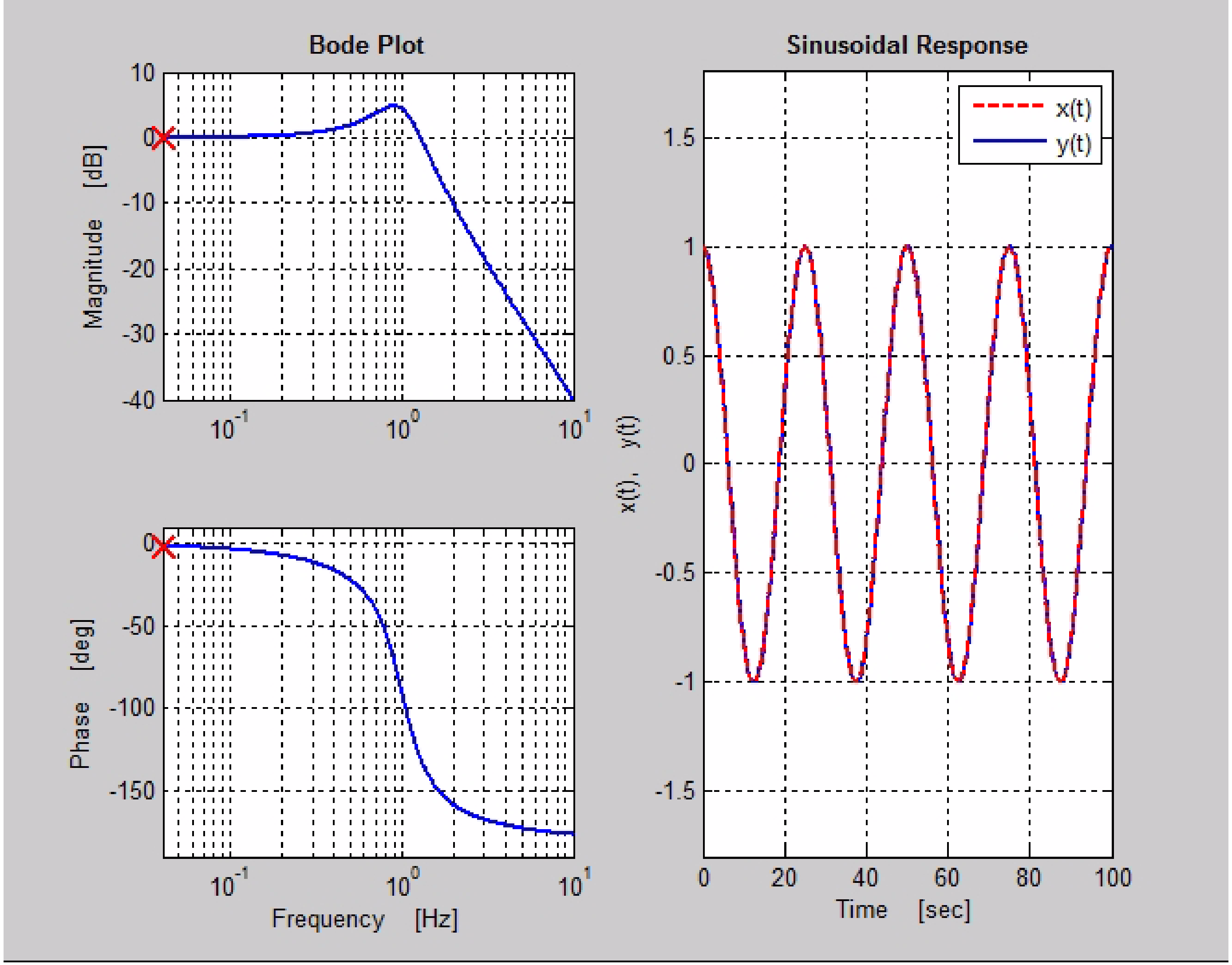
Bode plots tell you the gain and phase shift at all frequencies:
choose a frequency, read gain and phase values from the plot

For a 10KHz sinusoidal input, the gain is 0dB (1) and the phase shift is 0°.



For a 10MHz sinusoidal input, the gain is -32dB (0.025), and the phase shift is -176°.

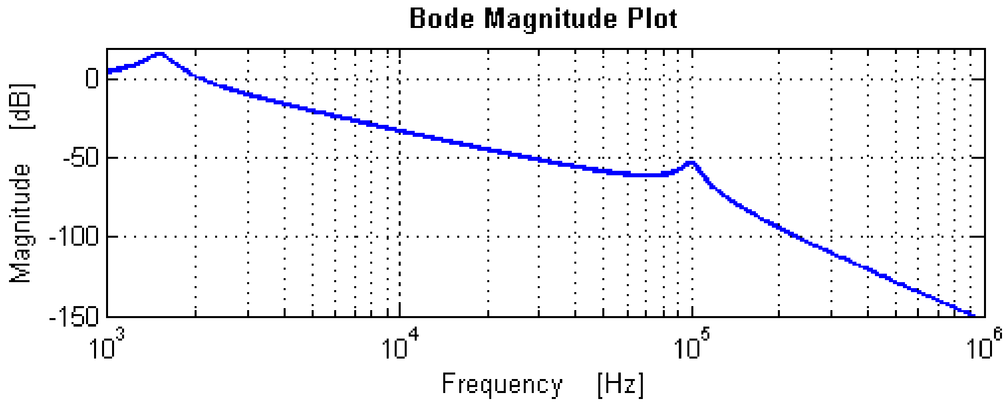
Interpreting Bode Plots



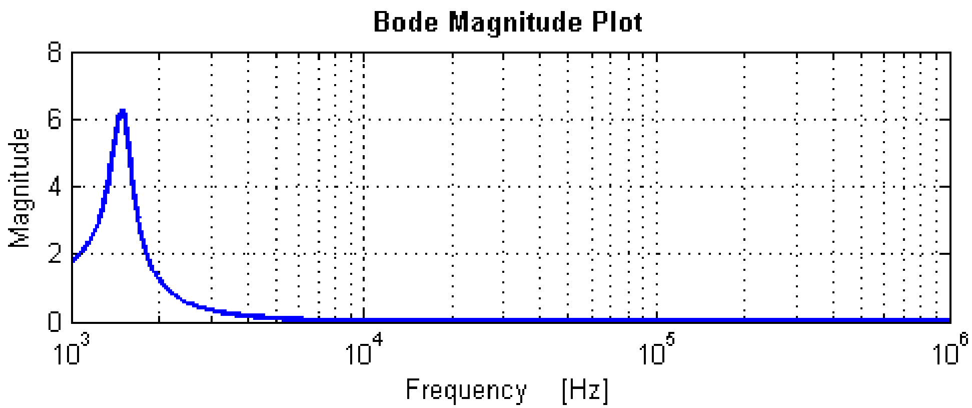
Value of Logarithmic Axes - Gain

- Gain axis is linear in dB
  - A logarithmic scale
  - Allows for displaying detail at very large and very small levels on the same plot

- Gain plotted in dB
  - Two resonant peaks clearly visible



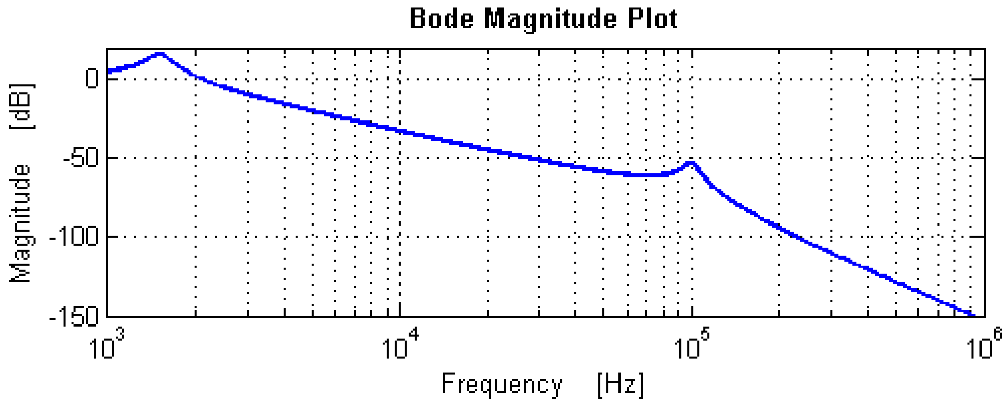
- Linear gain scale
  - Smaller peak has disappeared



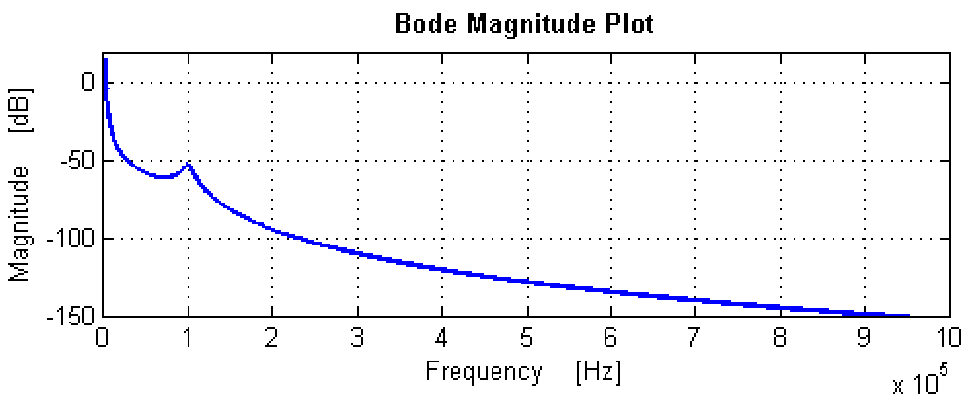
Value of Logarithmic Axes - Frequency

- Frequency axis is logarithmic
  - Allows for displaying detail at very low and very high frequencies on the same plot

- Log frequency axis
  - Can resolve frequency of both resonant peaks



- Linear frequency axis
  - Lower resonant frequency is unclear

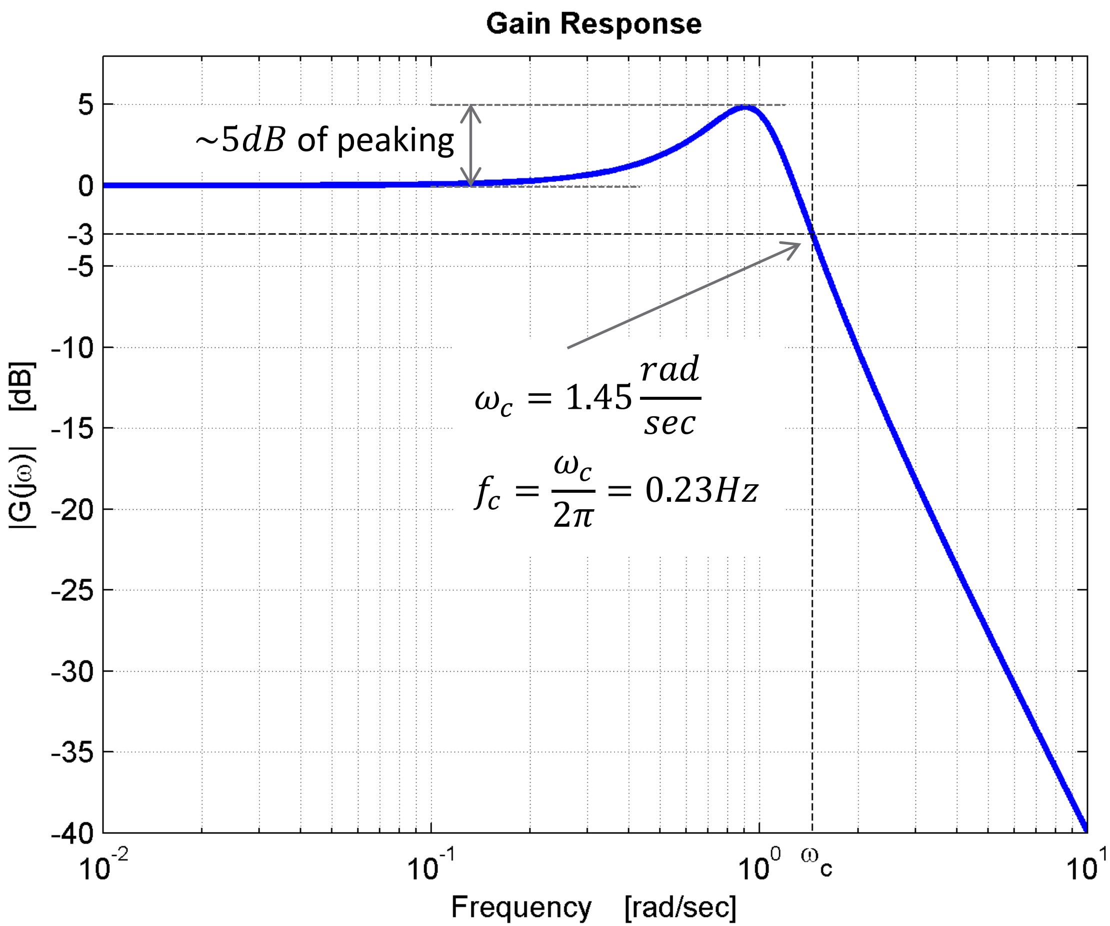


Gain Response – Terminology

- ***Corner frequency, cut off frequency, -3dB frequency***:
  - Frequency at which gain is 3dB below its low-frequency value

$$f_c = \frac{\omega_c}{2\pi}$$

  - This is the ***bandwidth*** of the system
- ***Peaking***
  - Any increase in gain above the low frequency gain



Response of 1st- and 2nd-Order Factors

This section examines the frequency responses of first- and second-order transfer function factors.

Transfer Function Factors

- We've already seen that a transfer function denominator can be factored into first- and second-order terms

$$G(s) = \frac{Num(s)}{(s-p_1)(s-p_2)\cdots(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)(s^2+2\zeta_2\omega_{n2}s+\omega_{n2}^2)\cdots}$$

- The same is true of the numerator

$$G(s) = \frac{(s-z_1)(s-z_2)\cdots(s^2+2\zeta_a\omega_{na}s+\omega_{na}^2)\left(s^2+2\zeta_2\omega_{nb}s+\omega_{nb}^2\right)\cdots}{(s-p_1)(s-p_2)\cdots(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)(s^2+2\zeta_2\omega_{n2}s+\omega_{n2}^2)\cdots}$$

- Can think of the transfer function as a product of the individual factors
- For example, consider the following system

$$G(s) = \frac{(s-z_1)}{(s-p_1)(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)}$$

- Can rewrite as

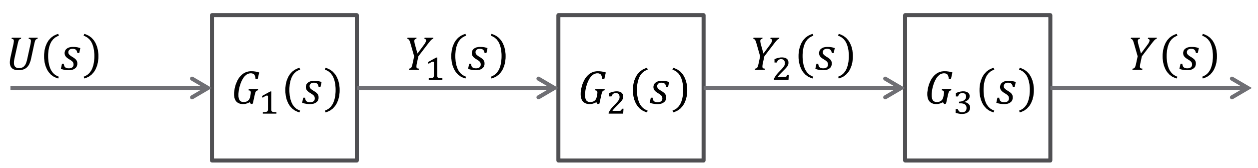
$$G(s) = (s-z_1)\cdot\frac{1}{(s-p_1)}\cdot\frac{1}{(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)}$$

Transfer Function Factors

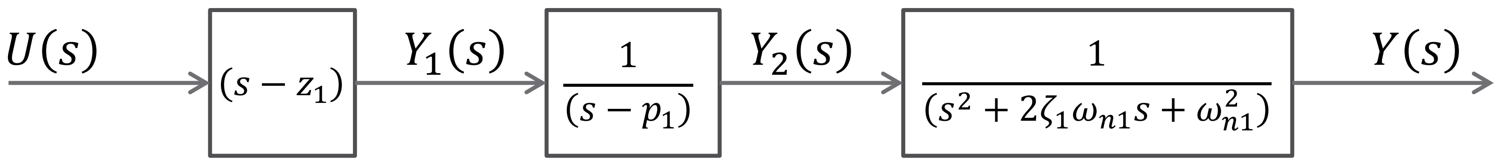
$$G(s) = (s - z_1) \cdot \frac{1}{(s - p_1)} \cdot \frac{1}{(s^2 + 2\zeta_1 \omega_{n1} s + \omega_{n1}^2)}$$

- Think of this as three cascaded transfer functions

$$G_1(s) = (s - z_1), \quad G_2(s) = \frac{1}{(s-p_1)}, \quad G_3(s) = \frac{1}{(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)}$$

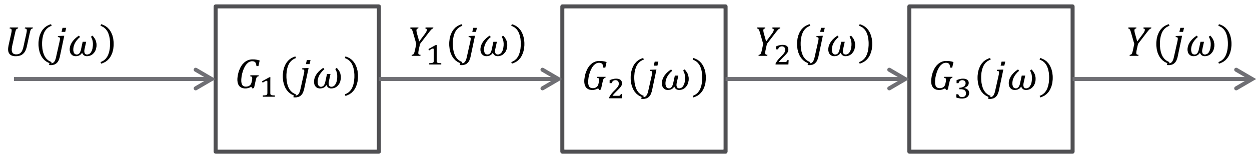


or



Transfer Function Factors

- Overall transfer function – and therefore, frequency response – is the product of individual first- and second-order factors
- Instructive, therefore, to understand the responses of the individual factors
  - ***First- and second-order poles and zeros***



First-Order Factors

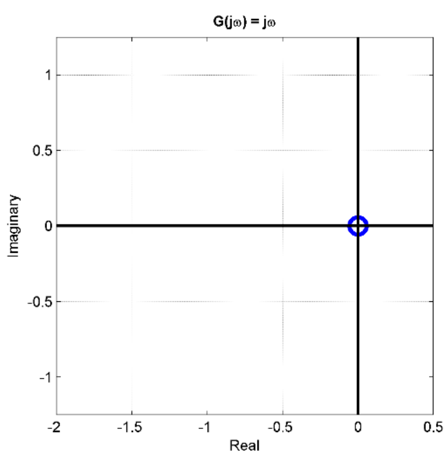
- First-order factors
  - Single, real poles or zeros
- In the Laplace domain:

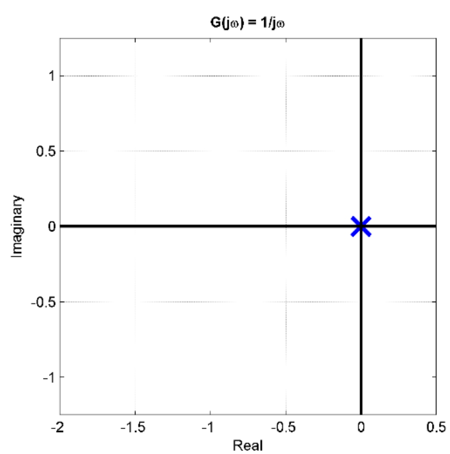
$$G(s) = s, \quad G(s) = \frac{1}{s}, \quad G(s) = s + a, \quad G(s) = \frac{1}{s+a}$$

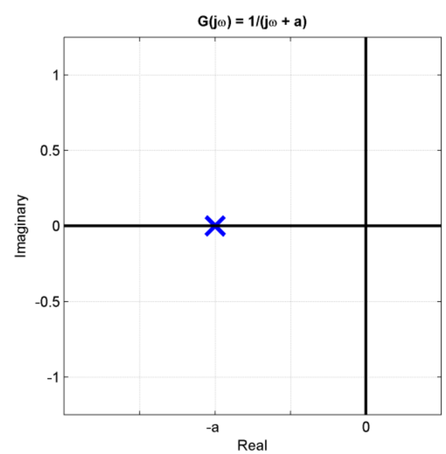
- In the frequency domain

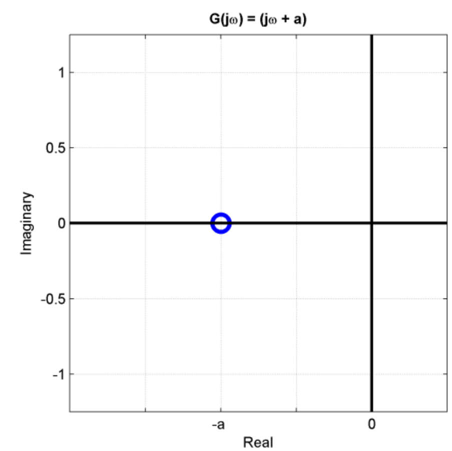
$$G(j\omega) = j\omega, \quad G(j\omega) = \frac{1}{j\omega}, \quad G(j\omega) = j\omega + a, \quad G(j\omega) = \frac{1}{j\omega+a}$$

- Pole/zero plots:









First-Order Factors – Zero at the Origin

- A ***differentiator***

$$G(s) = s$$

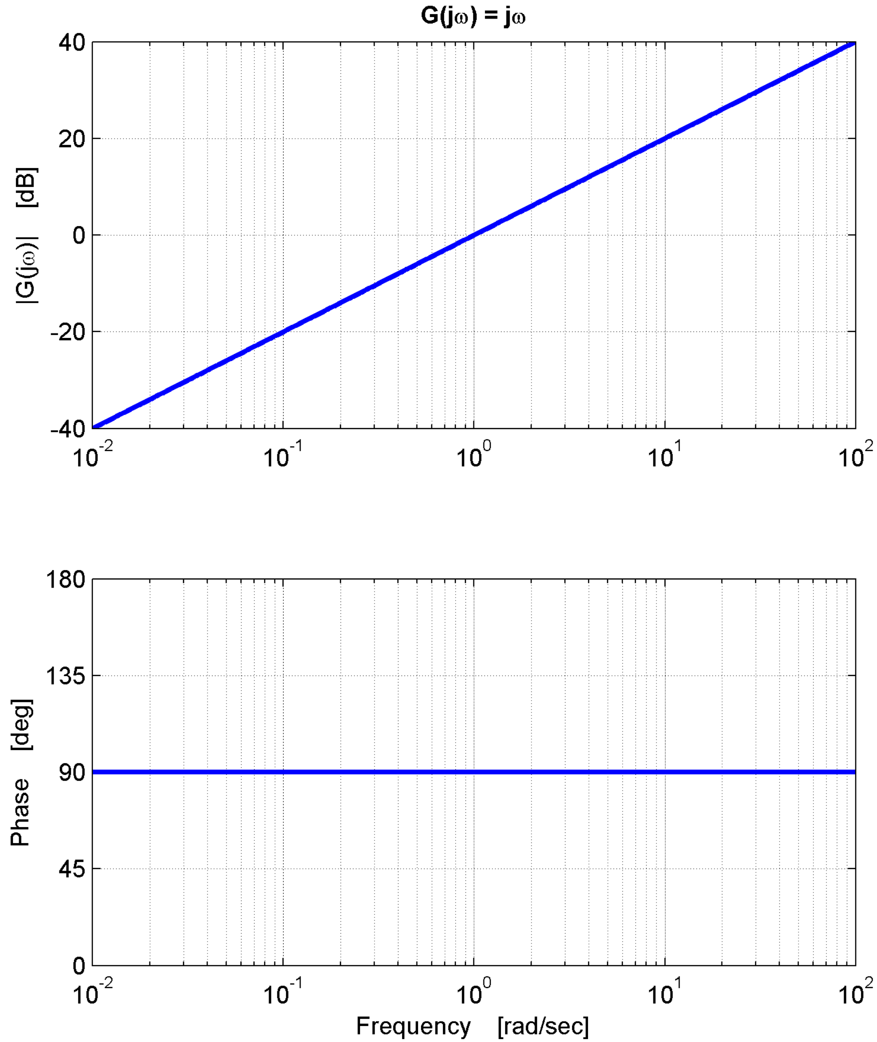
$$G(j\omega) = j\omega$$

- **<u>Gain</u>:**

$$|G(j\omega)| = |j\omega| = \omega$$

- **<u>Phase</u>:**

$$\angle G(j\omega) = +90°, \quad \forall\omega$$



First-Order Factors – Pole at the Origin

- An ***integrator***

$$G(s) = \frac{1}{s}$$

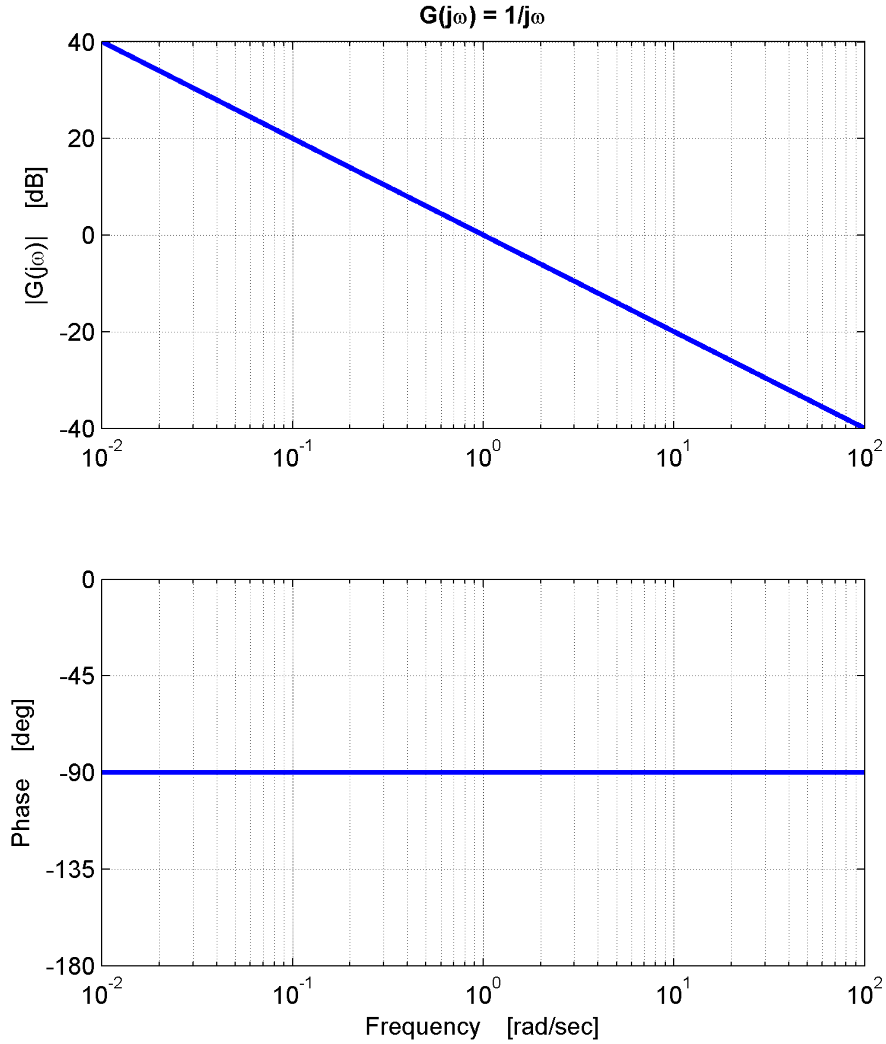
$$G(j\omega) = \frac{1}{j\omega}$$

- **Gain**:

$$|G(j\omega)| = \left|\frac{1}{j\omega}\right| = \frac{1}{\omega}$$

- **Phase**:

$$\angle G(j\omega) = \angle - j\frac{1}{\omega} = -90°, \quad \forall\omega$$



First-Order Factors – Single, Real Zero

- Single, real zero at $s = -a$

$$G(j\omega) = j\omega + a$$

- **Gain**:

$$|G(j\omega)| = \sqrt{\omega^2 + a^2}$$

for $\omega \ll a$

$$|G(j\omega)| \approx a$$

for $\omega \gg a$

$$|G(j\omega)| \approx \omega$$

- **Phase**:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$$

for $\omega \ll a$

$$\angle G(j\omega) \approx \angle a = 0°$$

for $\omega \gg a$

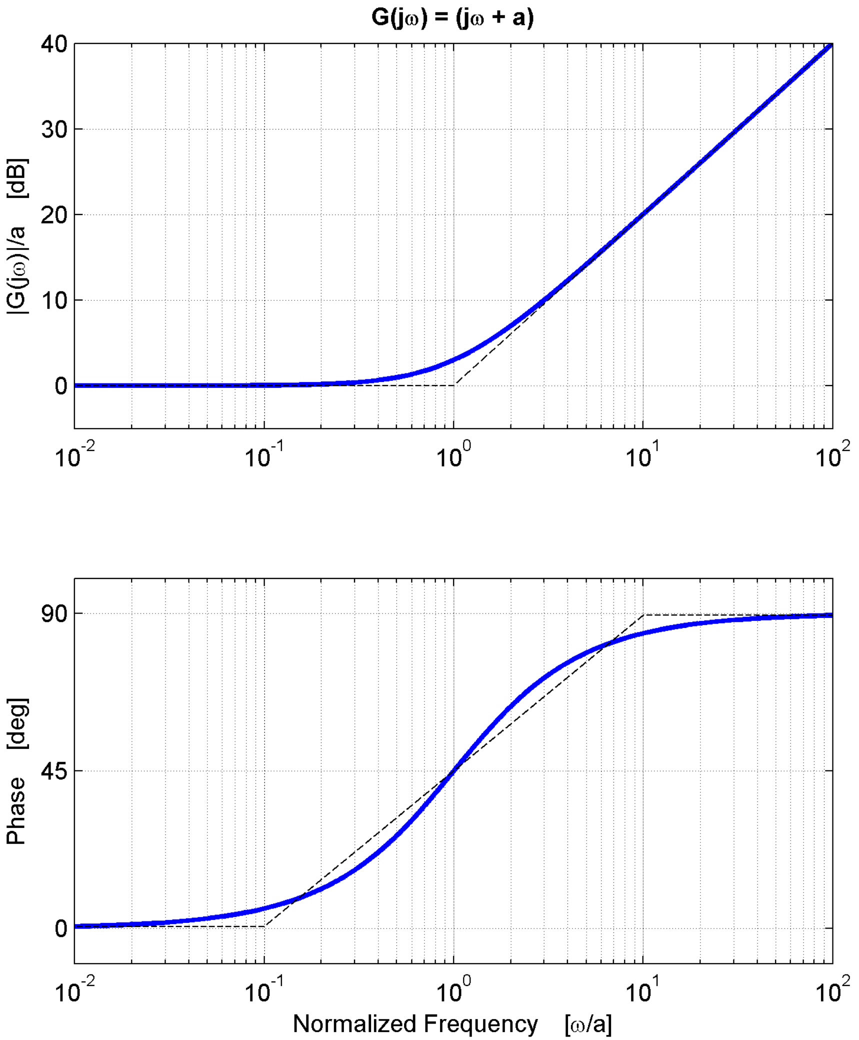
$$\angle G(j\omega) \approx \angle j\omega = 90°$$

First-Order Factors – Single, Real Zero

- Corner frequency:

$$\omega_c = a$$

  - $|G(j\omega_c)| = a\sqrt{2} = 1.414 \cdot a$
  - $|G(j\omega_c)|_{dB} = (a)_{dB} + 3dB$
  - $\angle G(j\omega_c) = +45°$
- For $\omega \gg \omega_c$, gain increases at:
  - $20dB/dec$
  - $6dB/oct$
- From $\sim 0.1\omega_c$ to $\sim 10\omega_c$, phase increases at a rate of:
  - $\sim 45°/dec$
  - Rough approximation



First-Order Factors – Single, Real Pole

- Single, real pole at $s = -a$

$$G(j\omega) = \frac{1}{j\omega + a}$$

- **Gain**:

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2 + a^2}}$$

for $\omega \ll a$

$$|G(j\omega)| \approx \frac{1}{a}$$

for $\omega \gg a$

$$|G(j\omega)| \approx \frac{1}{\omega}$$

- **Phase**:

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

for $\omega \ll a$

$$\angle G(j\omega) \approx \angle\frac{1}{a} = 0°$$

for $\omega \gg a$

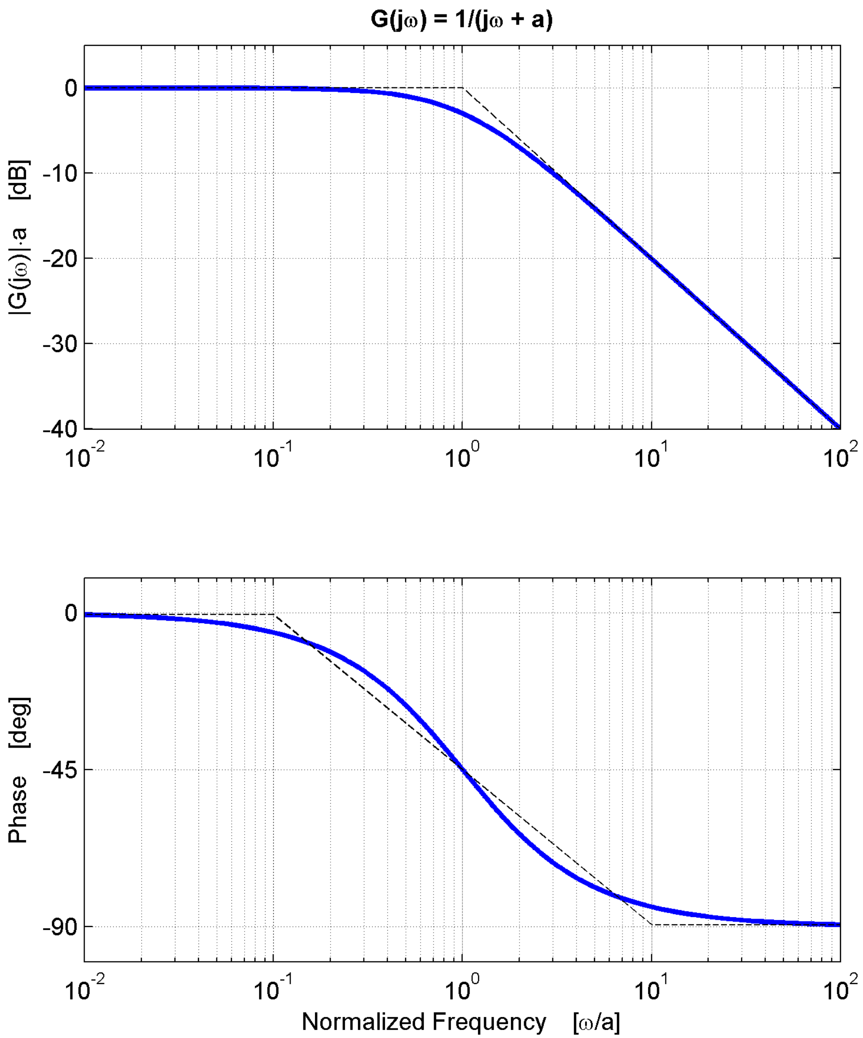
$$\angle G(j\omega) \approx \angle\frac{1}{j\omega} = -90°$$

First-Order Factors – Single, Real Pole

- Corner frequency:

  $$\omega_c = a$$

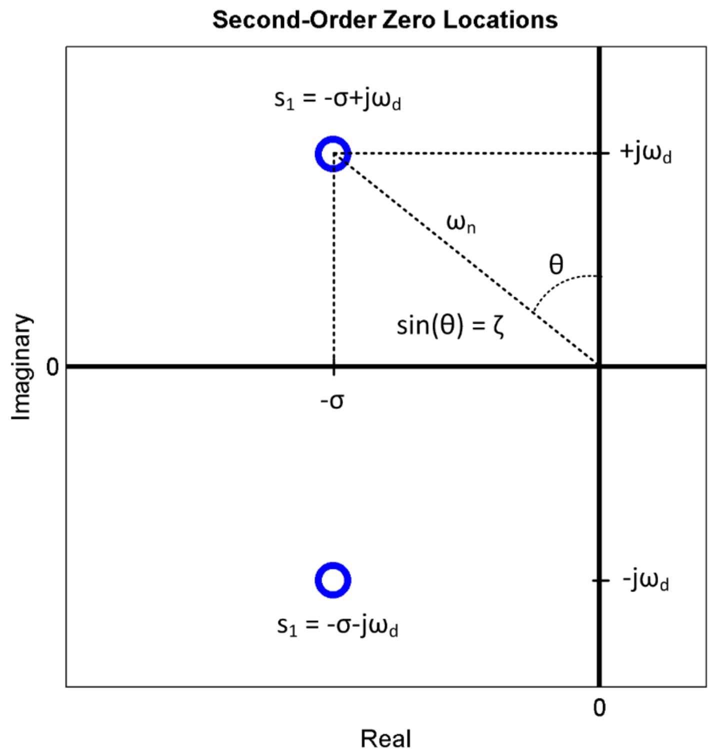
  - $|G(j\omega_c)| = \frac{1}{a\sqrt{2}} = 0.707 \cdot \frac{1}{a}$
  - $|G(j\omega_c)|_{dB} = \left(\frac{1}{a}\right)_{dB} - 3dB$
  - $\angle G(j\omega_c) = -45°$
- For $\omega \gg \omega_c$, gain decreases at:
  - $-20dB/dec$
  - $-6dB/oct$
- From $\sim 0.1\omega_c$ to $\sim 10\omega_c$, phase decreases at a rate of:
  - $\sim -45°/dec$
  - Rough approximation



Second-Order Factors

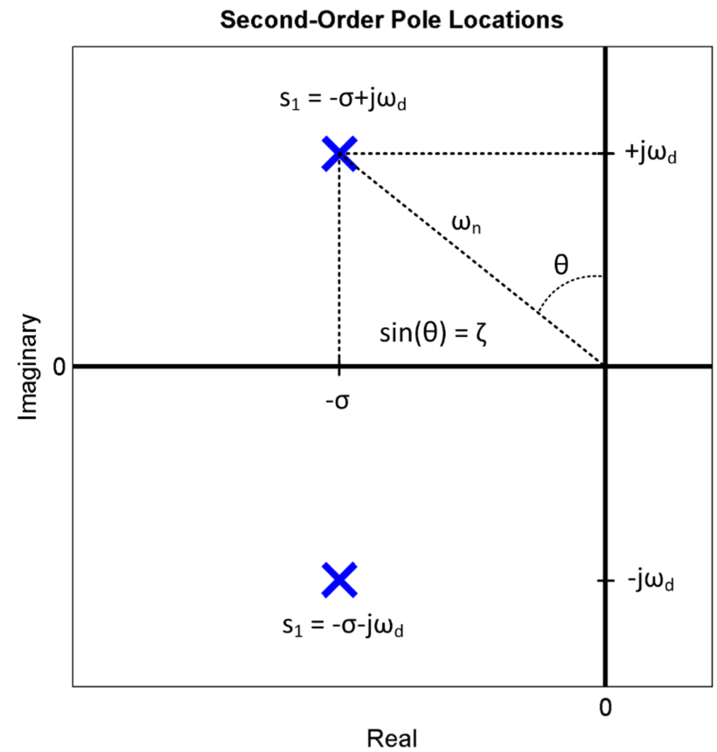
- Complex-conjugate zeros

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$



- Complex-conjugate poles

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$\sigma = \zeta\omega_n,\ \omega_d = \omega_n\sqrt{1-\zeta^2}$$

2nd-Order Factors – Complex-Conjugate Zeros

- Complex-conjugate zeros at $s = -\sigma \pm j\omega_d$

$$G(j\omega) = (j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2$$

- **Gain**:

for $\omega \ll \omega_n$

$$|G(j\omega)| \approx \omega_n^2$$

for $\omega = \omega_n$

$$|G(j\omega)| = 2\zeta\omega_n^2$$

for $\omega \gg \omega_n$

$$|G(j\omega)| \approx \omega^2$$

- **Phase**:

for $\omega \ll \omega_n$

$$\angle G(j\omega) \approx \angle \omega_n^2 = 0°$$

for $\omega = \omega_n$

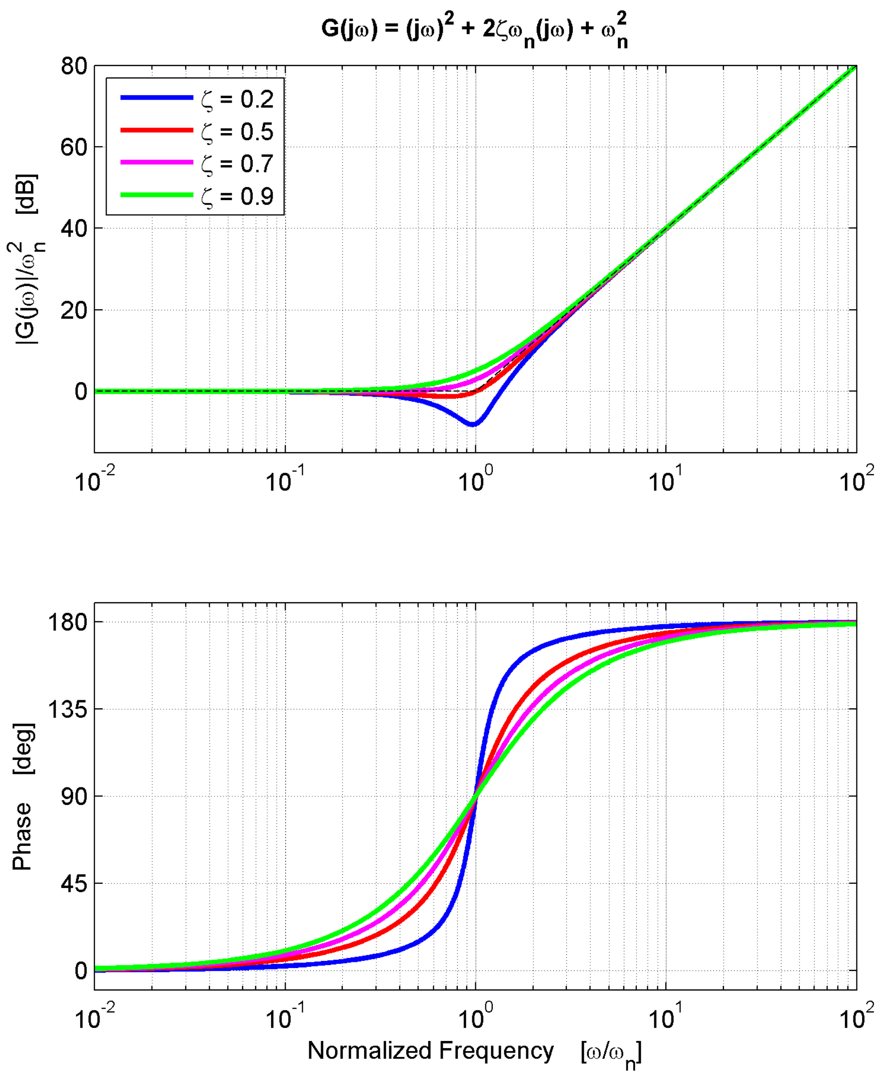
$$\angle G(j\omega) = \angle j2\zeta\omega_n = +90°$$

for $\omega \gg \omega_n$

$$\angle G(j\omega) \approx \angle -\omega^2 = +180°$$

2^{nd}-Order Factors – Complex-Conjugate Zeros

- Response may dip below low-freq. value near ω_n
  - Peaking increases as ζ decreases
- Gain increases at $+40dB/dec$ or $+12dB/oct$ for $\omega \gg \omega_n$
- Corner frequency depends on damping ratio, ζ
  - ω_c increases as ζ decreases
- At $\omega = \omega_c, \angle G(j\omega) = 90°$
- Phase transition abruptness depends on ζ



2nd-Order Factors – Complex-Conjugate Poles

- Complex-conjugate zeros at $s = -\sigma \pm j\omega_d$

$$G(j\omega) = \frac{1}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

- **Gain**:

for $\omega \ll \omega_n$

$$|G(j\omega)| \approx \frac{1}{\omega_n^2}$$

for $\omega = \omega_n$

$$|G(j\omega)| = \frac{1}{2\zeta\omega_n^2}$$

for $\omega \gg \omega_n$

$$|G(j\omega)| \approx \frac{1}{\omega^2}$$

- **Phase**:

for $\omega \ll \omega_n$

$$\angle G(j\omega) \approx \angle \frac{1}{\omega_n^2} = 0°$$

for $\omega = \omega_n$

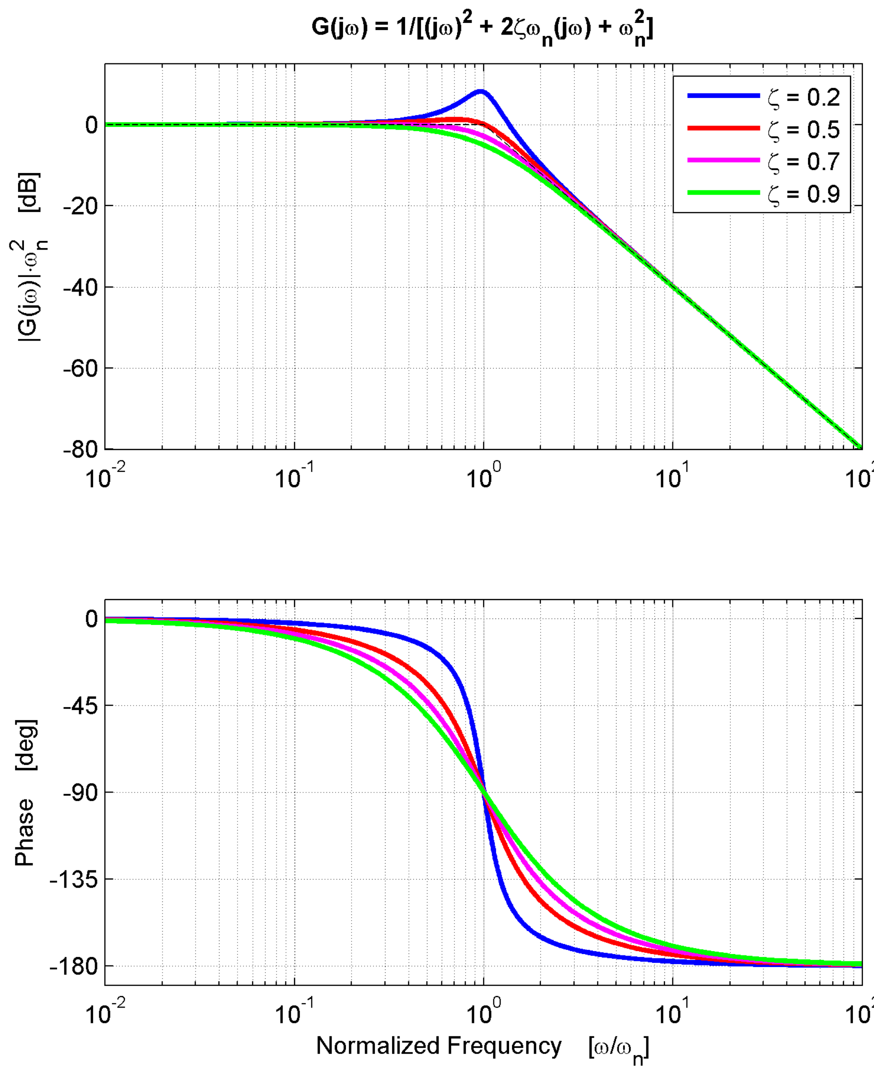
$$\angle G(j\omega) = \angle \frac{1}{j2\zeta\omega_n} = -90°$$

for $\omega \gg \omega_n$

$$\angle G(j\omega) \approx \angle - \frac{1}{\omega^2} = -180°$$

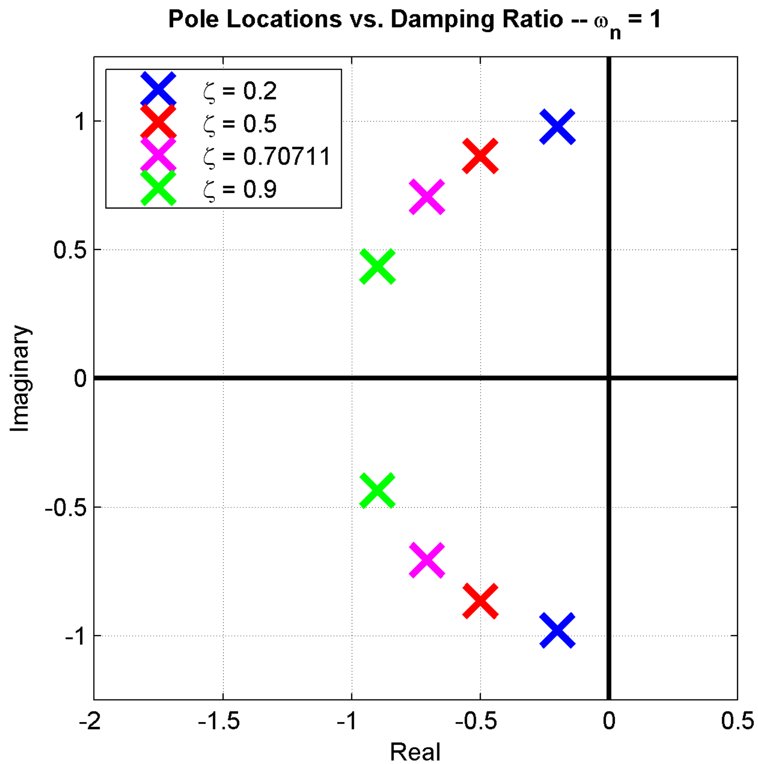
2^{nd}-Order Factors – Complex-Conjugate Poles

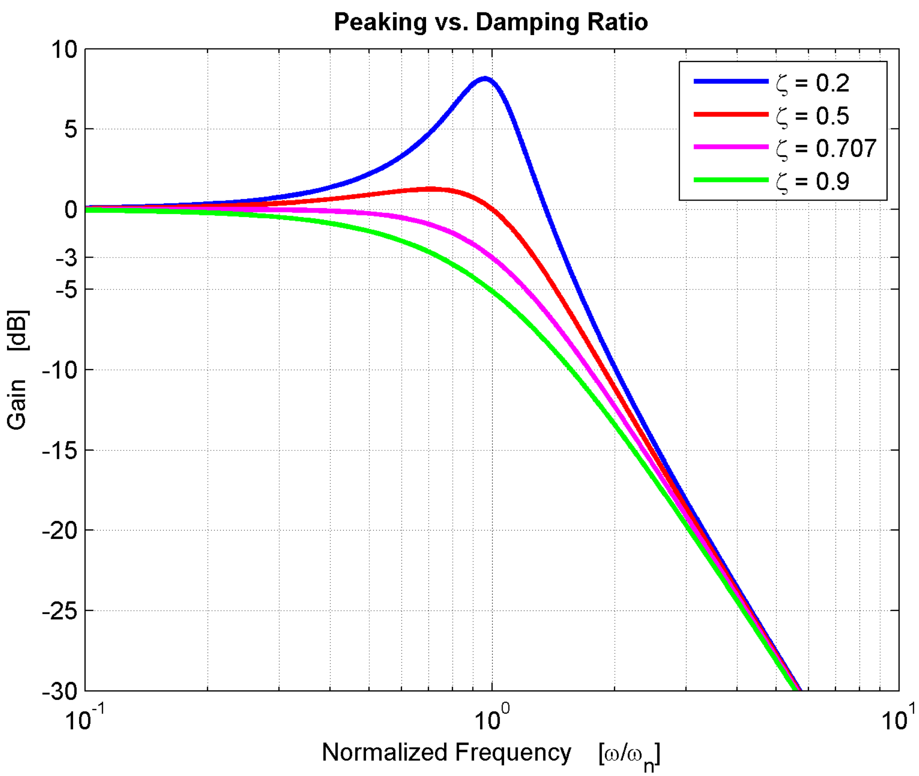
- Response may peak above low-freq. value near ω_n
  - Peaking increases as ζ decreases
- Gain decreases at $-40dB/dec$ or $-12dB/oct$ for $\omega \gg \omega_n$
- Corner frequency depends on damping ratio, ζ
  - ω_c increases as ζ decreases
- At $\omega = \omega_c$, $\angle G(j\omega) = -90°$
- Phase transition abruptness depends on ζ



Pole Location and Peaking

- Peaking is dependent on ζ – pole locations
  - No peaking at all for $\zeta \geq 1/\sqrt{2} = 0.707$
  - $\zeta = 0.707$ – ***maximally-flat*** or ***Butterworth*** response





Frequency Response Components - Example

- Consider the following system

$$G(s) = \frac{20(s+20)}{(s+1)(s+100)}$$

- The system's frequency response function is

$$G(j\omega) = \frac{20(j\omega+20)}{(j\omega+1)(j\omega+100)}$$

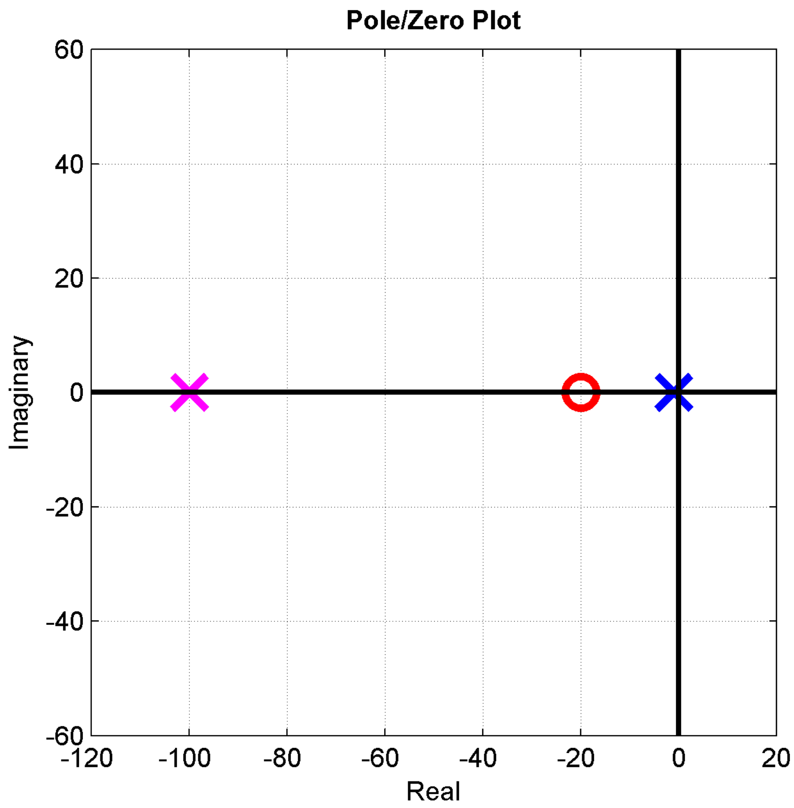
- As we've seen we can consider this a product of individual frequency response factors

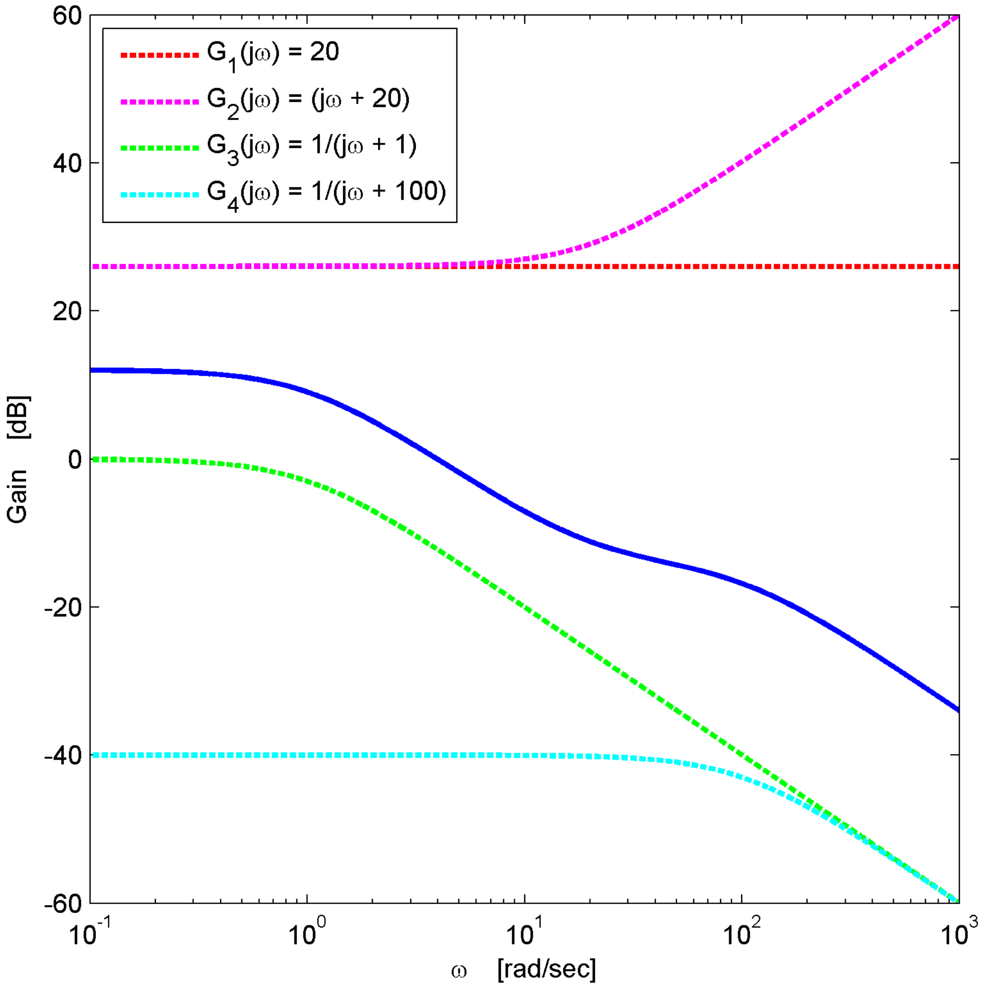
$$G(j\omega) = 20 \cdot (j\omega+20) \cdot \frac{1}{(j\omega+1)} \cdot \frac{1}{(j\omega+100)}$$

- Overall response is the composite of the individual responses
  - Product of individual gain responses – sum in dB
  - Sum of individual phase responses

Frequency Response Components - Example

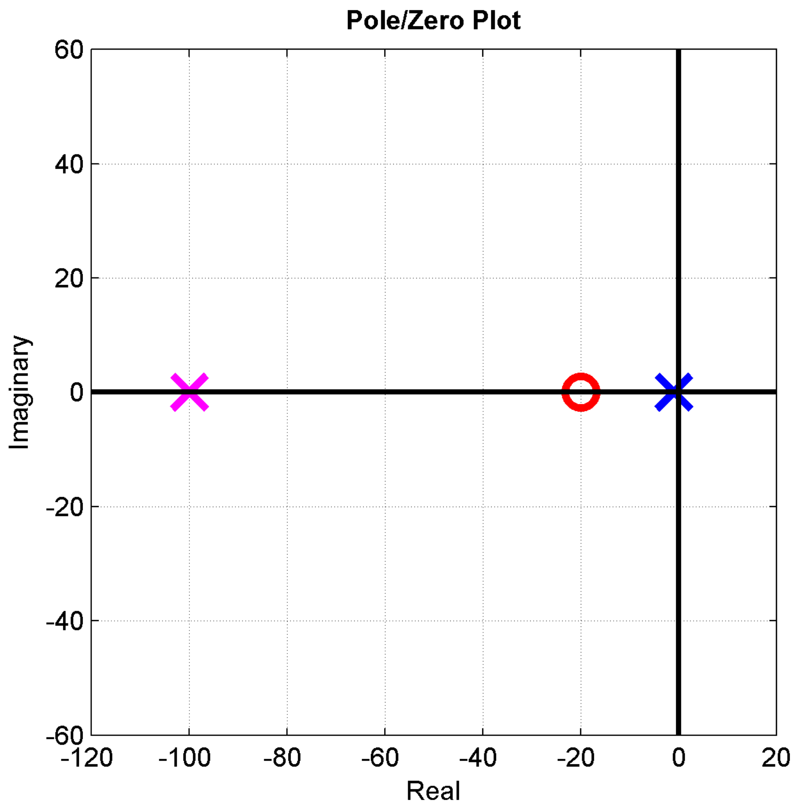
- Gain response

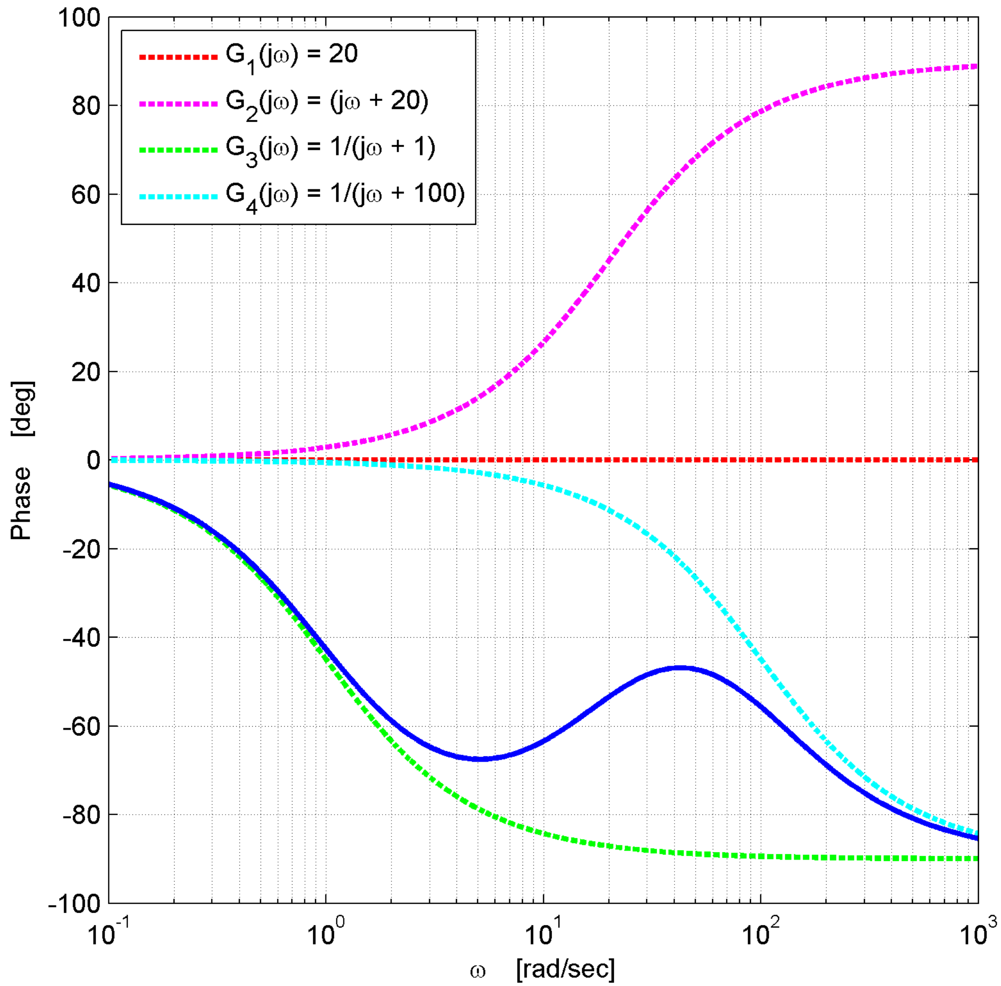




Frequency Response Components - Example

- Phase response





Bode Plot Construction

In this section, we'll look at a method for sketching, by hand, a straight-line, asymptotic approximation for a Bode plot.

Bode Plot Construction

- We've just seen that a system's transfer function can be factored into first- and second-order terms
  - Each factor contributes a component to the overall gain and phase responses
- Now, we'll look at a technique for ***manually sketching a system's Bode plot***
  - In practice, you'll almost always plot with a computer
  - But, learning to do it by hand provides valuable insight
- We'll look at how to approximate Bode plots for each of the different factors

Bode Form of the Transfer function

- Consider the general transfer function form:

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s^2+2\zeta_a\omega_{na}s+\omega_{na}^2)\cdots}{(s-p_1)(s-p_2)\cdots(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)\cdots}$$

- We first want to put this into ***Bode form***:

$$G(s) = K_0\frac{\left(\frac{s}{\omega_{ca}}+1\right)\left(\frac{s}{\omega_{cb}}+1\right)\cdots\left(\frac{s^2}{\omega_{na}^2}+\frac{2\zeta_a}{\omega_{na}}s+1\right)\cdots}{\left(\frac{s}{\omega_{c1}}+1\right)\left(\frac{s}{\omega_{c2}}+1\right)\cdots\left(\frac{s^2}{\omega_{n1}^2}+\frac{2\zeta_1}{\omega_{n1}}s+1\right)\cdots}$$

- Putting $G(s)$ into Bode form requires putting each of the ***first- and second-order factors into Bode form***

First-Order Factors in Bode Form

- ***First-order transfer function factors*** include:

$$G(s) = s^n, \quad G(s) = s + \sigma, \quad G(s) = \frac{1}{s+\sigma}$$

- For the first factor, $G(s) = s^n$, n is a positive or negative integer
  - Already in Bode form
- For the second two, divide through by σ, giving

$$G(s) = \sigma\left(\frac{s}{\sigma} + 1\right) \quad \text{and} \quad G(s) = \frac{1}{\sigma\left(\frac{s}{\sigma}+1\right)}$$

- Here, $\sigma = \omega_c$, the ***corner frequency*** associated with that zero or pole, so

$$G(s) = \omega_c\left(\frac{s}{\omega_c} + 1\right) \quad \text{and} \quad G(s) = \frac{1}{\omega_c\left(\frac{s}{\omega_c}+1\right)}$$

Second-Order Factors in Bode Form

- ***Second-order transfer function factors*** include:

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 \quad \text{and} \quad G(s) = \frac{1}{(s)^2+2\zeta\omega_n s+\omega_n^2}$$

- Again, normalize the s^0 coefficient, giving

$$G(s) = \omega_n^2\left[\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1\right] \quad \text{and} \quad G(s) = \frac{1/\omega_n^2}{\frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1}$$

- Putting each factor into its Bode form involves factoring out any DC gain component
- Lump all of ***DC gains*** together into a single gain constant, K_0

$$G(s) = K_0\frac{\left(\frac{s}{\omega_{ca}}+1\right)\left(\frac{s}{\omega_{cb}}+1\right)\cdots\left(\frac{s^2}{\omega_{na}^2}+\frac{2\zeta_a}{\omega_{na}}s+1\right)\cdots}{\left(\frac{s}{\omega_{c1}}+1\right)\left(\frac{s}{\omega_{c2}}+1\right)\cdots\left(\frac{s^2}{\omega_{n1}^2}+\frac{2\zeta_1}{\omega_{n1}}s+1\right)\cdots}$$

Bode Plot Construction

- Transfer function in Bode form

$$G(s) = K_0 \frac{\left(\frac{s}{\omega_{ca}}+1\right)\left(\frac{s}{\omega_{cb}}+1\right)\cdots\left(\frac{s^2}{\omega_{na}^2}+\frac{2\zeta_a}{\omega_{na}}s+1\right)\cdots}{\left(\frac{s}{\omega_{c1}}+1\right)\left(\frac{s}{\omega_{c2}}+1\right)\cdots\left(\frac{s^2}{\omega_{n1}^2}+\frac{2\zeta_1}{\omega_{n1}}s+1\right)\cdots}$$

- Product of a constant DC gain factor, K_0, and first- and second-order factors
- Plot the frequency response of each factor individually, then combine graphically
  - Overall response is the product of individual factors
    - Product of gain responses – sum on a dB scale
    - Sum of phase responses

Bode Plot Construction

- **<u>Bode plot construction procedure</u>**:
  1. Put the transfer function into ***Bode form***
  2. Draw a ***straight-line asymptotic approximation*** for the gain and phase response of each individual factor
  3. ***Graphically add*** all individual response components and sketch the result

- Note that we are really plotting the frequency response function, $G(j\omega)$
  - We use the transfer function, $G(s)$, to simplify notation

- Next, we'll look at the straight-line asymptotic approximations for the Bode plots for each of the transfer function factors

Bode Plot – Constant Gain Factor

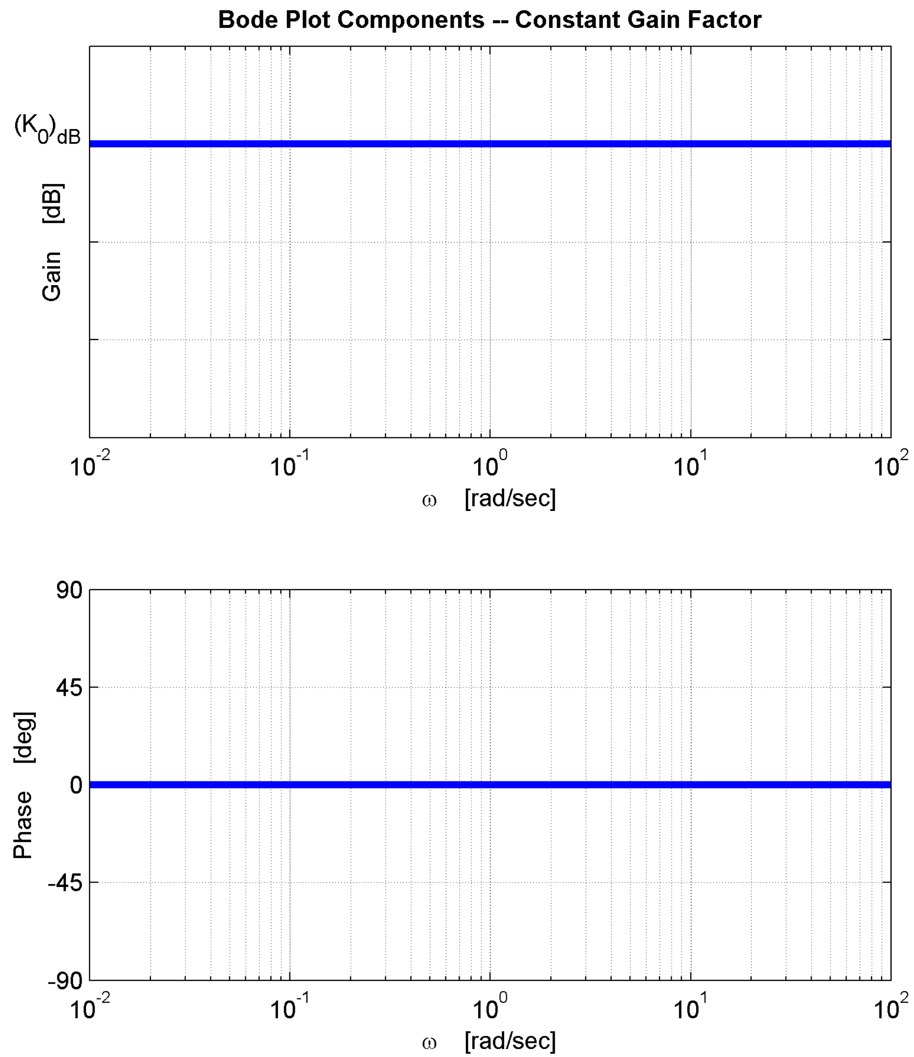
$$G(s) = K_0$$

- Constant gain

$$|G(s)| = K_0$$

- Constant Phase

$$\angle G(s) = 0°$$

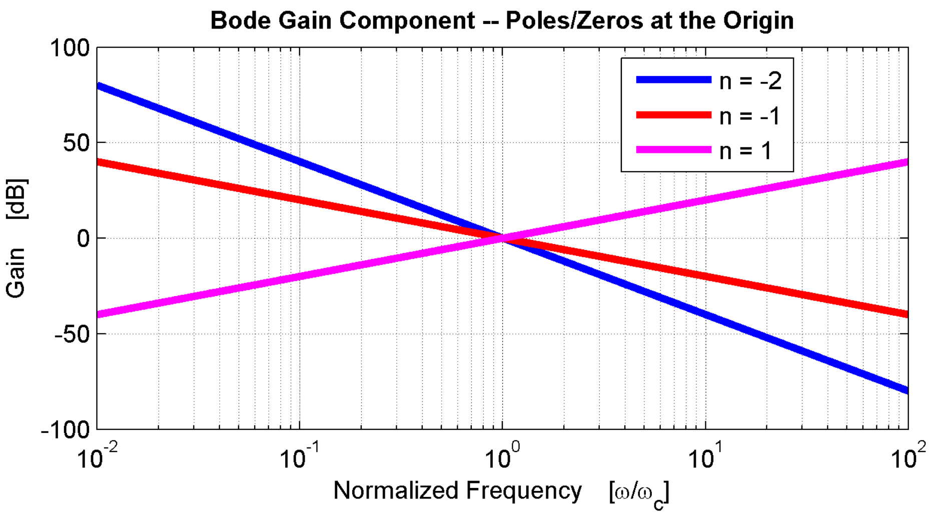


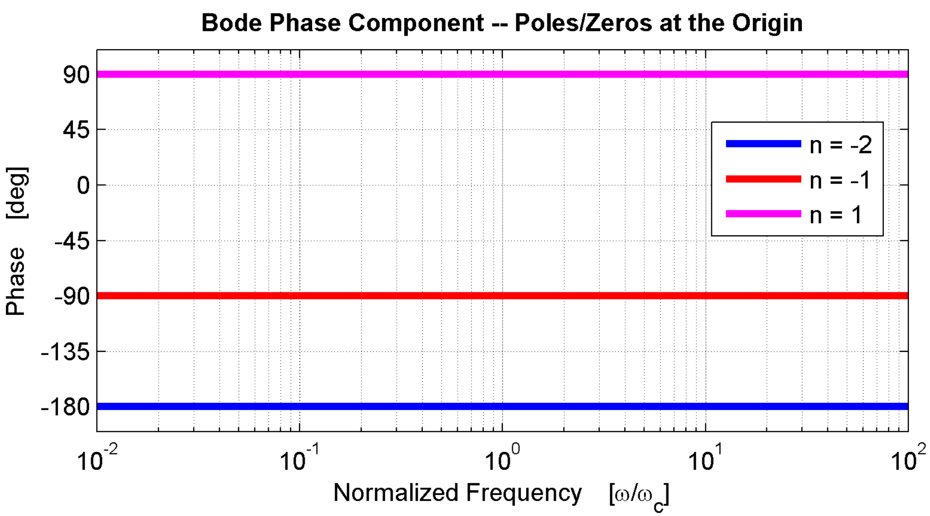
Bode Plot – Poles/Zeros at the Origin

$$G(s) = s^n$$

- $n > 0$:
  - n zeros at the origin
- $n < 0$:
  - n poles at the origin
- **Gain**:
  - Straight line
  - Slope $= n \cdot 20 \frac{dB}{dec} = n \cdot 6 \frac{dB}{oct}$
  - $0dB$ at $\omega = 1$
- **Phase**:

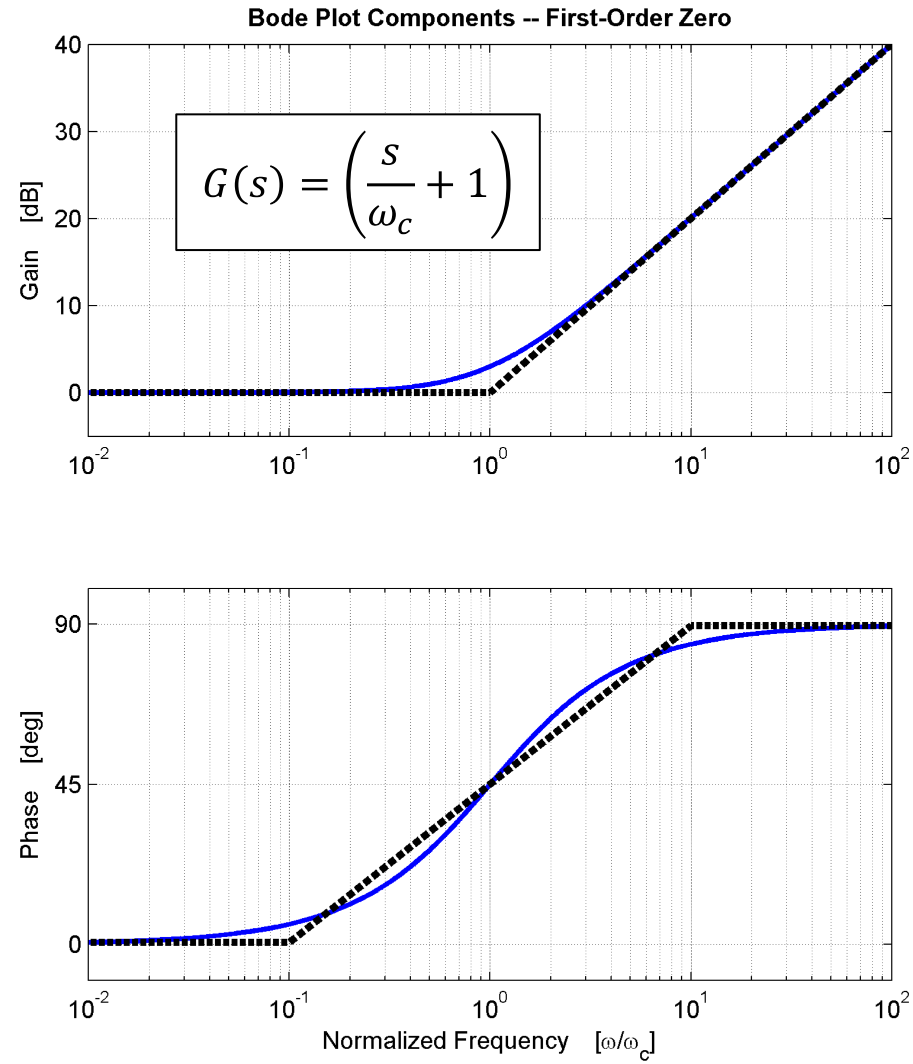
$$\angle G(s) = n \cdot 90°$$





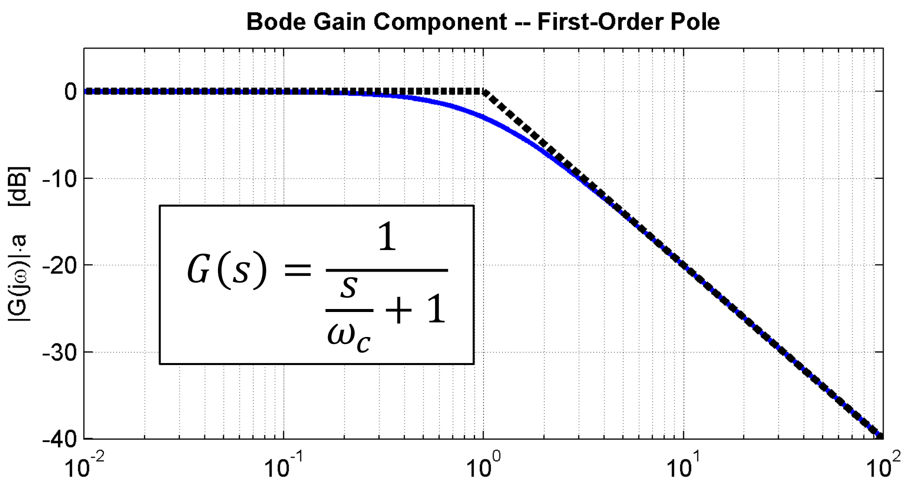
Bode Plot – First-Order Zero

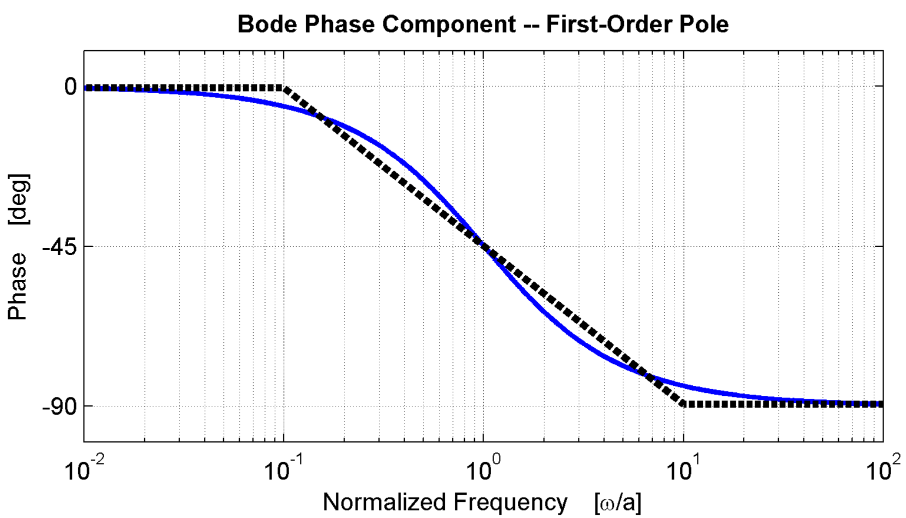
- Single real zero at $s = -\omega_c$
- **Gain**:
  - $0dB$ for $\omega < \omega_c$
  - $+20\frac{dB}{dec} = +6\frac{dB}{oct}$ for $\omega > \omega_c$
  - Straight-line asymptotes intersect at $(\omega_c, 0dB)$
- **Phase**:
  - $0°$ for $\omega \leq 0.1\omega_c$
  - $45°$ for $\omega = \omega_c$
  - $90°$ for $\omega \geq 10\omega_c$
  - $\frac{+45°}{dec}$ for $0.1\omega_c \leq \omega \leq 10\omega_c$



Bode Plot – First-Order Pole

- Single real pole at $s = -\omega_c$
- **Gain**:
  - $0dB$ for $\omega < \omega_c$
  - $-20\frac{dB}{dec} = -6\frac{dB}{oct}$ for $\omega > \omega_c$
  - Straight-line asymptotes intersect at $(\omega_c, 0dB)$
- **Phase**:
  - $0°$ for $\omega \leq 0.1\omega_c$
  - $-45°$ for $\omega = \omega_c$
  - $-90°$ for $\omega \geq 10\omega_c$
  - $\frac{-45°}{dec}$ for $0.1\omega_c \leq \omega \leq 10\omega_c$



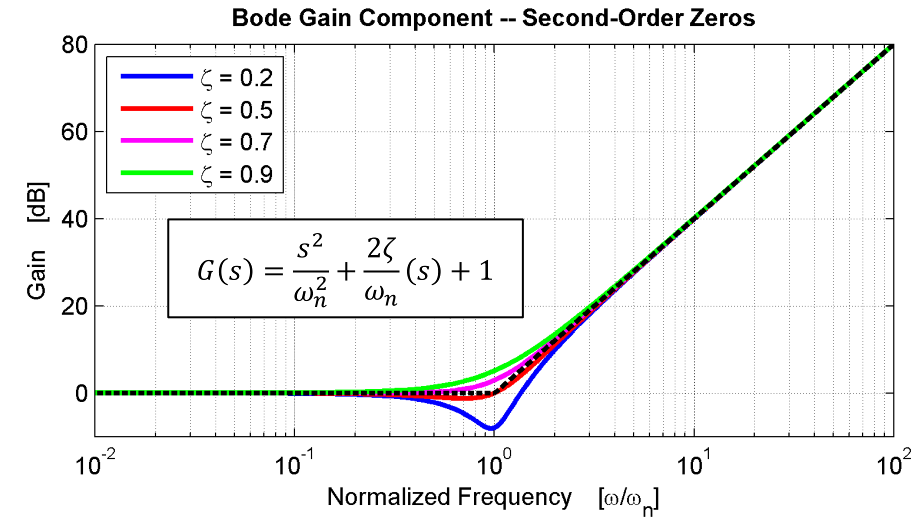


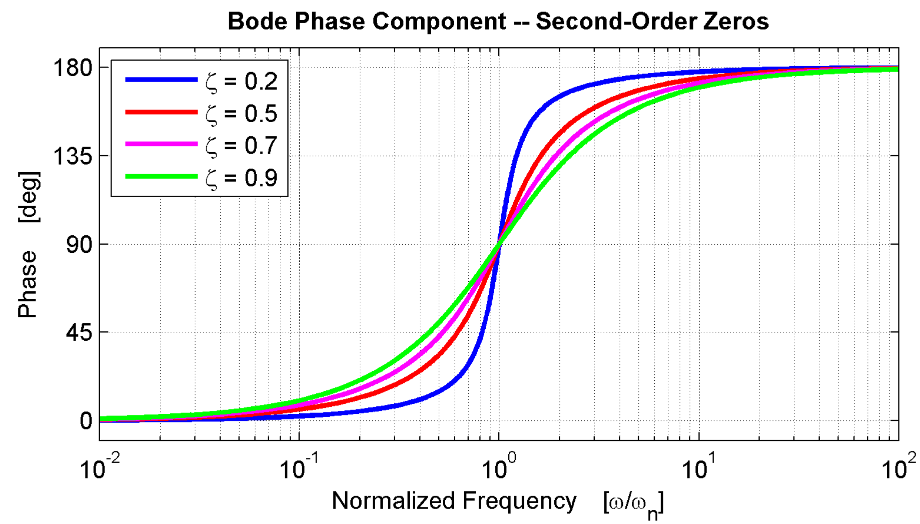
Bode Plot – Second-Order Zero

- Complex-conjugate zeros:

$$s_{1,2} = -\sigma \pm j\omega_d$$

- **Gain**:
  - $0dB$ for $\omega \leq \omega_n$
  - $+40\frac{dB}{dec} = +12\frac{dB}{oct}$ for $\omega > \omega_n$
  - Straight-line asymptotes intersect at $(\omega_n, 0dB)$
  - ζ-dependent peaking around ω_n
- **Phase**:
  - $0°$ for $\omega \leq 0.1 \cdot \omega_n$
  - $90°$ for $\omega = \omega_n$
  - $180°$ for $\omega \geq 10 \cdot \omega_n$
  - $\frac{+90°}{dec}$ for $0.1\omega_c \leq \omega \leq 10\omega_c$



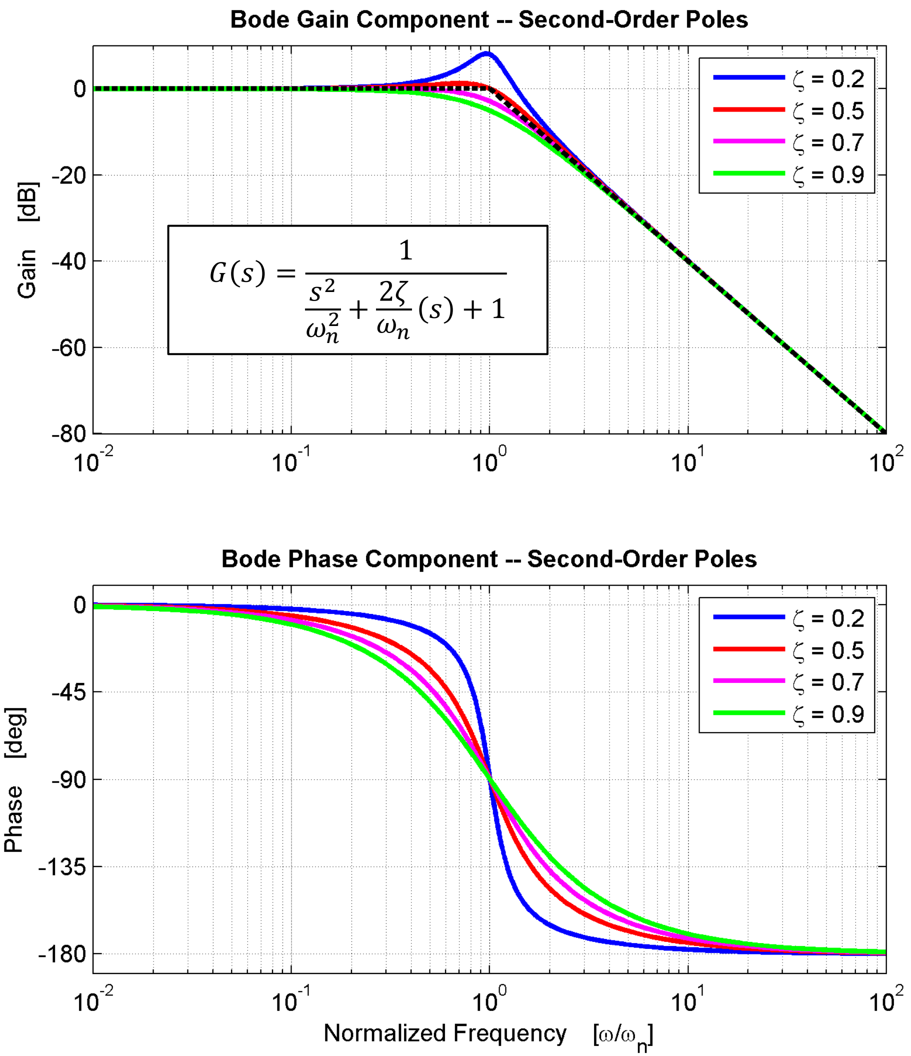


Bode Plot – Second-Order Pole

- Complex-conjugate poles:

$$s_{1,2} = -\sigma \pm j\omega_d$$

- **Gain**:
  - $0dB$ for $\omega \leq \omega_n$
  - $-40\frac{dB}{dec} = -12\frac{dB}{oct}$ for $\omega > \omega_n$
  - Straight-line asymptotes intersect at $(\omega_n, 0dB)$
  - ζ-dependent peaking around ω_n
- **Phase**:
  - $0°$ for $\omega \leq 0.1 \cdot \omega_n$
  - $-90°$ for $\omega = \omega_n$
  - $-180°$ for $\omega \geq 10 \cdot \omega_n$
  - $\frac{-90°}{dec}$ for $0.1\omega_c \leq \omega \leq 10\omega_c$



Bode Plot Construction – Example

- Consider a system with the following ***transfer function***

$$G(s)=\frac{10(s+20)}{s(s+400)}$$

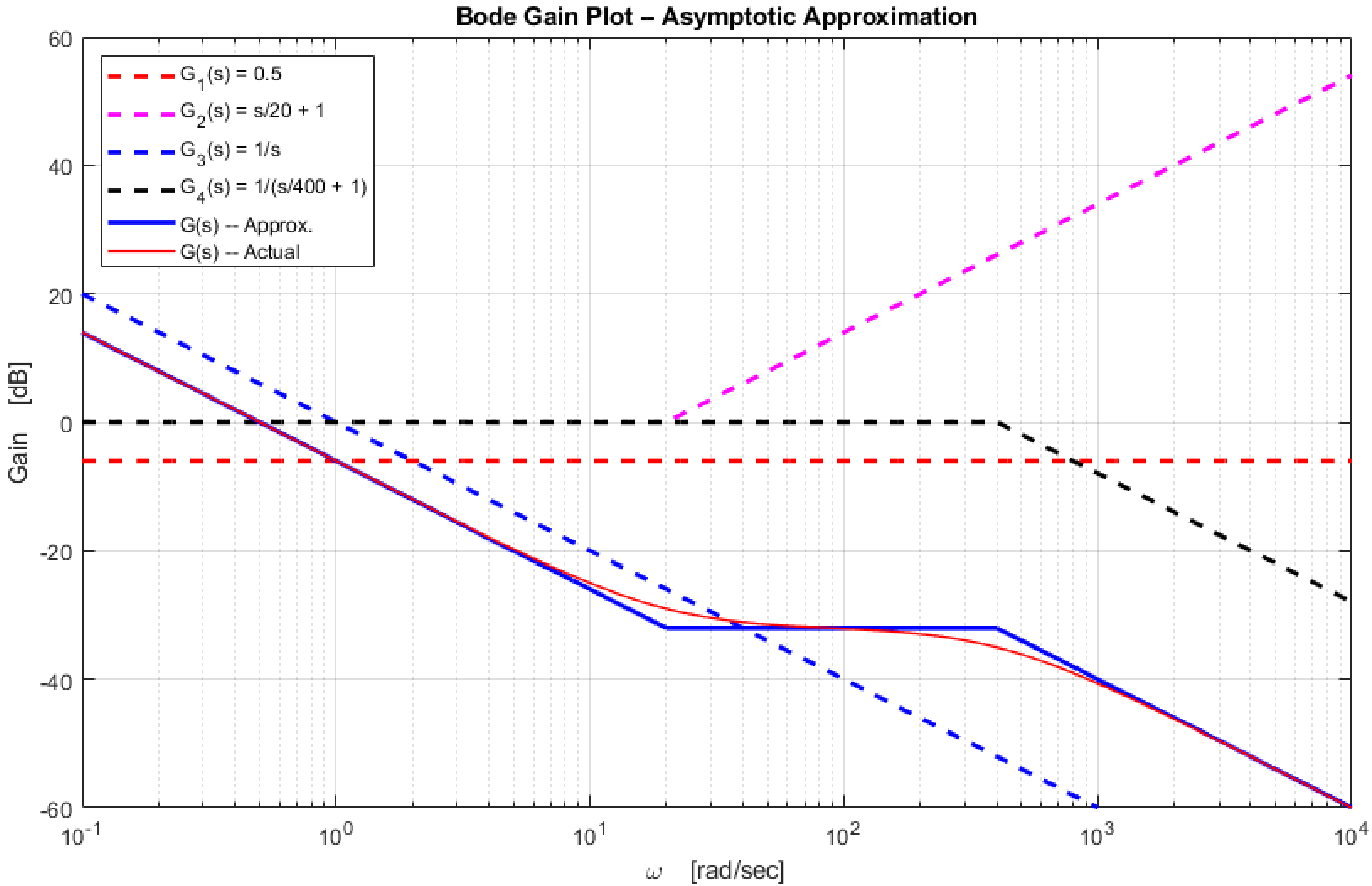
- Put it into ***Bode form***

$$G(s)=\frac{10\cdot 20\left(\frac{s}{20}+1\right)}{s\cdot 400\left(\frac{s}{400}+1\right)}=\frac{0.5\left(\frac{s}{20}+1\right)}{s\cdot\left(\frac{s}{400}+1\right)}$$

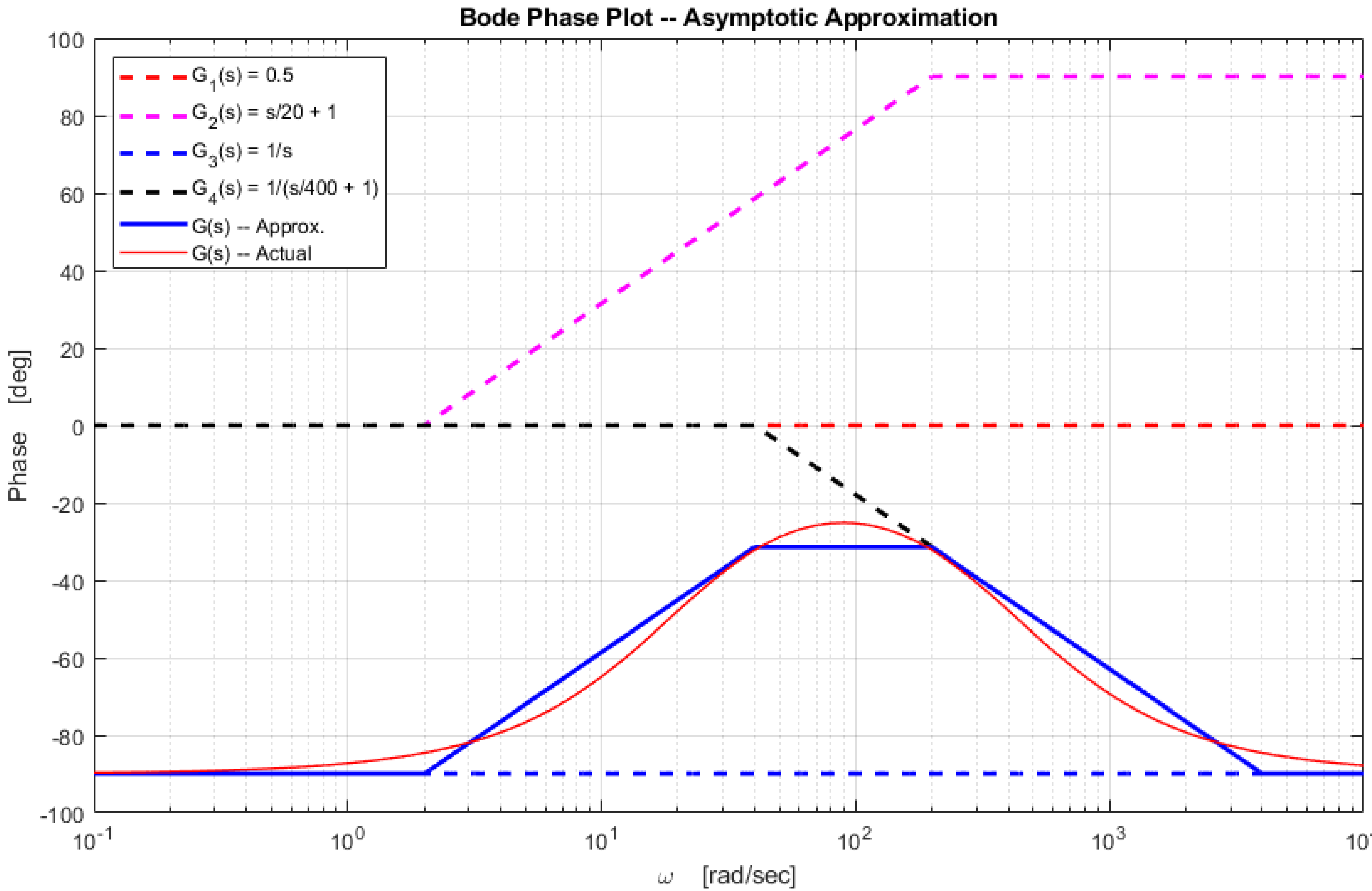
- Represent as a ***product of factors***

$$G(s)=0.5\cdot\left(\frac{s}{20}+1\right)\cdot\frac{1}{s}\cdot\frac{1}{\left(\frac{s}{400}+1\right)}$$

Bode Plot Construction – Example



Bode Plot Construction – Example



Relationship between Pole/Zero Plots and Bode Plots

It is also possible to calculate a system's frequency response directly from that system's pole/zero plot.

Bode Construction from Pole/Zero Plots

- Transfer function can be expressed as

$$G(s) = \frac{\prod_i (s - z_i)}{\prod_i (s - p_i)} \xrightarrow{s \to j\omega} G(j\omega) = \frac{\prod_i (j\omega - z_i)}{\prod_i (j\omega - p_i)}$$

  - Numerator is a product of first-order zero terms
  - Denominator is a product of first-order pole terms
  - $j\omega$ is a point on the imaginary axis
  - $(j\omega - z_i)$ represents a ***vector*** from z_i to $j\omega$
  - $(j\omega - p_i)$ represents a ***vector*** from p_i to $j\omega$
- ***Gain*** is given by

$$|G(j\omega)| = \frac{|\prod_i (j\omega - z_i)|}{|\prod_i (j\omega - p_i)|}$$

- ***Phase*** can be calculated as

$$\angle G(j\omega) = \Sigma \angle (j\omega - z_i) - \Sigma \angle (j\omega - p_i)$$

- Possible to evaluate the frequency response graphically from a pole/zero diagram
  - Not done in practice, but provides useful insight

Bode Construction from Pole/Zero Plots

- Consider the following system:

$$G(j\omega) = \frac{(j\omega + 3)}{(j\omega + 2 + j1.75)(j\omega + 2 - j1.75)}$$

- Evaluate at $\omega = 2.5 rad/sec$
- **Gain**:

$$|G(j2.5)| = \frac{|3+j2.5|}{|2+j4.25||2+j0.75|}$$

$$|G(j2.5)| = \frac{3.9}{4.7 \cdot 2.1}$$

$$|G(j2.5)| = 0.389 \ \rightarrow -8.2dB$$

- **Phase**:

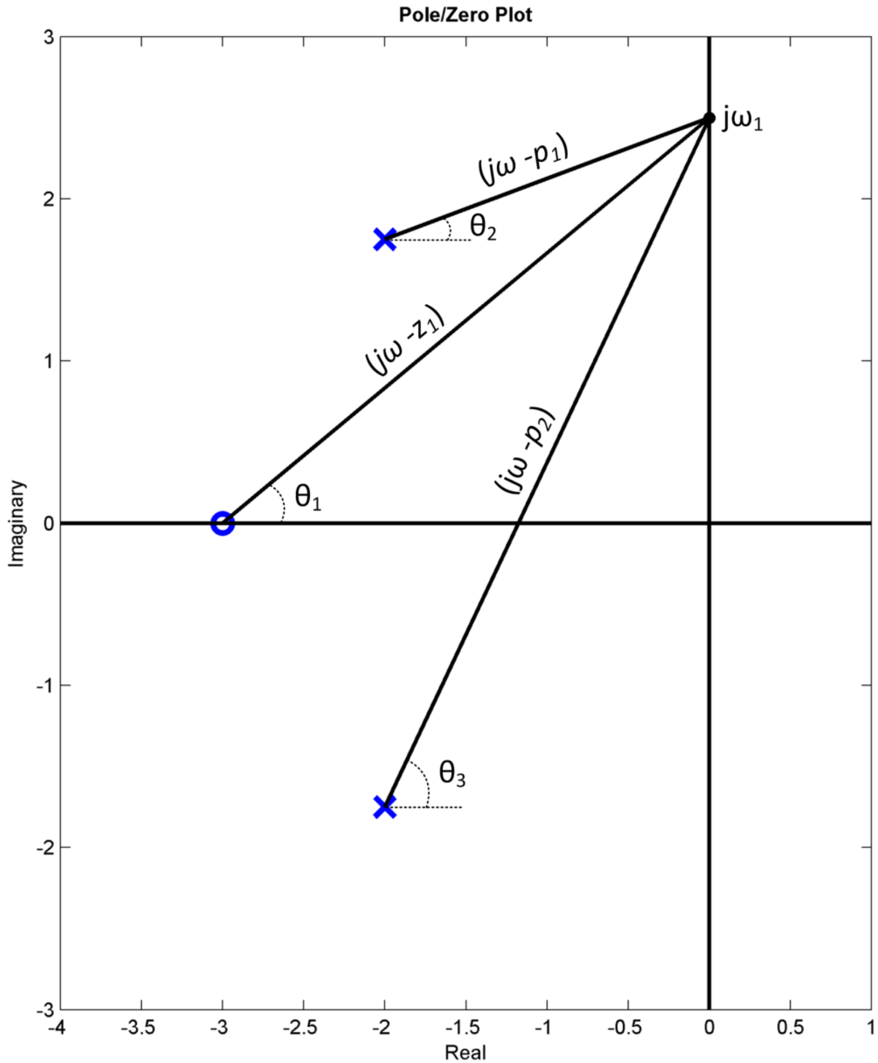
$$\angle G(j2.5) = \theta_1 - \theta_2 - \theta_3$$

$$\theta_1 = \angle(3 + j2.5) = 39.8°$$

$$\theta_2 = \angle(2 + j0.75) = 20.6°$$

$$\theta_3 = \angle(2 + j4.25) = 64.8°$$

$$\angle G(j2.5) = -45.5°$$



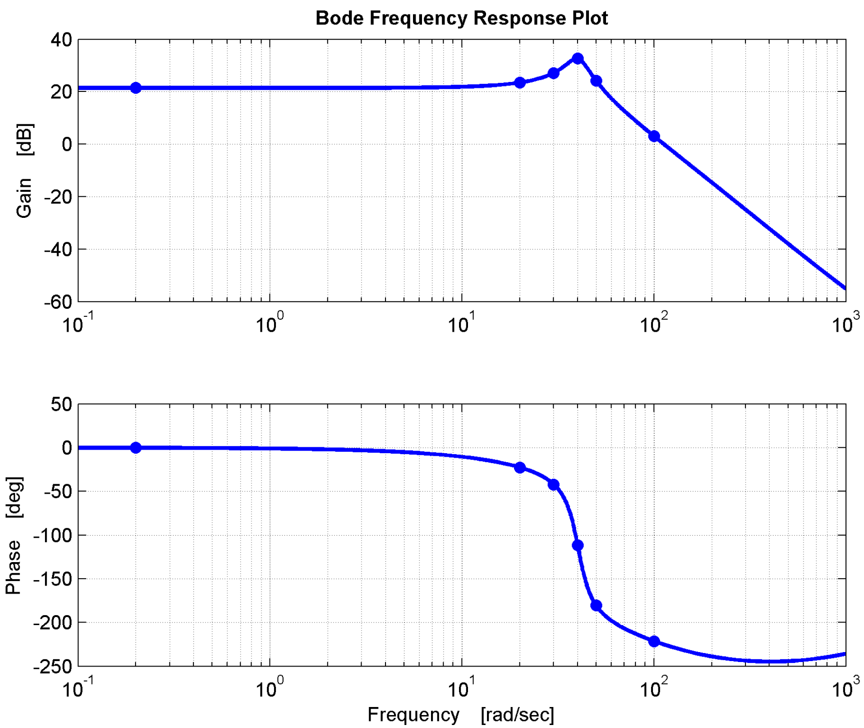
Polar Frequency Response Plots

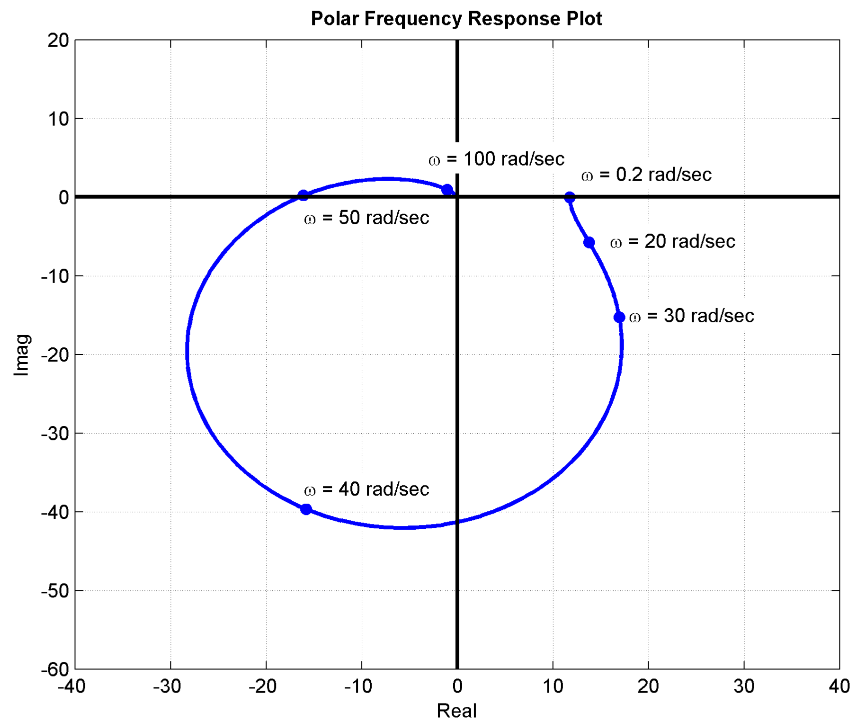
Polar Frequency Response Plots

- $G(j\omega)$ is a complex function of frequency
  - Typically plot as Bode plots
    - Magnitude and phase plotted separately
    - Aids visualization of system behavior
- A ***real*** and an ***imaginary part*** at each value of ω
  - A point in the complex plane at each frequency
  - Defines a curve in the complex plane
  - A ***polar plot***
  - ***Parametrized*** by frequency – not as easy to distinguish frequency as on a Bode plot
- Polar plots are not terribly useful as a means of displaying a frequency response
  - Useful in control system design – Nyquist stability criterion

Polar Frequency Response Plots

- Identical frequency responses plotted two ways:
  - Bode plot and polar plot
- Note uneven frequency spacing along polar plot curve
  - Dependent on frequency rates of change of gain and phase





Frequency and Time Domains

A system's frequency response and it's various time-domain responses are simply different perspectives on the same dynamic behavior.

Frequency and Time Domains

- We've seen many ways we can represent a system
  - n^{th}-order differential equation
  - Bond-graph model
  - State-variable model
  - Impulse response
  - Step response
  - Transfer function
  - Frequency response/Bode plot

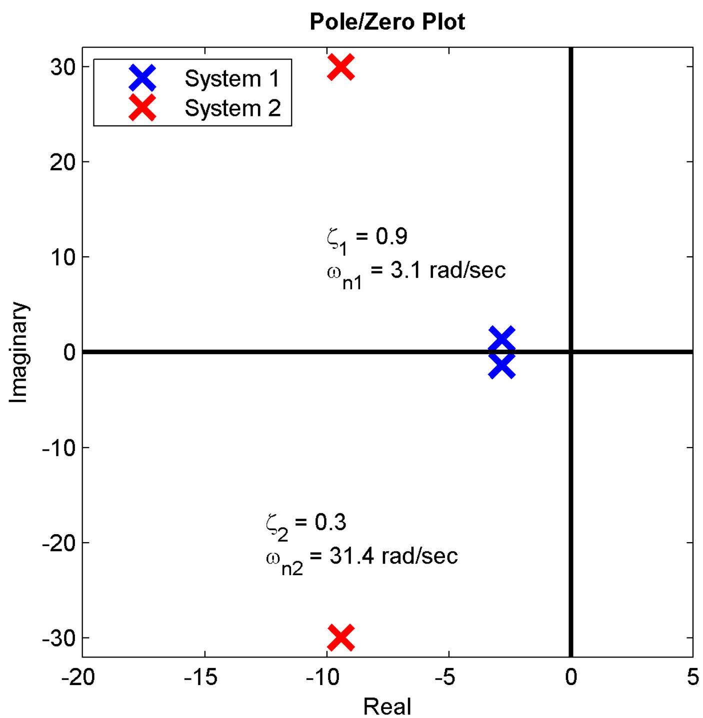
(Time-domain representations: n^{th}-order differential equation, Bond-graph model, State-variable model, Impulse response, Step response)

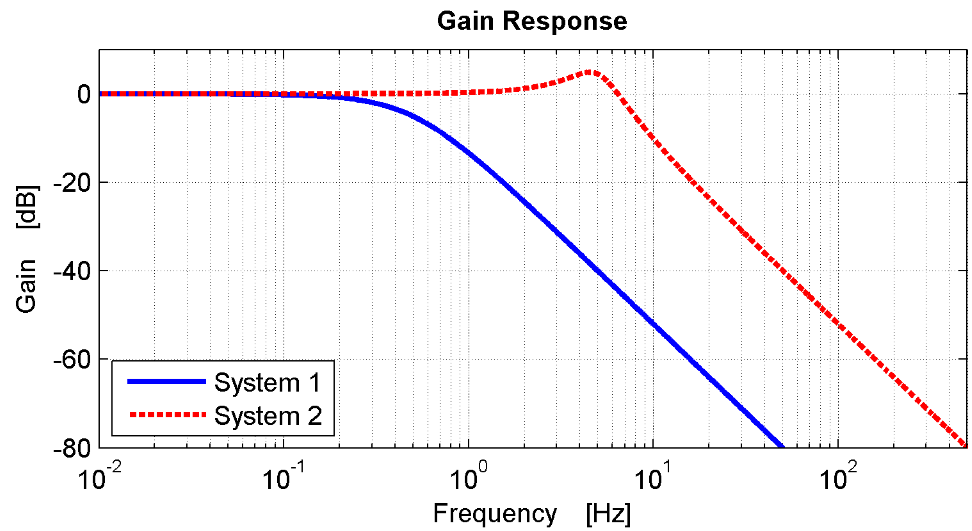
(Frequency-domain representations: Transfer function, Frequency response/Bode plot)

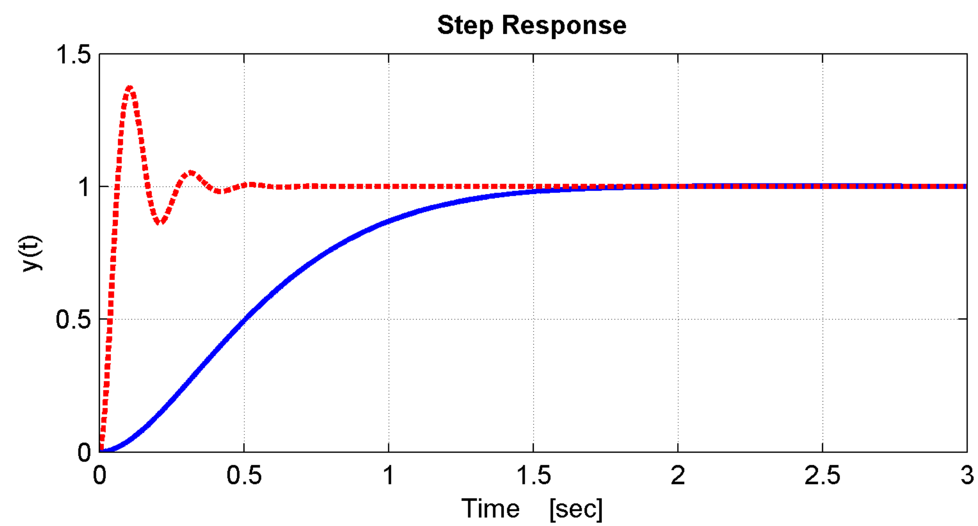
- All are valid and complete models
  - They all contain the same information in different forms
  - Different ways of looking at the same thing

Time/Frequency Domain Correlation

- $$G_1(s) = \frac{9.87}{s^2+5.655s+9.87}$$
- $$G_2(s) = \frac{987}{s^2+18.85s+987}$$







Frequency-Domain Analysis in MATLAB

As was the case for time-domain simulation, MATLAB has some useful functions for simulating system behavior in the frequency domain as well.

System Objects

- MATLAB has data types dedicated to linear system models
- Two primary system model objects:
  - ***State-space model***
  - ***Transfer function model***
- Objects created by calling MATLAB functions
  - `ss.m` – creates a state-space model
  - `tf.m` – creates a transfer function model

State-Space Model – `ss(…)`


```
sys = ss(A,B,C,D)
```

- `A`: system matrix - $n \times n$
- `B`: input matrix - $n \times m$
- `C`: output matrix - $p \times n$
- `D`: feed-through matrix - $p \times m$
- `sys`: state-space model object

- State-space model object will be used as an input to other MATLAB functions

Transfer Function Model – `tf(…)`


```
sys = tf(Num,Den)
```

- `Num`: vector of numerator polynomial coefficients
- `Den`: vector of denominator polynomial coefficients
- `sys`: transfer function model object

- Transfer function is assumed to be of the form

$$G(s) = \frac{b_1 s^r + b_2 s^{r-1} + \cdots + b_r s + b_{r+1}}{a_1 s^n + a_2 s^{n-1} + \cdots + a_n s + a_{n+1}}$$

- Inputs to `tf(…)` are
  - `Num = [b1,b2,…,br+1];`
  - `Den = [a1,a2,…,an+1];`

Frequency Response Simulation – `bode(…)`


```
[mag,phase] = bode(sys,w)
```

- `sys`: system model – state-space, transfer function, or other
- `w`: *optional* frequency vector – in rad/sec
- `mag`: system gain response vector
- `phase`: system phase response vector – in degrees

- If no outputs are specified, bode response is automatically plotted – preferable to plot yourself
- Frequency vector input is optional
  - If not specified, MATLAB will generate automatically

- May need to do: `squeeze(mag)` and `squeeze(phase)` to eliminate singleton dimensions of output matrices

Log-spaced Vectors – `logspace(…)`


```
f=logspace(x0,x1,N)
```

- `x0`: first point in `f` is 10^{x_0}
- `x1`: last point in `f` is 10^{x_1}
- `N`: number of points in `f`
- `f`: vector of logarithmically-spaced points

- Generates N logarithmically-spaced points between 10^{x_0} and 10^{x_1}
- Useful for generating independent-variable vectors for log plots (e.g., frequency vectors for bode plots)
  - Linearly spaced on a logarithmic axis