# SECTION 3: STABILITY

ESE 430 – Feedback Control Systems

# Introduction

# Stability



- Consider the following 2nd-order systems

$$G_1(s) = \frac{15}{(s+3)(s+5)} \quad \text{and} \quad G_2(s) = \frac{8}{s^2+4s+8}$$

- $G_1(s)$ has two real poles:

$$s_1 = -3 \quad \text{and} \quad s_2 = -5$$

- $G_2(s)$ has a complex-conjugate pair of poles:

$$s_{1,2} = -2 \pm j2$$

- The step response of each system is:

$$y_1(t) = 1.5e^{-5t} - 2.5e^{-3t} + 1$$
$$y_2(t) = -e^{-2t}[\cos(2t) + \sin(2t)] + 1$$

# Stability



- Both step responses are a superposition of:
  - ***Natural response*** (transient)
  - ***Driven*** or ***forced response*** (steady-state)

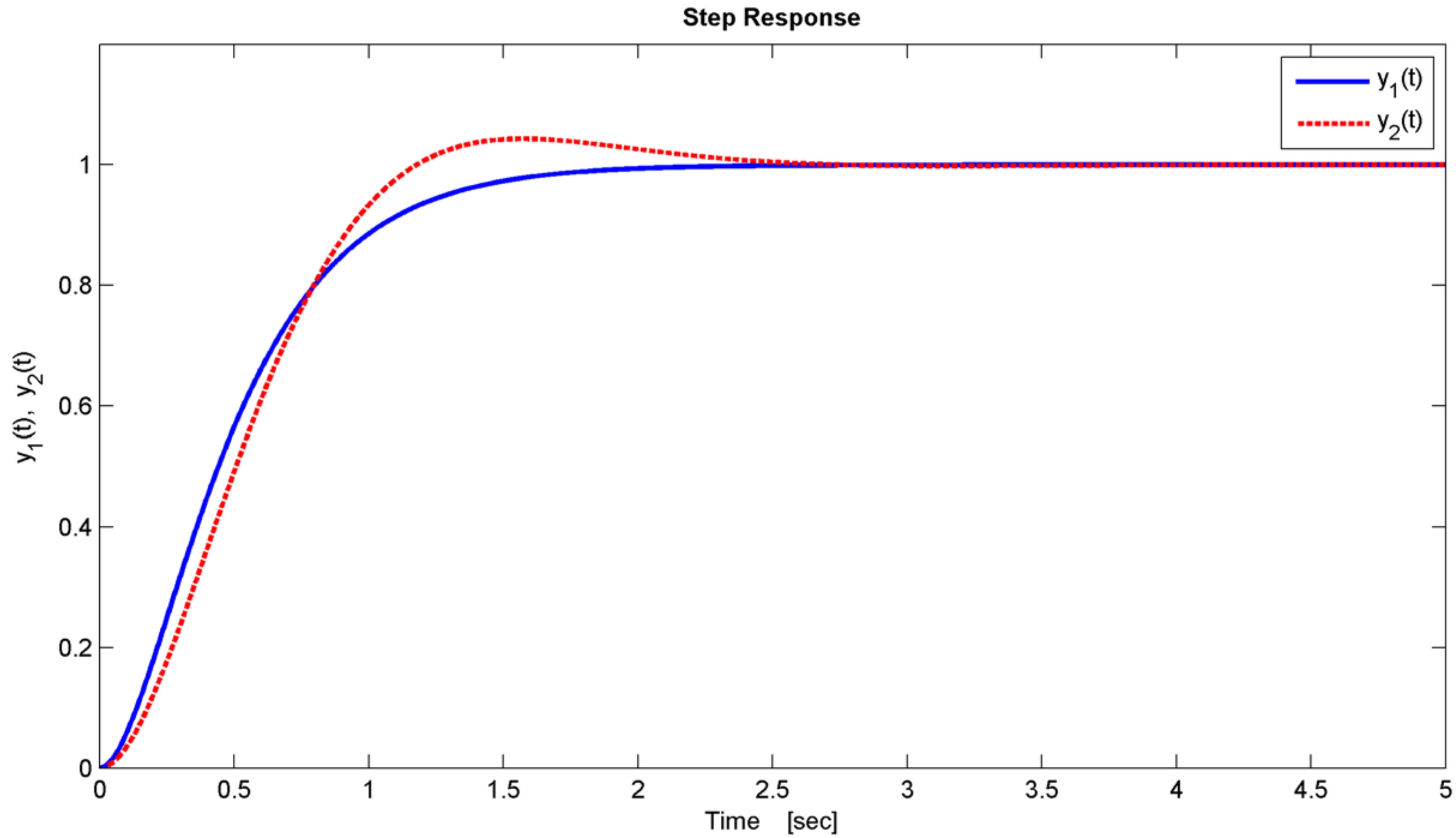
| Natural Response | Driven Response |
|---|---|
| $y_1(t) = 1.5e^{-5t} - 2.5e^{-3t}$ | $+1$ |
| $y_2(t) = -e^{-2t}[\cos(2t) + \sin(2t)]$ | $+1$ |

- In both cases, the natural response decays to zero as $t \to \infty$

# Stability



- Both step responses are characteristic of ***stable*** systems



# Stability



- Now, consider the following similar-looking systems:

$$G_3(s) = \frac{15}{(s-3)(s-5)} \quad \text{and} \quad G_4(s) = \frac{8}{s^2-4s+8}$$

- $G_3(s)$ has two real poles

$$s_1 = 3 \quad \text{and} \quad s_2 = 5$$

- $G_4(s)$ has a complex-conjugate pair of poles

$$s_{1,2} = 2 \pm j2$$

- The step responses of these systems are:

$$y_3(t) = 1.5e^{5t} - 2.5e^{3t} + 1$$

$$y_4(t) = -e^{2t}[\cos(2t) + \sin(2t)] + 1$$

# Stability



- Again, step responses consist of a natural response component and a driven component

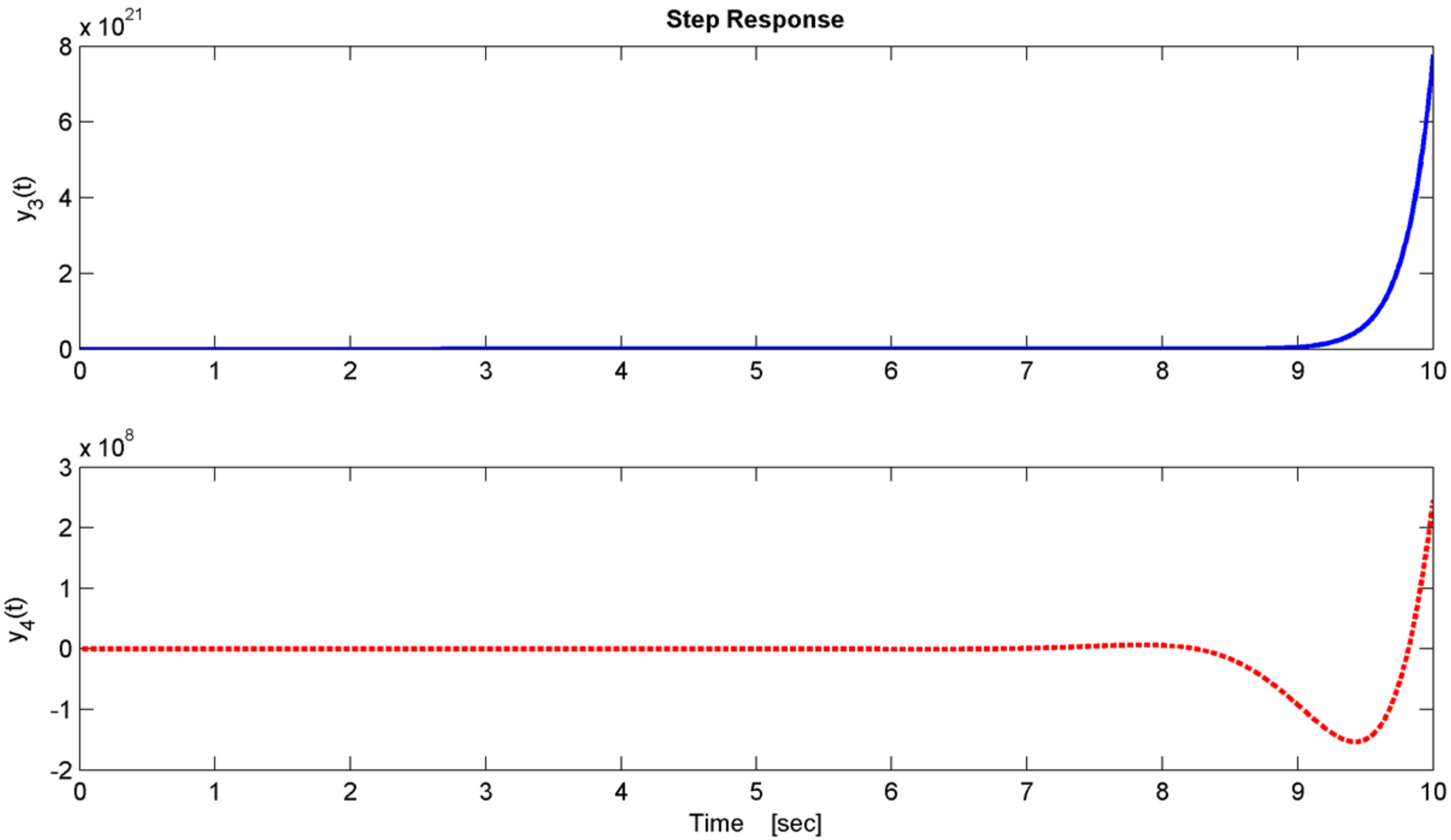
| Natural Response | Driven Response |
|---|---|
| $y_1(t) = 1.5e^{5t} - 2.5e^{3t}$ | $+1$ |
| $y_2(t) = -e^{2t}[\cos(2t) + \sin(2t)]$ | $+1$ |

- Now, as $t \to \infty$, the natural responses do not decay to zero
  - They blow up – why?
  - ***Exponential terms are positive***

# Stability



- Step responses characteristic of ***unstable*** systems



# Stability



- Why are the exponential terms positive?
  - Determined by the system poles
- For the over-damped system, the poles are

$$s_1 = \sigma_1 \quad \text{and} \quad s_2 = \sigma_2$$

- And, the step response is

$$y(t) = r_1 e^{\sigma_1 t} + r_2 e^{\sigma_2 t} + r_3$$

- For the under-damped system, the poles are

$$s_{1,2} = \sigma \pm j\omega_d$$

- The step response is

$$y(t) = r_1 e^{\sigma t} \cos(\omega_d t) + r_2 e^{\sigma t} \sin(\omega_d t) + r_3$$

# Stability and System Poles



- Sign of the exponentials determined by $\sigma$, the ***real part of the system poles***

- If $\sigma < 0$
  - Pole is in the ***left half-plane*** (LHP)
  - Natural response $\rightarrow 0$ as $t \rightarrow \infty$
  - System is ***stable***

- If $\sigma > 0$
  - Pole is in the ***right half-plane*** (RHP)
  - Natural response $\rightarrow \infty$ as $t \rightarrow \infty$
  - System is ***unstable***

# Purely-Imaginary Poles



- ***LHP poles*** correspond to ***stable*** systems
- ***RHP poles*** correspond to ***unstable*** systems
- It seems that the imaginary axis is the boundary for stability
- What if poles are on the imaginary axis?
- Consider the following system

$$G_5(s) = \frac{4}{s^2 + 4}$$

- Two purely-imaginary poles

$$s_{1,2} = \pm j2$$

# Marginal Stability



- Step response for this ***undamped system*** is

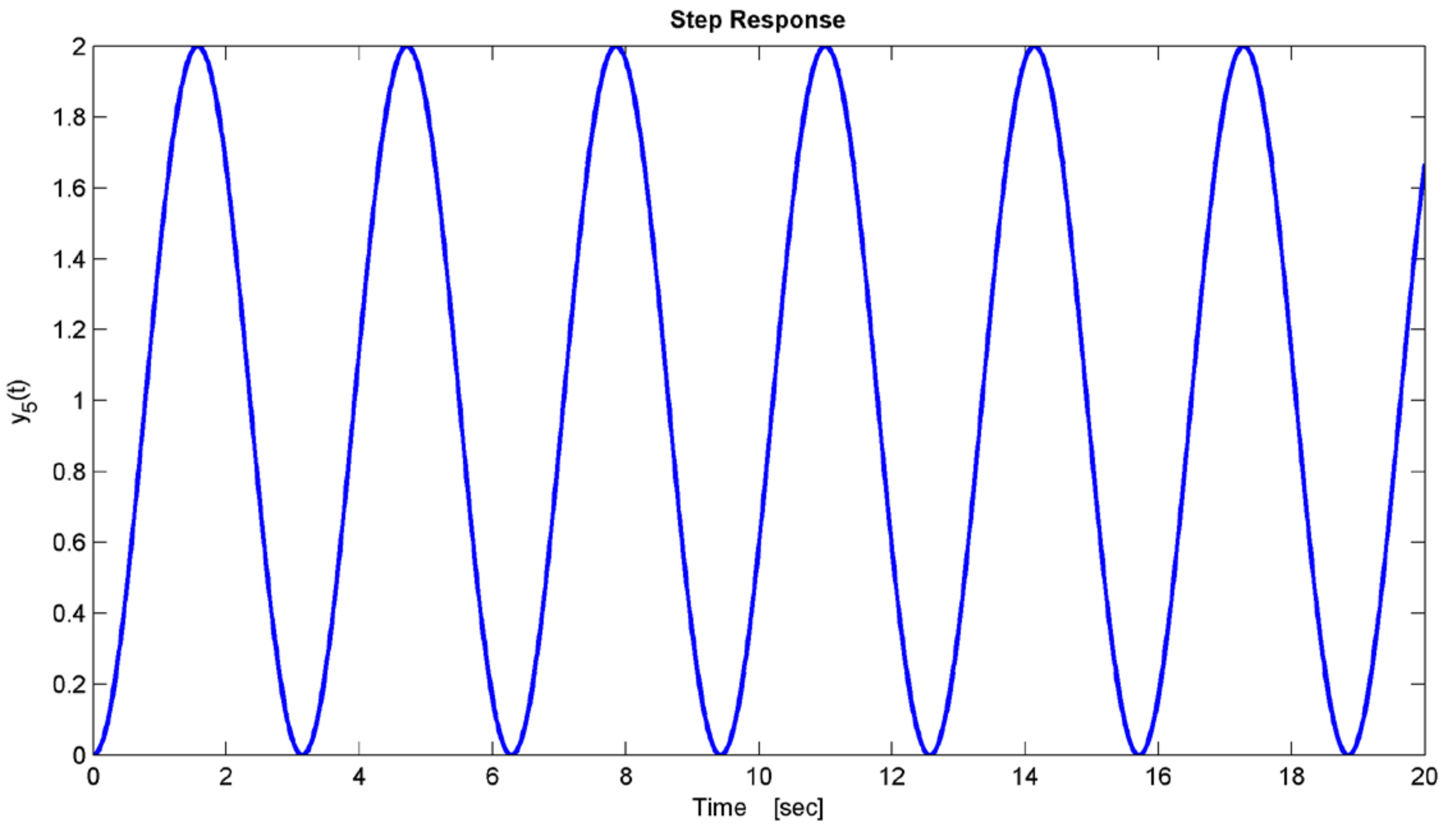
| Natural Response | Driven Response |
|---|---|
| $y_5(t) = -\cos(2t)$ | $+1$ |

- Natural response neither decays to zero, nor grows without bound
  - Oscillates indefinitely
  - System is ***marginally stable***

# Marginal Stability



- Step response is characteristic of a ***marginally-stable*** system



# Repeated Imaginary Poles



- We'll look at one more interesting case before presenting a formal definition for stability
- Consider the following system

$$G_6(s) = \frac{16}{s^4 + 8s^2 + 16} = \frac{16}{(s^2+4)^2}$$

- Repeated poles on the imaginary axis

$$s_{1,2} = \pm j2 \quad \text{and} \quad s_{3,4} = \pm j2$$

- The step response for this system is

| **Natural Response** | **Driven Response** |
|---|---|
| $y_6(t) = -\cos(2t) - t \cdot \sin(2t)$ | $+1$ |

# Repeated Imaginary Poles



$$y_6(t) = -\cos(2t) - t \cdot \sin(2t) + 1$$

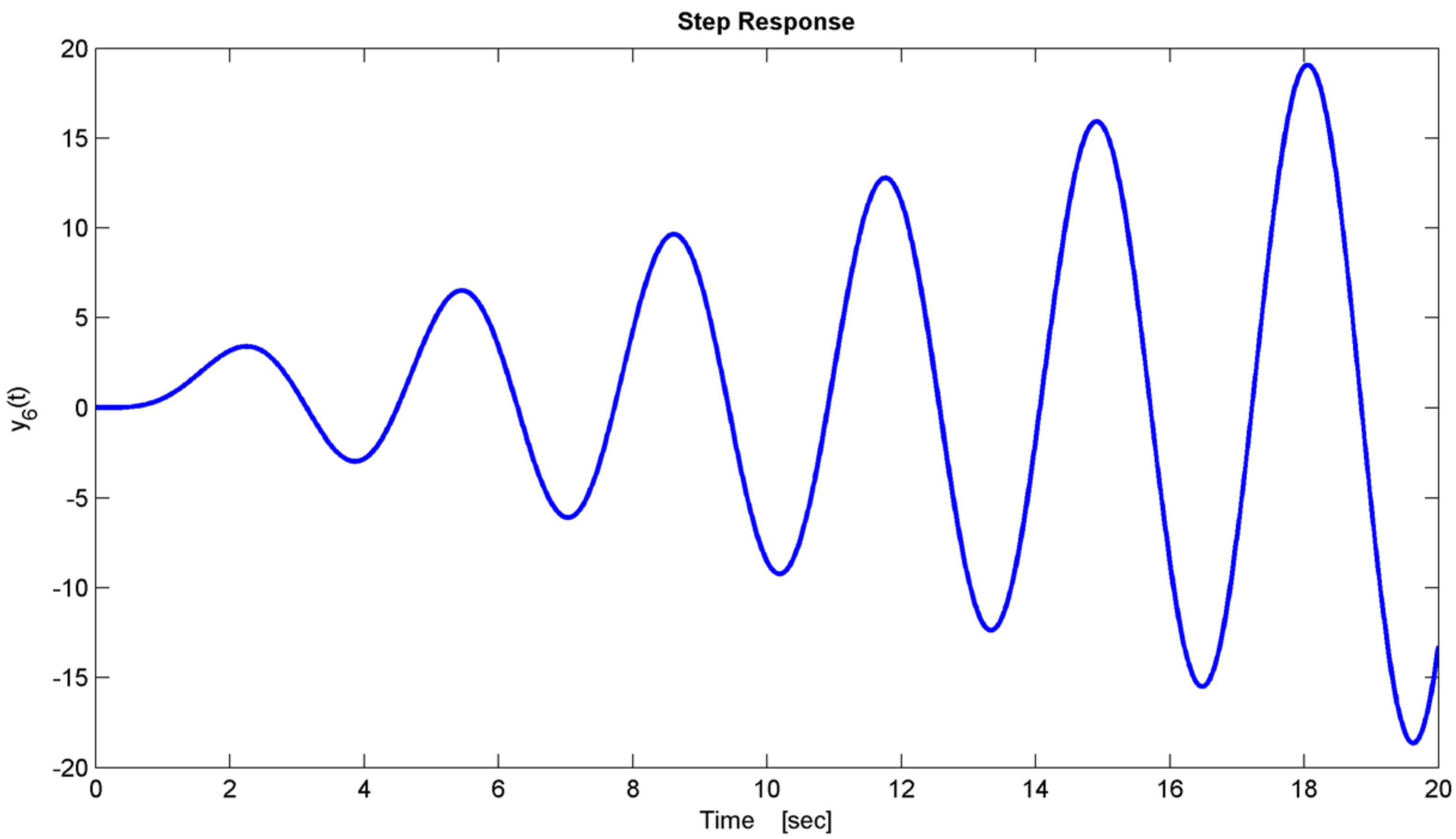
- Multiplying time factor causes the natural response to grow without bound
  - An ***unstable system***
  - Results from repeated poles

- ***Multiple identical poles on the imaginary axis implies an unstable system***

# Repeated Imaginary Poles



- Step response shows that the system is unstable



# Definitions of Stability

# Definitions of Stability – Natural Response



- We know that system response is the sum of a natural response and a driven response
- Can define the categories of stability based on the ***natural response***:
- ***<u>Stable</u>***
  - A system is stable if its natural response $\rightarrow 0$ as $t \rightarrow \infty$
- ***<u>Unstable</u>***
  - A system is unstable if its natural response $\rightarrow \infty$ as $t \rightarrow \infty$
- ***<u>Marginally Stable</u>***
  - A system is marginally stable if its natural response neither decays nor grows, but remains constant or oscillates

# BIBO Stability



- Alternatively, we can define stability based on the total response
- ***Bounded-input, bounded-output*** (***BIBO***) ***stability***
- ***<u>Stable</u>***
  - A system is stable if *every* bounded input yields a bounded output
- ***<u>Unstable</u>***
  - A system is unstable if *any* bounded input yields an unbounded output

# Closed-Loop Poles and Stability



- ***Stable***
  - A stable system has all of its closed-loop poles in the left-half plane
- ***Unstable***
  - An unstable system has at least one pole in the right half-plane and/or repeated poles on the imaginary axis
- ***Marginally Stable***
  - A marginally-stable system has non-repeated poles on the imaginary axis and (possibly) poles in the left half-plane

# Determining System Stability

# Determining Stability



- Stability determined by pole locations
  - Poles determined by the characteristic polynomial, $\Delta(s)$
- Factoring the characteristic polynomial will always tell us if a system is stable or not
  - Easily done with a computer or calculator
- Would like to be able to detect RHP poles without a computer
  - Form of $\Delta(s)$ may indicate RHP poles directly, or
  - Routh-Hurwitz Criterion

# Stability from $\Delta(s)$ Coefficients



- A stable system has all poles in the LHP

$$T(s) = \frac{Num(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)}$$

  - Poles: $p_i = -a_i$
  - For all LHP poles, $a_i > 0, \quad \forall i$
  - Result is that all coefficients of $\Delta(s)$ are ***positive***

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- If any coefficient of $\Delta(s)$ is ***negative***, there is at least one RHP pole, and the system is ***unstable***
- If any coefficient of $\Delta(s)$ is ***zero***, the system is ***unstable*** or, at best, ***marginally stable***
- If all coefficients of $\Delta(s)$ are ***positive***, the system may be ***stable*** or may be ***unstable***

# Routh-Hurwitz Criterion



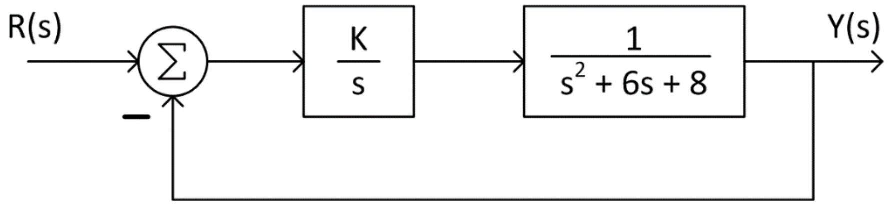
- Need a method to detect RHP poles if all coefficients of $\Delta(s)$ are positive:
  - ***Routh-Hurwitz criterion***

- General procedure:
  1. Generate a ***Routh table*** using the characteristic polynomial of the closed-loop system
  2. Apply the ***Routh-Hurwitz criterion*** to interpret the table and determine the *number* (not locations) of RHP poles

# Routh-Hurwitz – Utility



- Routh-Hurwitz was very useful for determining stability in the days before computers
  - Factoring polynomials by hand is difficult
- Still useful for ***design***, e.g.:




$$T(s) = \frac{K}{s^3 + 6s^2 + 8s + K}$$

- Stable for some range of gain, $K$, but unstable beyond that range
- Routh-Hurwitz allows us to determine that range

# Routh Table



- Consider a 4th-order closed-loop transfer function:

$$T(s) = \frac{Num(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

- Routh table has one row for each power of $s$ in $\Delta(s)$
  - First row contains coefficients of even powers of $s$ (odd if the order of $\Delta(s)$ is odd)
  - Second row contains coefficients of odd (even) powers of $s$
  - Fill in zeros if needed – if even order

| | | | |
|---|---|---|---|
| $s^4$ | $a_4$ | $a_2$ | $a_0$ |
| $s^3$ | $a_3$ | $a_1$ | $0$ |
| $s^2$ | | | |
| $s^1$ | | | |
| $s^0$ | | | |

# Routh Table



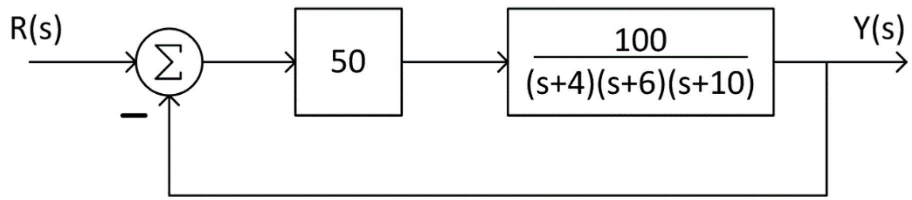
- Remaining table entries calculated using entries from two preceding rows as follows:

| | | | |
|---|---|---|---|
| $s^4$ | $a_4$ | $a_2$ | $a_0$ |
| $s^3$ | $a_3$ | $a_1$ | $0$ |
| $s^2$ | $-\dfrac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$ | $-\dfrac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$ | $-\dfrac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_3 = 0$ |
| $s^1$ | $-\dfrac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$ | $-\dfrac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_2 = 0$ | $-\dfrac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_3 = 0$ |
| $s^0$ | $-\dfrac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$ | $-\dfrac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_2 = 0$ | $-\dfrac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_3 = 0$ |

# Routh Table – Example



- Consider the following feedback system




- The closed-loop transfer function is

$$T(s) = \frac{5000}{s^3 + 20s^2 + 124s + 5240}$$

- The first two rows of the Routh table are

| | | |
|---|---|---|
| $s^3$ | 1 | 124 |
| $s^2$ | ~~20~~ 1 | ~~5240~~ 262 |

- Note that we can simplify by scaling an entire row by any factor

# Routh Table – Example



- Calculate the remaining table entries:

| | | |
|---|---|---|
| $s^3$ | $1$ | $124$ |
| $s^2$ | $\cancel{20}\ 1$ | $\cancel{5240}\ 262$ |
| $s^1$ | $-\dfrac{\begin{vmatrix} 1 & 124 \\ 1 & 262 \end{vmatrix}}{1} = -138$ | $-\dfrac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$ |
| $s^0$ | $-\dfrac{\begin{vmatrix} 1 & 262 \\ -138 & 0 \end{vmatrix}}{-138} = 262$ | $-\dfrac{\begin{vmatrix} 1 & 0 \\ -138 & 0 \end{vmatrix}}{1} = 0$ |

- How do we interpret this table?
  - ***Routh-Hurwitz criterion***

# Routh-Hurwitz Criterion



- ***Routh-Hurwitz Criterion***
  - *The number of poles in the RHP is equal to the number of sign changes in the first column of the Routh table*

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- Apply this criterion to our example:

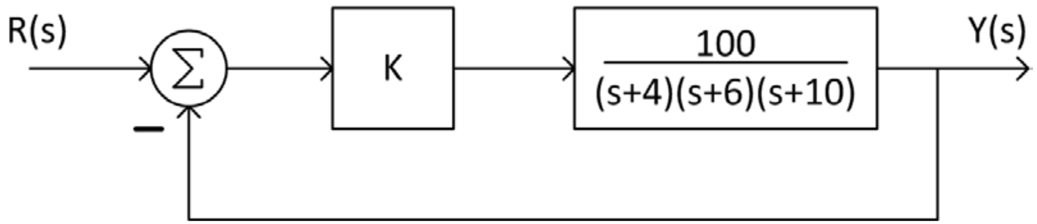
| | | |
|---|---|---|
| $s^3$ | $1$ | $124$ |
| $s^2$ | $1$ | $262$ |
| $s^1$ | $-138$ | $0$ |
| $s^0$ | $262$ | $0$ |

- Two sign changes in the first column indicate ***two RHP poles*** → system is ***unstable***

# Routh-Hurwitz – Stability Requirements



- Consider the same system, where controller gain is left as a parameter




- Closed-loop transfer function:

$$T(s) = \frac{100K}{s^3 + 20s^2 + 124s + 240 + 100K}$$

- Plant itself is stable
  - Presumably there is some range of gain, $K$, for which the closed-loop system is also stable
  - Use ***Routh-Hurwitz*** to determine this range

# Routh-Hurwitz – Stability Requirements



$$T(s) = \frac{100K}{s^3 + 20s^2 + 124s + 240 + 100K}$$

- Create the Routh table

| | | |
|---|---|---|
| $s^3$ | $1$ | $124$ |
| $s^2$ | ~~$20$~~ $1$ | ~~$240 + 100K$~~ $12 + 5K$ |
| $s^1$ | $-\frac{\begin{vmatrix} 1 & 124 \\ 1 & 12 + 5K \end{vmatrix}}{1} = 112 - 5K$ | $-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$ |
| $s^0$ | $-\frac{\begin{vmatrix} 1 & 12 + 5K \\ 112 - 5K & 0 \end{vmatrix}}{112 - 5K} = 12 + 5K$ | $-\frac{\begin{vmatrix} 1 & 0 \\ -138 & 0 \end{vmatrix}}{1} = 0$ |

# Routh-Hurwitz – Stability Requirements



| | | |
|---|---|---|
| $s^3$ | $1$ | $124$ |
| $s^2$ | $1$ | $12+5K$ |
| $s^1$ | $112-5K$ | $0$ |
| $s^0$ | $12+5K$ | $0$ |

- Since $K>0$, only the third element in the first column can be negative
- ***Stable*** for

$$112-5K>0$$

$$K<22.4$$

- ***Unstable*** (two RHP poles) for

$$112-5K<0$$

$$K>22.4$$

# Routh Table – Special Cases



- Two special cases can arise when creating a Routh table:

  1. ***A zero in only the first column of a row***
     - Divide-by-zero problem when forming the next row

  2. ***An entire row of zeros***
     - Indicates the presence of pairs of poles that are mirrored about the imaginary axis

- We'll next look at methods for dealing with each of these scenarios

# Routh Table – Zero in the First Column



- If a zero appears in the first column
  1. Replace the zero with $\pm\epsilon$
  2. Complete the Routh table as usual
  3. Take the limit as $\epsilon \rightarrow 0$
  4. Evaluate the sign of the first-column entries

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- For example:

$$T(s) = \frac{10}{s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9}$$

- First two rows in the Routh table:

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 2 | 6 |
| $s^4$ | ~~3~~ 1 | ~~6~~ 2 | ~~9~~ 3 |

# First-Column Zero – Example



| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $2$ | $6$ |
| $s^4$ | $1$ | $2$ | $3$ |
| $s^3$ | $-\frac{\begin{vmatrix}1 & 2\\ 1 & 2\end{vmatrix}}{1} = \cancel{0}\ \epsilon$ | $-\frac{\begin{vmatrix}1 & 6\\ 1 & 3\end{vmatrix}}{1} = 3$ | $-\frac{\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$ |

- Replace the first-column zero with $\epsilon$ and proceed as usual

| | | | |
|---|---|---|---|
| $s^2$ | $-\frac{\begin{vmatrix}1 & 2\\ \epsilon & 3\end{vmatrix}}{\epsilon} = \frac{2\epsilon - 3}{\epsilon}$ | $-\frac{\begin{vmatrix}1 & 3\\ \epsilon & 0\end{vmatrix}}{\epsilon} = 3$ | $-\frac{\begin{vmatrix}1 & 0\\ \epsilon & 0\end{vmatrix}}{\epsilon} = 0$ |
| $s^1$ | $-\frac{\begin{vmatrix}\epsilon & 3\\ \frac{2\epsilon - 3}{\epsilon} & 3\end{vmatrix}}{\frac{2\epsilon - 3}{\epsilon}} = 3\epsilon - \frac{3\epsilon^2}{2\epsilon - 3}$ | $-\frac{\begin{vmatrix}\epsilon & 0\\ \frac{2\epsilon - 3}{\epsilon} & 0\end{vmatrix}}{\frac{2\epsilon - 3}{\epsilon}} = 0$ | $-\frac{\begin{vmatrix}\epsilon & 0\\ \frac{2\epsilon - 3}{\epsilon} & 0\end{vmatrix}}{\frac{2\epsilon - 3}{\epsilon}} = 0$ |

- Continuing on the next page …

# First-Column Zero – Example



| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $2$ | $6$ |
| $s^4$ | $1$ | $2$ | $3$ |
| $s^3$ | $\epsilon$ | $3$ | $0$ |
| $s^2$ | $\frac{2\epsilon-3}{\epsilon}$ | $3$ | $0$ |
| $s^1$ | $3\epsilon-\frac{3\epsilon^2}{2\epsilon-3}$ | $0$ | $0$ |
| $s^0$ | $-\frac{\begin{vmatrix} \frac{2\epsilon-3}{\epsilon} & 3 \\ 3\epsilon-\frac{3\epsilon^2}{2\epsilon-3} & 0 \end{vmatrix}}{3\epsilon-\frac{3\epsilon^2}{2\epsilon-3}}=3$ | $-\frac{\begin{vmatrix} \frac{2\epsilon-3}{\epsilon} & 0 \\ 3\epsilon-\frac{3\epsilon^2}{2\epsilon-3} & 0 \end{vmatrix}}{3\epsilon-\frac{3\epsilon^2}{2\epsilon-3}}=0$ | $-\frac{\begin{vmatrix} \frac{2\epsilon-3}{\epsilon} & 0 \\ 3\epsilon-\frac{3\epsilon^2}{2\epsilon-3} & 0 \end{vmatrix}}{3\epsilon-\frac{3\epsilon^2}{2\epsilon-3}}=0$ |

- Next, take the limit as $\epsilon \to 0$

# First-Column Zero – Example



- Taking the limit as $\epsilon \to 0$ and looking at the first column:

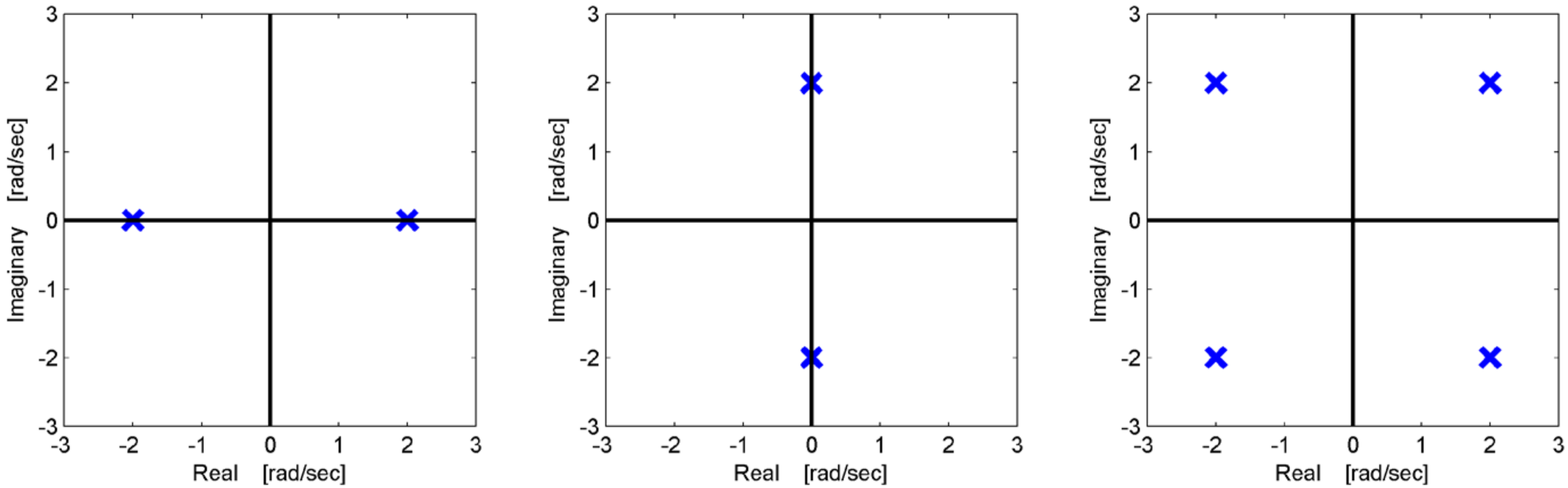
$$
\begin{array}{c|c}
s^5 & 1 \\
s^4 & 1 \\
s^3 & \epsilon \\
s^2 & \dfrac{2\epsilon - 3}{\epsilon} \\
s^1 & 3\epsilon - \dfrac{3\epsilon^2}{2\epsilon - 3} \\
s^0 & 3
\end{array}
\quad \xrightarrow{\lim_{\epsilon \to 0}} \quad
\begin{array}{c|c}
s^5 & 1 \\
s^4 & 1 \\
s^3 & 0 \\
s^2 & -\infty \\
s^1 & 0 \\
s^0 & 3
\end{array}
$$

- Two sign changes
  - Two RHP poles
  - System is ***unstable***

# Routh Table – Row of Zeros



- A whole row of zeros indicates the presence of pairs of poles that are mirrored about the imaginary axis:




- At best, the system is ***marginally stable***
- Use a Routh table to determine if it is ***unstable***

# Routh Table – Row of Zeros



- If an entire row of zeros appears in a Routh table
  1. Create an ***auxiliary polynomial*** from the row above the row of zeros, skipping every other power of $s$
  2. Differentiate the auxiliary polynomial w.r.t. $s$
  3. Replace the zero row with the coefficients of the resulting polynomial
  4. Complete the Routh table as usual
  5. Evaluate the sign of the first-column entries

# Row of Zeros – Example



- Consider the following system

$$T(s) = \frac{1}{s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12}$$

- The first few rows of the Routh table:

$$\begin{array}{c|ccc} s^5 & 1 & 11 & 28 \\ s^4 & 5 & 23 & 12 \\ s^3 & -\frac{\begin{vmatrix}1 & 11\\5 & 23\end{vmatrix}}{5} = \cancel{6.4}\ 1 & -\frac{\begin{vmatrix}1 & 28\\5 & 12\end{vmatrix}}{5} = \cancel{25.6}\ 4 & -\frac{\begin{vmatrix}1 & 0\\5 & 0\end{vmatrix}}{5} = 0 \\ s^2 & -\frac{\begin{vmatrix}5 & 23\\1 & 4\end{vmatrix}}{1} = \cancel{3}\ 1 & -\frac{\begin{vmatrix}5 & 12\\1 & 0\end{vmatrix}}{1} = \cancel{12}\ 4 & -\frac{\begin{vmatrix}5 & 0\\1 & 0\end{vmatrix}}{1} = 0 \end{array}$$

- Continuing on the next page …

# Row of Zeros – Example



$$
\begin{array}{c|ccc}
s^5 & 1 & 11 & 28 \\
s^4 & 5 & 23 & 12 \\
s^3 & 1 & 4 & 0 \\
s^2 & 1 & 4 & 0 \\
s^1 & -\frac{\begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}}{1} = 0 & -\frac{\begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}}{1} = 0 & -\frac{\begin{vmatrix} 5 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0
\end{array}
$$

- A row of zeros has appeared
  - Create an auxiliary polynomial from the $s^2$ row

$$P(s) = s^2 + 4$$

  - Differentiate

$$\frac{dP}{ds} = 2s$$

  - Replace the $s^1$ row with the $dP/ds$ coefficients

# Row of Zeros – Example



$$\frac{dP}{ds} = 2s$$

- Replacing the $s^1$ row with the coefficients of $dP/ds$

$$\begin{array}{c|ccc}
s^5 & 1 & 11 & 28 \\
s^4 & 5 & 23 & 12 \\
s^3 & 1 & 4 & 0 \\
s^2 & 1 & 4 & 0 \\
s^1 & \not{0}\ \ 2 & 0 & 0 \\
s^0 & -\frac{\begin{vmatrix}1 & 4\\2 & 0\end{vmatrix}}{2} = 4 & -\frac{\begin{vmatrix}1 & 0\\2 & 0\end{vmatrix}}{2} = 0 & -\frac{\begin{vmatrix}1 & 0\\2 & 0\end{vmatrix}}{2} = 0
\end{array}$$

- No sign changes, so RHP poles, *but*
  - Row of zeros indicates that system is ***marginally stable***

# Stability Evaluation – Summary



- If coefficients of $\Delta(s)$ have different signs
  - System is unstable
- If some coefficients of $\Delta(s)$ are zero
  - System is, at best, marginally stable
- If all $\Delta(s)$ coefficients have the same sign
  - System may be stable or unstable
  - Generate a Routh table and apply Routh-Hurwitz criterion
  - Replace any zero first-column entries with $\epsilon$ and let take the limit as $\epsilon \rightarrow 0$
  - Replace a row of zeros with coefficients from the derivative of the auxiliary polynomial
    - If no RHP poles are detected, the system is marginally stable