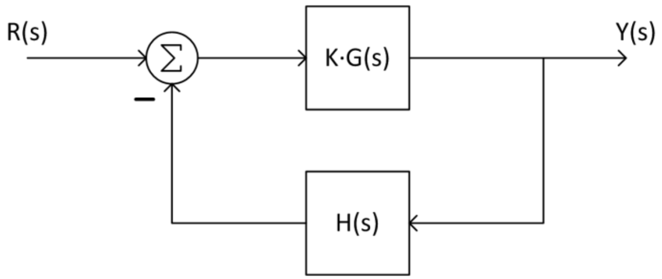
SECTION 5: ROOT-LOCUS ANALYSIS

ESE 430 – Feedback Control Systems

Introduction

Introduction

- Consider a general feedback system:



- Closed-loop transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- $G(s)$ is the forward-path transfer function
  - May include controller and plant
- $H(s)$ is the feedback-path transfer function
- Each are, in general, rational polynomials in s

$$G(s) = \frac{N_G(s)}{D_G(s)} \quad \text{and} \quad H(s) = \frac{N_H(s)}{D_H(s)}$$

Introduction

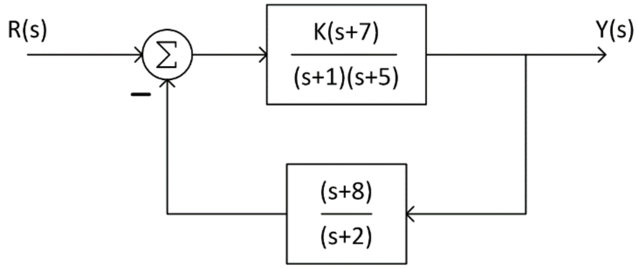
- So, the closed-loop transfer function is

$$T(s) = \frac{K\frac{N_G(s)}{D_G(s)}}{1 + K\frac{N_G(s)}{D_G(s)}\frac{N_H(s)}{D_H(s)}} = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

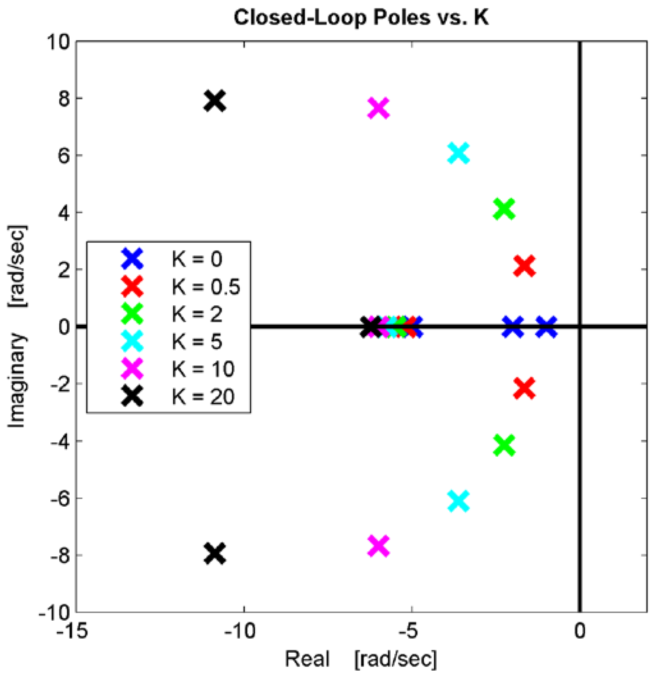
- ***Closed-loop zeros***:
  - Zeros of $G(s)$
  - Poles of $H(s)$
- ***Closed-loop poles***:
  - A function of gain, K
  - Consistent with what we’ve already seen – feedback moves poles

Closed-Loop Poles vs. Gain

- How do closed-loop poles vary as a function of K?
  - Plot for $K = 0, 0.5, 1, 2, 5, 10, 20$



- Trajectory of closed-loop poles vs. gain (or some other parameter): ***root locus***

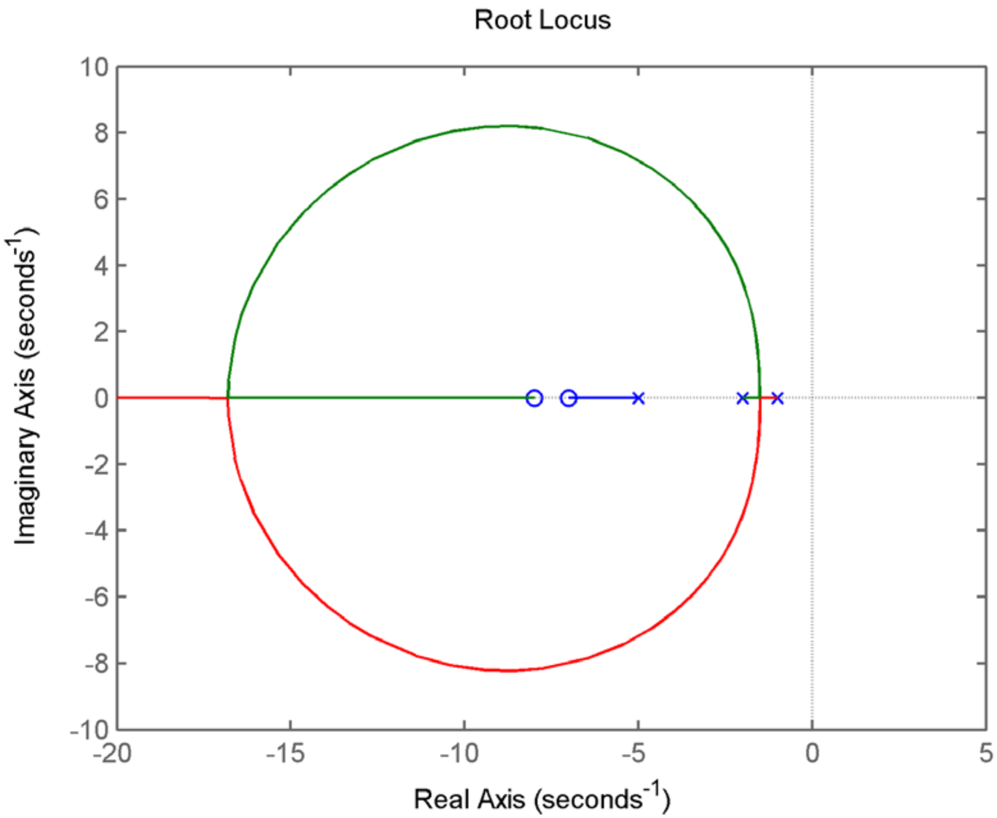


- Graphical tool to help determine the controller gain that will put poles where we want them

- We'll learn techniques for sketching this locus by hand

Root Locus

- An example of the type of root locus we'll learn to sketch by hand, as well as plot in MATLAB:



Evaluation of Complex Functions

Vector Interpretation of Complex Functions

- Consider a function of a complex variable s

$$G(s) = \frac{(s - z_1)(s - z_2)\cdots}{(s - p_1)(s - p_2)\cdots}$$

where z_i are the ***zeros*** of the function, and p_i are the ***poles*** of the function

- We can write the function as

$$G(s) = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

where m is the # of zeros, and n is the # of poles

Vector Interpretation of Complex Functions

- At any value of s, i.e. any point in the complex plane, $G(s)$ evaluates to a complex number
  - Another point in the complex plane with magnitude and phase

$$G(s) = M\angle\theta$$

where

$$M = |G(s)| = \frac{\left|\prod_{i=1}^{m}(s - z_i)\right|}{\left|\prod_{i=1}^{n}(s - p_i)\right|}$$

and

$$\theta = \angle\left[\prod_{i=1}^{m}(s - z_i)\right] - \angle\left[\prod_{i=1}^{n}(s - p_i)\right]$$

$$\theta = \sum_{i=1}^{m}\angle(s - z_i) - \sum_{i=1}^{n}\angle(s - p_i)$$

Vector Interpretation of Complex Functions

- Each term $(s - z_i)$ represents a ***vector*** from z_i to the point, s, at which we're evaluating $G(s)$
- Each $(s - p_i)$ represents a ***vector*** from p_i to s
- For example:

$$G(s) = \frac{(s+3)}{(s+4)(s^2+2s+5)}$$

  - Zero at: $s = -3$
  - Poles at: $s_{1,2} = -1 \pm j2$ and $s_3 = -4$
- Evaluate $G(s)$ at $s = -2 + j$

$$G(s)\Big|_{s=-2+j}$$

Vector Interpretation of Complex Functions

- First, evaluate the magnitude

$$|G(s)| = \frac{|s - z_1|}{|s - p_1||s - p_2||s - p_3|}$$

$$|s - z_1| = |1 + j| = \sqrt{2}$$

$$|s - p_1| = |-1 - j| = \sqrt{2}$$

$$|s - p_2| = |-1 + j3| = \sqrt{10}$$

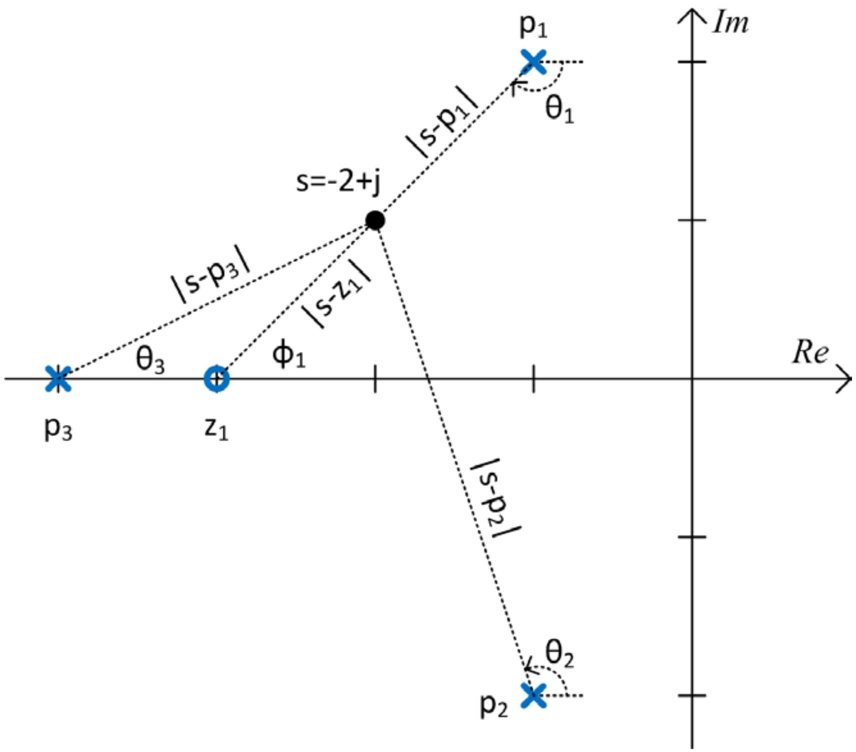
$$|s - p_3| = |2 + j| = \sqrt{5}$$

- The resulting magnitude:

$$|G(s)| = \frac{\sqrt{2}}{\sqrt{2}\sqrt{10}\sqrt{5}} = \frac{\sqrt{2}}{10}$$

$$|G(s)| = 0.1414$$

$$G(s) = |G(s)|\angle G(s)$$



Vector Interpretation of Complex Functions

- Next, evaluate the angle

$$\angle G(s) = \angle(s - z_1) - \angle(s - p_1)$$
$$-\angle(s - p_2) - \angle(s - p_3)$$

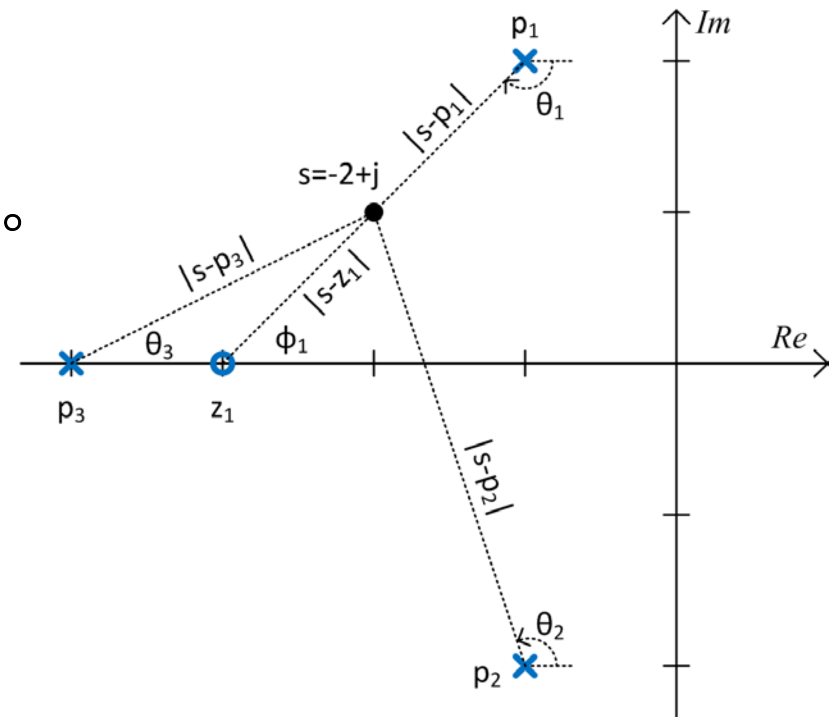
$$\angle(s - z_1) = \angle(1 + j) = 45°$$

$$\angle(s - p_1) = \angle(-1 - j) = -135°$$

$$\angle(s - p_2) = \angle(-1 + j3) = 108.4°$$

$$\angle(s - p_3) = \angle(2 + j) = 26.6°$$

$$G(s) = |G(s)|\angle G(s)$$



- The result:

$$G(s)\Big|_{s=-2+j} = 0.1414\angle 45°$$

Finite vs. Infinite Poles and Zeros

- Consider the following transfer function

$$G(s) = \frac{(s+8)}{s(s+3)(s+10)}$$

  - One ***finite zero***: $s = -8$
  - Three ***finite poles***: $s = 0$, $s = -3$, and $s = -10$
- But, as $s \to \infty$

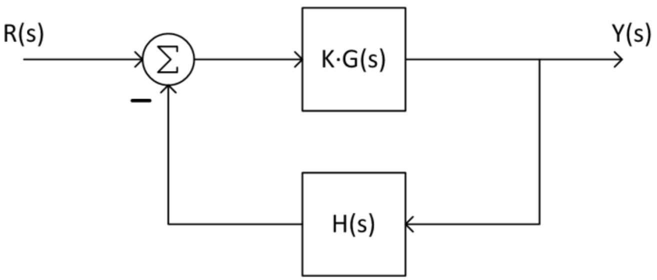
$$\lim_{s \to \infty} G(s) = \frac{\infty}{\infty^3} = 0$$

  - This implies there must be a zero at $s = \infty$
- ***All functions have an equal number of poles and zeros***
- If $G(s)$ has n poles and m zeros, where $n \geq m$, then $G(s)$ has $(n - m)$ zeros at $s = C^{\infty}$
  - C^{∞} is an infinite complex number – infinite magnitude and *some* angle

The Root Locus

Root Locus – Definition

- Consider a general feedback system:



- Closed-loop transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- Closed-loop poles are roots of

$$1 + KG(s)H(s)$$

- That is, the solutions to

$$1 + KG(s)H(s) = 0$$

- Or, the values of s for which

$$KG(s)H(s) = -1 \qquad (1)$$

Root Locus – Definition

- Because $G(s)$ and $H(s)$ are complex functions, (1) is really two equations:

$$\angle G(s)H(s) = (2i+1)180°$$

that is, the angle is an odd multiple of $180°$, and

$$|KG(s)H(s)| = 1$$

- So, if a certain value of s satisfies the ***angle criterion***

$$\boxed{\angle G(s)H(s) = (2i+1)180°}$$

then that value of s is a closed-loop pole for some value of K

- And, that value of K is given by the ***magnitude criterion***

$$\boxed{K = \left|\frac{1}{G(s)H(s)}\right|}$$

Root Locus – Definition

- The ***root locus*** is the set of all points in the s-plane that satisfy the ***angle criterion***

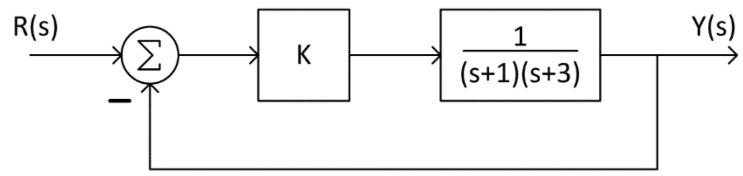
$$\angle G(s)H(s) = (2i+1)180°$$

  - The set of all ***closed-loop poles*** for $0 \leq K \leq \infty$

- We'll use the angle criterion to sketch the root locus
  - We will derive rules for sketching the root locus
  - Not necessary to test all possible s-plane points

Angle Criterion – Example

- Determine if $s_1 = -3 + j2$ is on this system's root locus



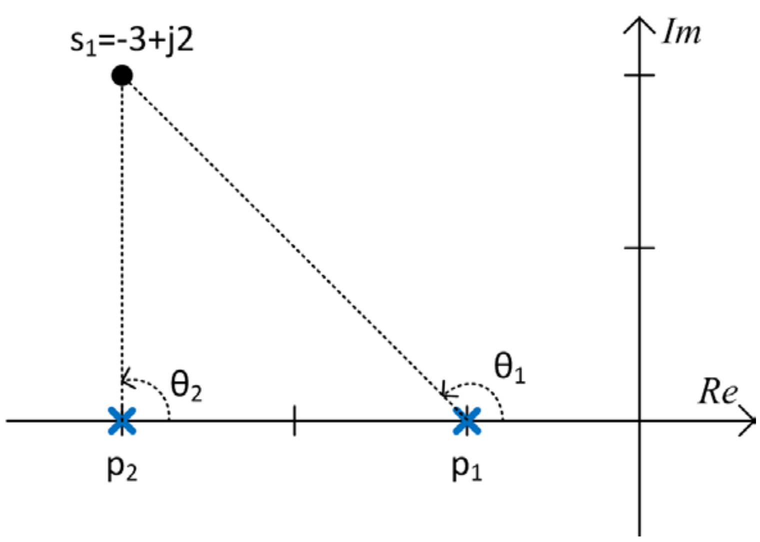
- s_1 is on the root locus if it satisfies the angle criterion

$$\angle G(s_1) = (2i + 1)180°$$

- From the pole/zero diagram

$$\angle G(s_1) = -(135° + 90°)$$
$$\angle G(s_1) = -225° \neq (2i + 1)180°$$

- s_1 does not satisfy the angle criterion
  - It is not on the root locus

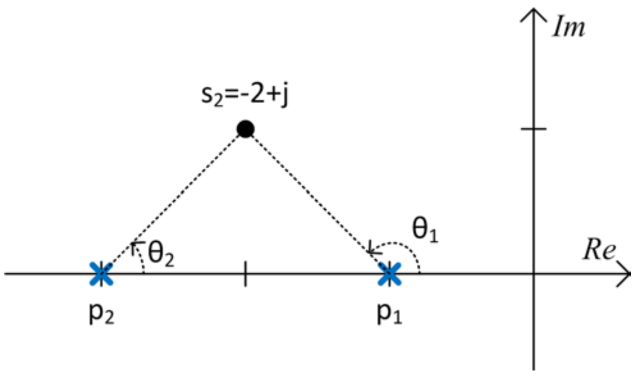


Angle Criterion – Example

- Is $s_2 = -2 + j$ on the root locus?
- Now we have

$$\angle G(s_2) = -(135° + 45°) = -180°$$

  - s_2 is on the root locus



- What gain results in a closed-loop pole at s_2?
  - Use the magnitude criterion to determine K

$$K = \left|\frac{1}{G(s_2)}\right| = |(s_2+1)(s_2+3)| = \sqrt{2}\cdot\sqrt{2} = 2$$

- $K = 2$ yields a closed-loop pole at $s_2 = -2 + j$
  - And at its complex conjugate, $\bar{s}_2 = -2 - j$

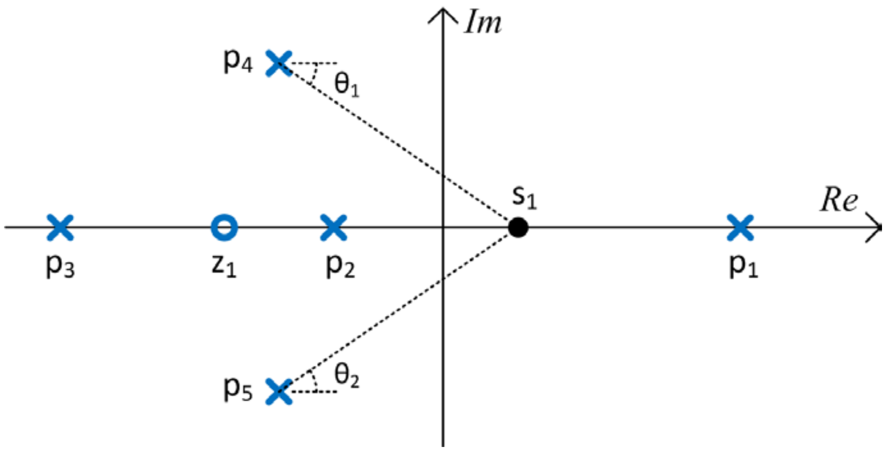
Root Locus – Real-axis segments

Real-Axis Root-Locus Segments

- We'll first consider points on the real axis, and whether or not they are on the root locus
- Consider a system with the following open-loop poles
  - Is s_1 on the root locus? I.e., does it satisfy the angle criterion?
- Angle contributions from complex poles cancel
- Pole to the ***right*** of s_1:

$$-\angle(s_1 - p_1) = -180°$$

- All poles/zeros to the ***left*** of s_1:

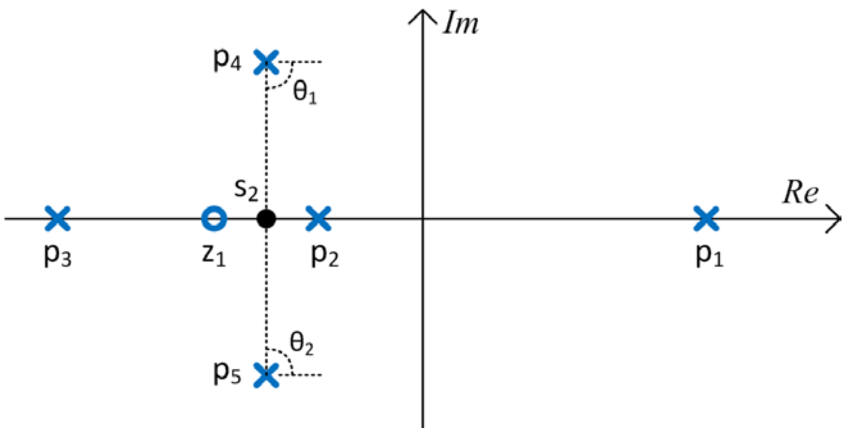


$$-\angle(s_1 - p_2) = -\angle(s_1 - p_3) = \angle(s_1 - z_1) = 0°$$

- s_1 satisfies the angle criterion, $\angle G(s_1) = -180°$, so it *is* on the root locus

Real-Axis Root-Locus Segments

- Now, determine if point s_2 is on the root locus
- Again ***angles from complex poles cancel***
  - Always true for real-axis points
- Pole and zero to the ***left*** of s_2 contribute 0°
  - Always true for real-axis points



- Two poles to the ***right*** of s_1:

$$-\angle(s_2 - p_1) - \angle(s_2 - p_2) = -360°$$

- Angle criterion is not satisfied

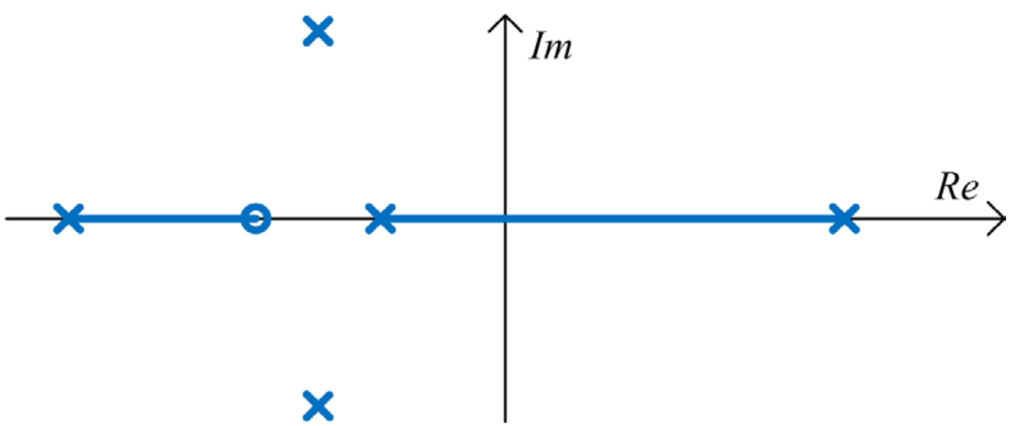
$$\angle G(s_2) = -360° \neq (2i + 1)180°$$

- s_2 is *not* on the root locus

Real-Axis Root-Locus Segments

- From the preceding development, we can conclude the following concerning real-axis segments of the root locus:

> ***All points on the real axis to the left of an odd number of open-loop poles and/or zeros are on the root locus***



Root Locus – Non-Real-Axis Segments

Non-Real-Axis Root-Locus Segments

- Transfer functions of physically-realizable systems are rational polynomials with *real-valued* coefficients
  - Complex poles/zeros come in complex-conjugate pairs

Root locus is symmetric about the real axis

- Root locus is a plot of closed loop poles as K varies from $0 \rightarrow \infty$
- Where does the locus ***start***? Where does it ***end***?

Non-Real-Axis Root-Locus Segments

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- We've seen that we can represent this closed-loop transfer function as

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

- The closed-loop poles are the roots of the closed-loop characteristic polynomial

$$\Delta(s) = D_G(s)D_H(s) + KN_G(s)N_H(s)$$

- As $K \rightarrow 0$

$$\Delta(s) \rightarrow D_G(s)D_H(s)$$

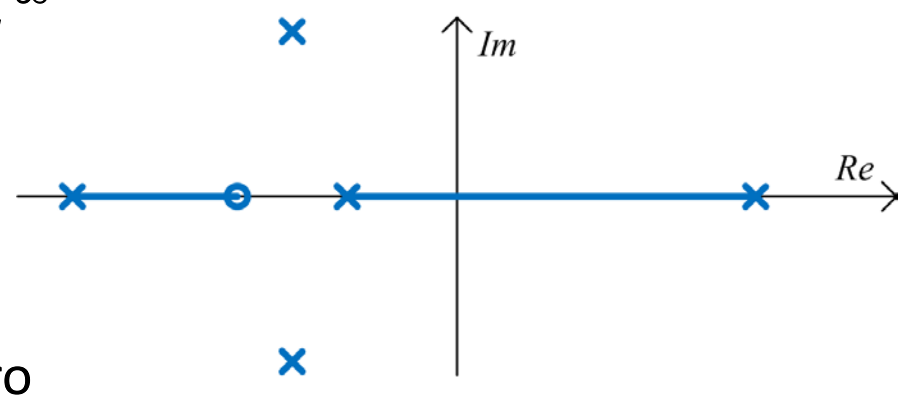
- Closed-loop poles approach the open-loop poles
  - ***Root locus starts at the open-loop poles for*** $K = 0$

Non-Real-Axis Root-Locus Segments

- As $K \to \infty$

$$\Delta(s) \to KN_G(s)N_H(s)$$

- So, as $K \to \infty$, the closed-loop poles approach the open-loop zeros
  - ***Root locus ends at the open-loop zeros for*** $K = \infty$
  - Including the $n - m$ zeros at C^{∞}



- Previous example:
  - $n = 5$ poles, $m = 1$ zero
  - One pole goes to the finite zero
  - Remaining poles go to the $(n - m) = 4$ zeros at C^{∞}
  - Where are those zeros? (what angles?)
  - How do the poles get there as $K \to \infty$?

Non-Real-Axis Root-Locus Segments

- As $K \rightarrow \infty$, m of the n poles approach the m finite zeros
- The remaining $(n-m)$ poles are at C^{∞}
- Looking back from C^{∞}, it appears that these $(n-m)$ poles all came from the same point on the real axis, σ_a
- Considering only these $(n-m)$ poles, the corresponding root locus equation is

$$G_a = 1 + K\frac{1}{(s-\sigma_a)^{n-m}} = 0$$

- These poles travel from σ_a (approximately) to C^{∞} along $(n-m)$ ***asymptotes*** at angles of $\theta_{a,i}$

Root Locus – Asymptote Angles

Asymptote Angles – $\theta_{a,i}$

- To determine the angles of the $(n-m)$ asymptotes, consider a point, s_1, very far from σ_a
- If s_1 is on the root locus, then

$$\angle G_a(s_1) = (2i+1)180°$$

- That is, the $(n-m)$ angles from σ_a to s_1 sum to an odd multiple of $180°$

$$(n-m)\theta_{a,i} = (2i+1)180°$$

- Therefore, the angles of the asymptotes are

$$\boxed{\theta_{a,i} = \frac{(2i+1)180°}{n-m}}$$

Asymptote Angles – $\theta_{a,i}$

- For example
  - $n = 5$ poles and $m = 3$ zeros
  - $(n - m) = 2$ poles go to C^∞ as $K \to \infty$
  - Poles approach C^∞ along asymptotes at angles of

$$\theta_{a,0} = \frac{(2 \cdot 0 + 1)180°}{5 - 3} = \frac{180°}{2} = 90°$$

$$\theta_{a,1} = \frac{540°}{2} = 270°$$

- If $(n - m) = 3$

$$\theta_{a,0} = \frac{180°}{3} = 60°, \quad \theta_{a,1} = \frac{540°}{3} = 180°, \quad \theta_{a,2} = \frac{900°}{3} = 300°$$

Root Locus – Asymptote Origin

Asymptote Origin

- The $(n-m)$ asymptotes come from a point, σ_a, on the real axis - where is σ_a located?
- The root locus equation can be written

$$1+K\frac{b(s)}{a(s)}=0$$

where

$$b(s)=s^m+b_1s^{m-1}+\cdots+b_m$$

$$a(s)=s^n+a_1s^{n-1}+\cdots+a_n$$

- According to a property of *monic* polynomials:

$$a_1=-\Sigma\, p_i$$

$$b_1=-\Sigma\, z_i$$

where p_i are the *open-loop poles*, and z_i are the *open-loop zeros*

Asymptote Origin

- The closed-loop characteristic polynomial is

$$s^n + a_1 s^{n-1} + \cdots + a_n + K(s^m + b_1 s^{m-1} + \cdots + b_m)$$

- If $m < (n-1)$, i.e. at least two more poles than zeros, then

$$a_1 = -\Sigma\, r_i$$

where r_i are the *closed-loop poles*

- The sum of the closed-loop poles is:
  - Independent of K
  - Equal to the sum of the open-loop poles

$$-\Sigma\, p_i = -\Sigma\, r_i = a_1$$

- The *equivalent* open-loop location for the $(n-m)$ poles going to infinity is σ_a
  - These poles, similarly, have a constant sum:

$$(n-m)\sigma_a$$

Asymptote Origin

- As $K \to \infty$, m of the closed-loop poles go to the open loop zeros
  - Their sum is the sum of the open-loop zeros
- The remainder of the poles go to C^{∞}
  - Their sum is $(n-m)\sigma_a$
- The sum of all closed-loop poles is equal to the sum of the open-loop poles

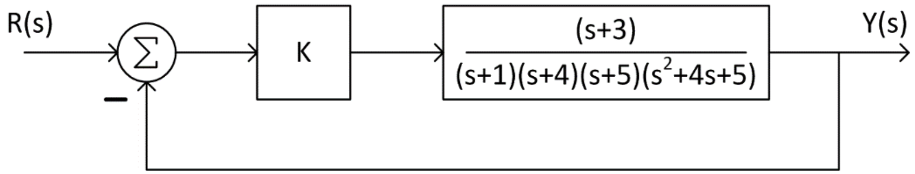
$$\Sigma\, r_i = \Sigma\, z_i + (n-m)\sigma_a = \Sigma\, p_i$$

- The ***origin of the asymptotes*** is

$$\sigma_a = \frac{\Sigma\, p_i - \Sigma\, z_i}{n-m}$$

Root Locus Asymptotes – Example

- Consider the following system



- $m = 1$ open-loop zero and $n = 5$ open-loop poles
- As $K \rightarrow \infty$:
  - One pole approaches the open-loop zero
  - Four poles go to C^{∞} along asymptotes at angles of:

$$\theta_{a,0} = \frac{180°}{4} = 45°, \qquad \theta_{a,1} = \frac{540°}{4} = 135°$$

$$\theta_{a,2} = \frac{900°}{4} = 225°, \qquad \theta_{a,3} = \frac{1260°}{4} = 315°$$

Root Locus Asymptotes – Example

- The origin of the asymptotes is

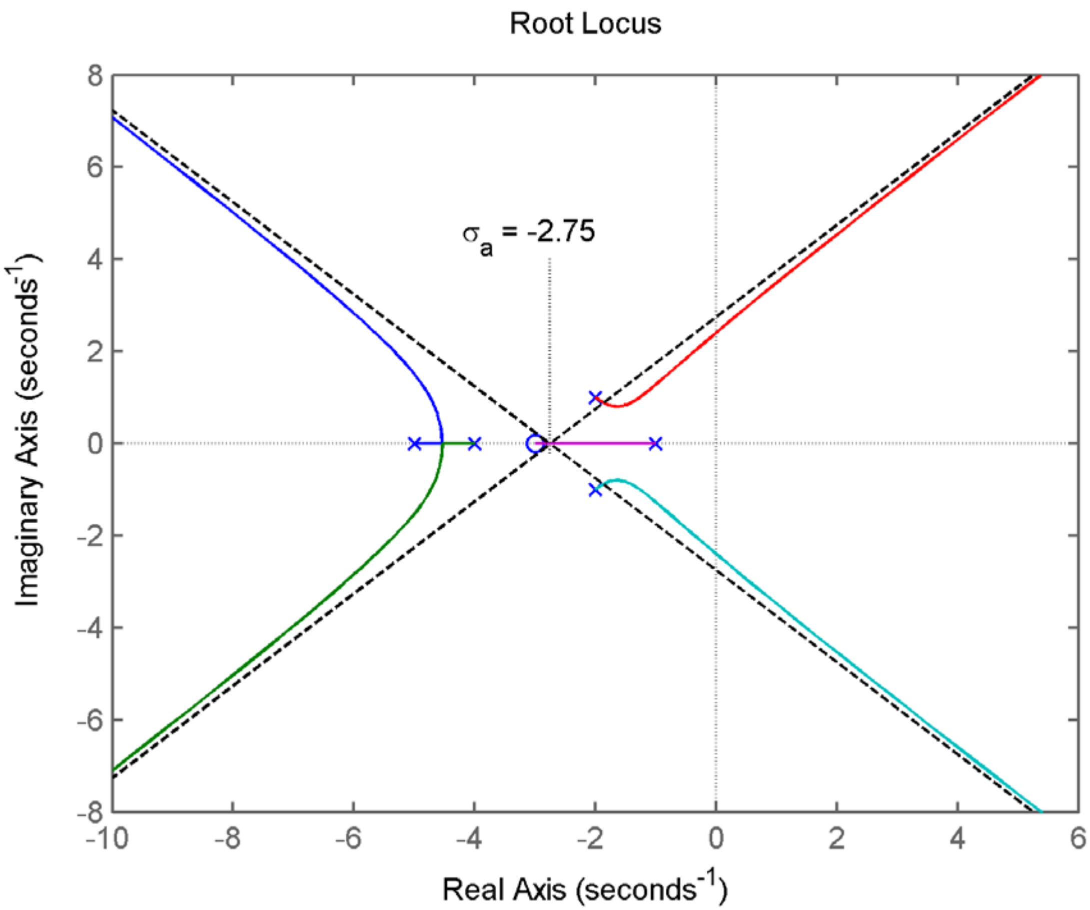
$$\sigma_a = \frac{\Sigma\, p_i - \Sigma\, z_i}{n-m}$$

$$\sigma_a = \frac{\big((-1)+(-4)+(-5)+(-2+j)+(-2-j)\big)-(-3)}{5-1}$$

$$\sigma_a = \frac{-14+3}{4} = -2.75$$

- As $K \rightarrow \infty$, four poles approach C^{∞} along four asymptotes emanating from $s=-2.75$ at angles of $45°$, $135°$, $225°$, and $315°$

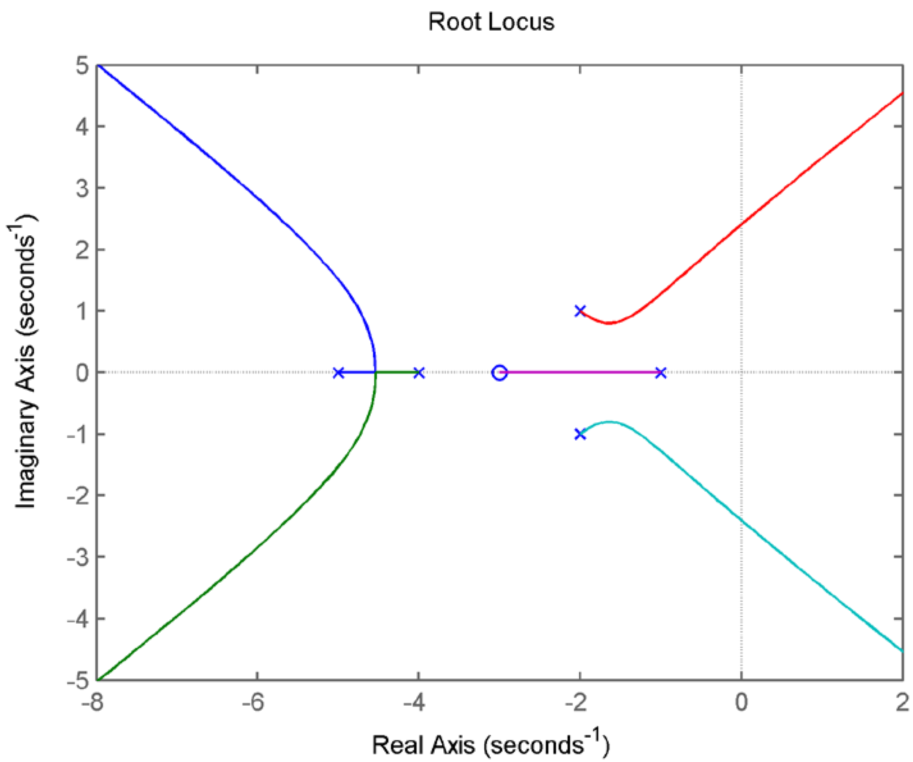
Root Locus Asymptotes – Example



Refining the Root Locus

Refining the Root Locus

- So far we've learned how to accurately sketch:
  - Real-axis root locus segments
  - Root locus segments heading toward C^{∞}, but only far from σ_a
- Root locus from previous example illustrates additional characteristics we must address:
  - Real-axis ***breakaway/break-in points***
  - ***Angles of departure/arrival*** at complex poles/zeros
  - $j\omega$-axis crossing locations



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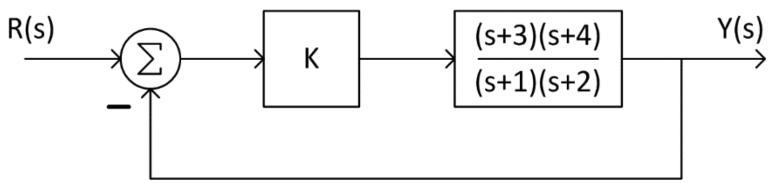
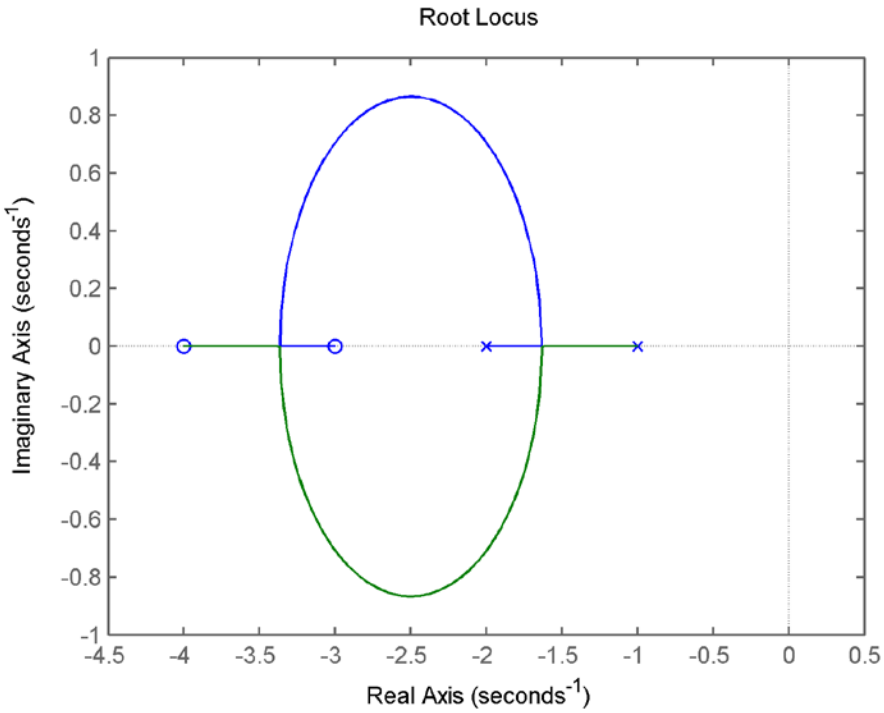
Real-Axis Breakaway/Break-In Points

Real-Axis Breakaway/Break-In Points

- Consider the following system and its root locus





- Two finite poles approach two finite zeros as $K \rightarrow \infty$
  - Where do they leave the real axis?
    - ***Breakaway point***
  - Where do they re-join the real axis?
    - ***Break-in point***

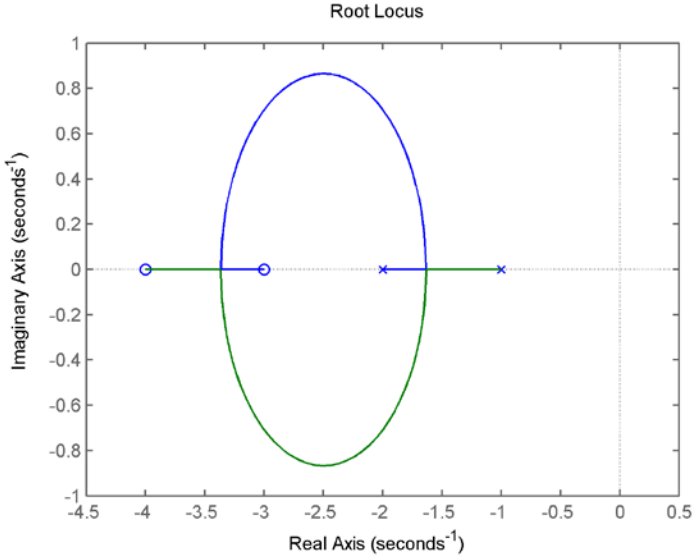
Real-Axis Breakaway Points

- Breakaway point occurs somewhere between $s = -1$ and $s = -2$
  - Breakaway angle:

$$\theta_{breakaway} = \frac{180°}{n}$$

    where n is the number of poles that come together – here, $\pm 90°$

  - As gain increases, poles come together then leave the real axis
  - Along the real-axis segment, maximum gain occurs at the breakaway point



- To calculate the breakaway point:
  - Determine an expression for gain, K, as a function of s
  - Differentiate w.r.t. s
  - Find s for $dK/ds = 0$ to locate the maximum gain point

Real-Axis Breakaway Points

- All points on the root locus satisfy

$$K = -\frac{1}{G(s)H(s)}$$

- On the segment containing the ***breakaway point***, $s = \sigma$, so

$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

- The breakaway point is a ***maximum gain point***, so

$$\frac{dK}{d\sigma} = \frac{d}{d\sigma}\left(-\frac{1}{G(\sigma)H(\sigma)}\right) = 0$$

- Solving for σ yields the breakaway point

Real-Axis Breakaway Points

- For our example, along the real axis

$$K = -\frac{1}{G(\sigma)} = -\frac{(\sigma+1)(\sigma+2)}{(\sigma+3)(\sigma+4)} = -\frac{\sigma^2+3\sigma+2}{\sigma^2+7\sigma+12}$$

- Differentiating w.r.t. σ

$$\frac{dK}{d\sigma} = -\frac{(\sigma^2+7\sigma+12)(2\sigma+3)-(\sigma^2+3\sigma+2)(2\sigma+7)}{(\sigma^2+7\sigma+12)^2} = 0$$

- Setting the derivative to zero

$$(\sigma^2+7\sigma+12)(2\sigma+3)-(\sigma^2+3\sigma+2)(2\sigma+7) = 0$$

$$4\sigma^2+20\sigma+22 = 0$$

$$\sigma = -1.63, -3.37$$

- The breakaway point occurs at $s = -1.63$

Real-Axis Break-In Points

- The poles re-join the real axis at a ***break-in point***
  - A ***minimum gain point***
  - As gain increases, poles move apart
  - Break-in angles are the same as breakaway angles

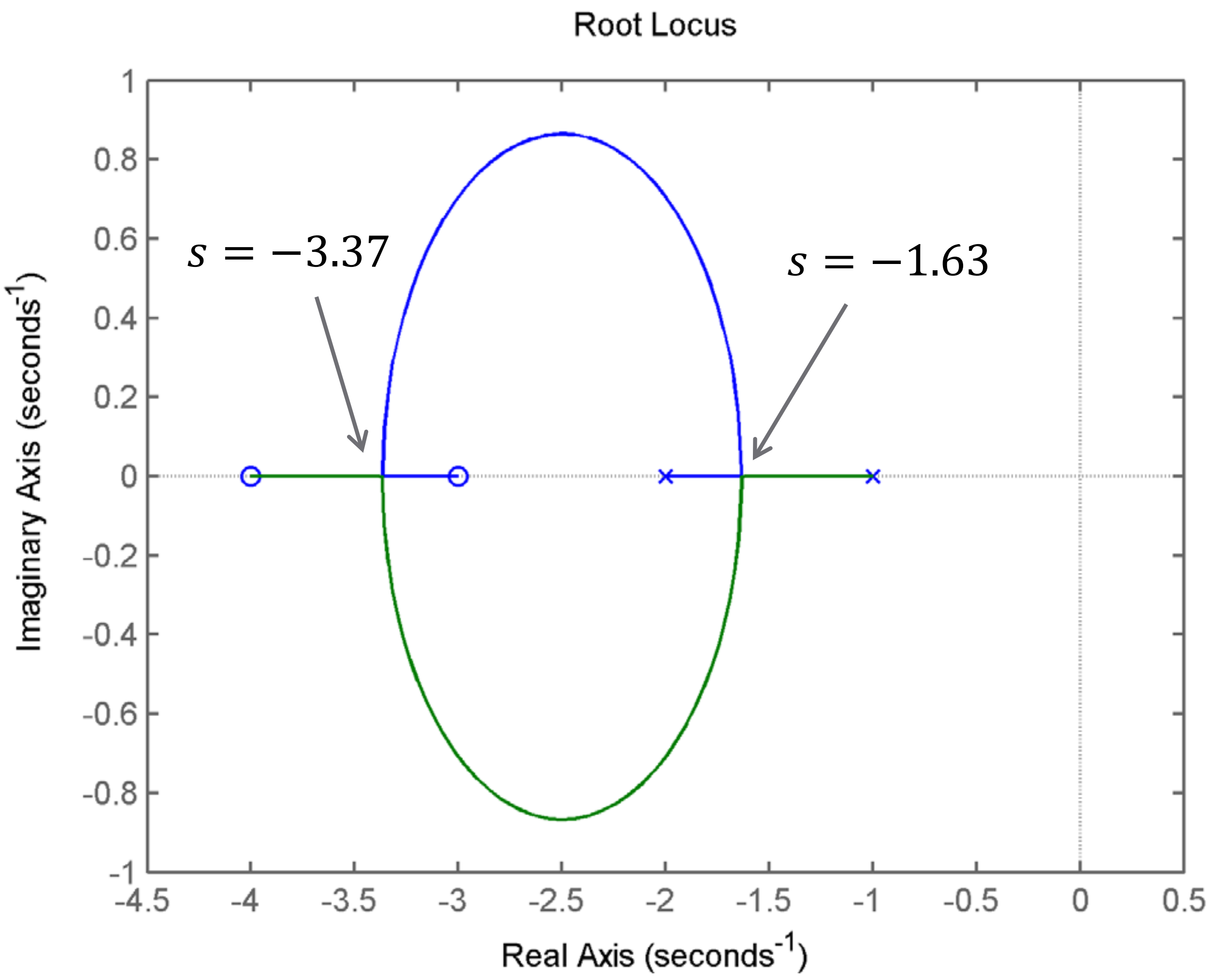
$$\theta_{break-in} = \frac{180°}{n}$$

- As for the breakaway point, the break-in point satisfies

$$\frac{dK}{d\sigma} = \frac{d}{d\sigma}\left(-\frac{1}{G(s)H(s)}\right) = 0$$

- In fact, this yields both breakaway and break-in points
- For our example, we had $\sigma = -1.63,\ -3.37$
  - ***Breakaway point***: $s = -1.63$
  - ***Break-in point***: $s = -3.37$

Real-Axis Breakaway/Break-In Points

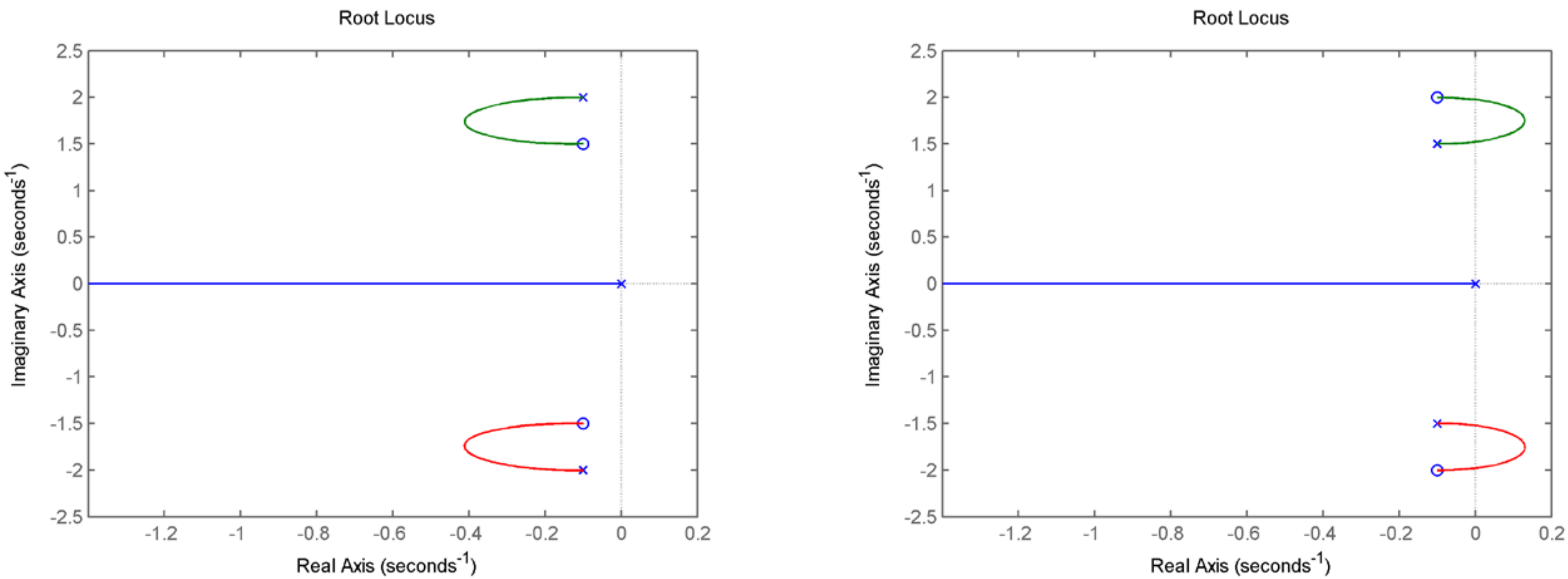


Angles of Departure/Arrival

Angles of Departure/Arrival

- Consider the following two systems

$$G_1(s) = \frac{s^2+0.2s+2.26}{s^2+0.2s+4.01} \qquad G_2(s) = \frac{s^2+0.2s+4.01}{s^2+0.2s+2.26}$$



- Similar systems, with very different stability behavior
  - Understanding how to determine ***angles of departure*** from complex poles and ***angles of arrival*** at complex zeros will allow us to predict this

Angle of Departure

- To find the angle of departure from a pole, p_1:
  - Consider a test point, s_0, very close to p_1
  - The angle from p_1 to s_0 is ϕ_1
  - The angle from all other poles/zeros, ϕ_i/ψ_i, to s_0 are approximated as the angle from p_i or z_i to p_1
  - Apply the angle criterion to find ϕ_1

$$\sum_{i=1}^{m} \psi_i - \phi_1 - \sum_{i=2}^{n} \phi_i = (2i+1)180°$$

- Solving for the departure angle, ϕ_1:

$$\phi_1 = \sum_{i=1}^{m} \psi_i - \sum_{i=2}^{n} \phi_i - 180°$$

- In words:

$$\phi_{depart} = \Sigma\angle(zeros) - \Sigma\angle(other\ poles) - 180°$$

Angle of Departure

- If we have complex-conjugate open-loop poles with multiplicity q, then

$$\sum_{i=1}^{m}\psi_i - q\phi_1 - \sum_{i=q+1}^{n}\phi_i = (2i+1)180°$$

- The q different angles of departure from the multiple poles are

$$\boxed{\phi_{1,i} = \frac{\sum_{i=1}^{m}\psi_i - \sum_{i=q+1}^{n}\phi_i - (2i+1)180°}{q}}$$

where $i = 1, 2, \ldots q$

Angle of Arrival

- Following the same procedure, we can derive an expression for the ***angle of arrival*** at a complex zero of multiplicity q

$$\psi_{1,i} = \frac{\sum_{i=1}^{n} \phi_i - \sum_{i=q+1}^{m} \psi_i + (2i+1)180°}{q}$$

- In summary

$$\phi_{depart,i} = \frac{\Sigma\angle(zeros) - \Sigma\angle(other\ poles) - (2i+1)180°}{multiplicity}$$

$$\psi_{arrive,i} = \frac{\Sigma\angle(poles) - \Sigma\angle(other\ zeros) + (2i+1)180°}{multiplicity}$$

Departure/Arrival Angles – Example

- Angle of departure from p_1

$$\phi_1 = \sum_{i=1}^{m} \psi_i - \sum_{i=2}^{n} \phi_i - 180°$$

$$\phi_1 = [90° + 90°] - [90° + 92.9°] - 180°$$

$$\phi_1 = -182.9°$$

- Due to symmetry:

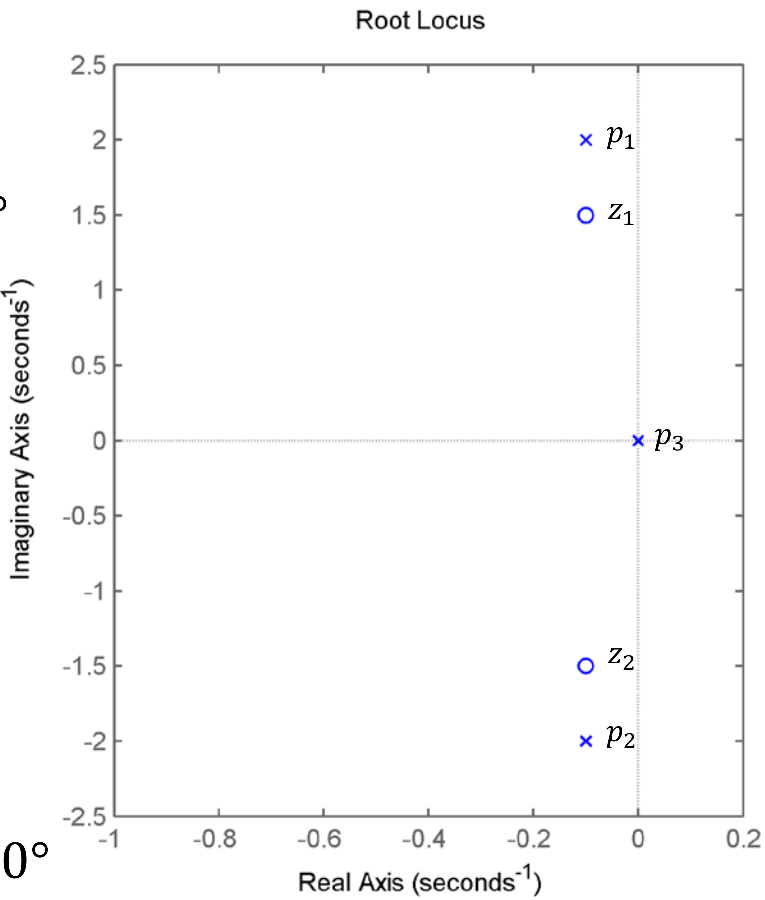
$$\phi_2 = -\phi_1 = 182.9°$$

- Angle of arrival at z_1

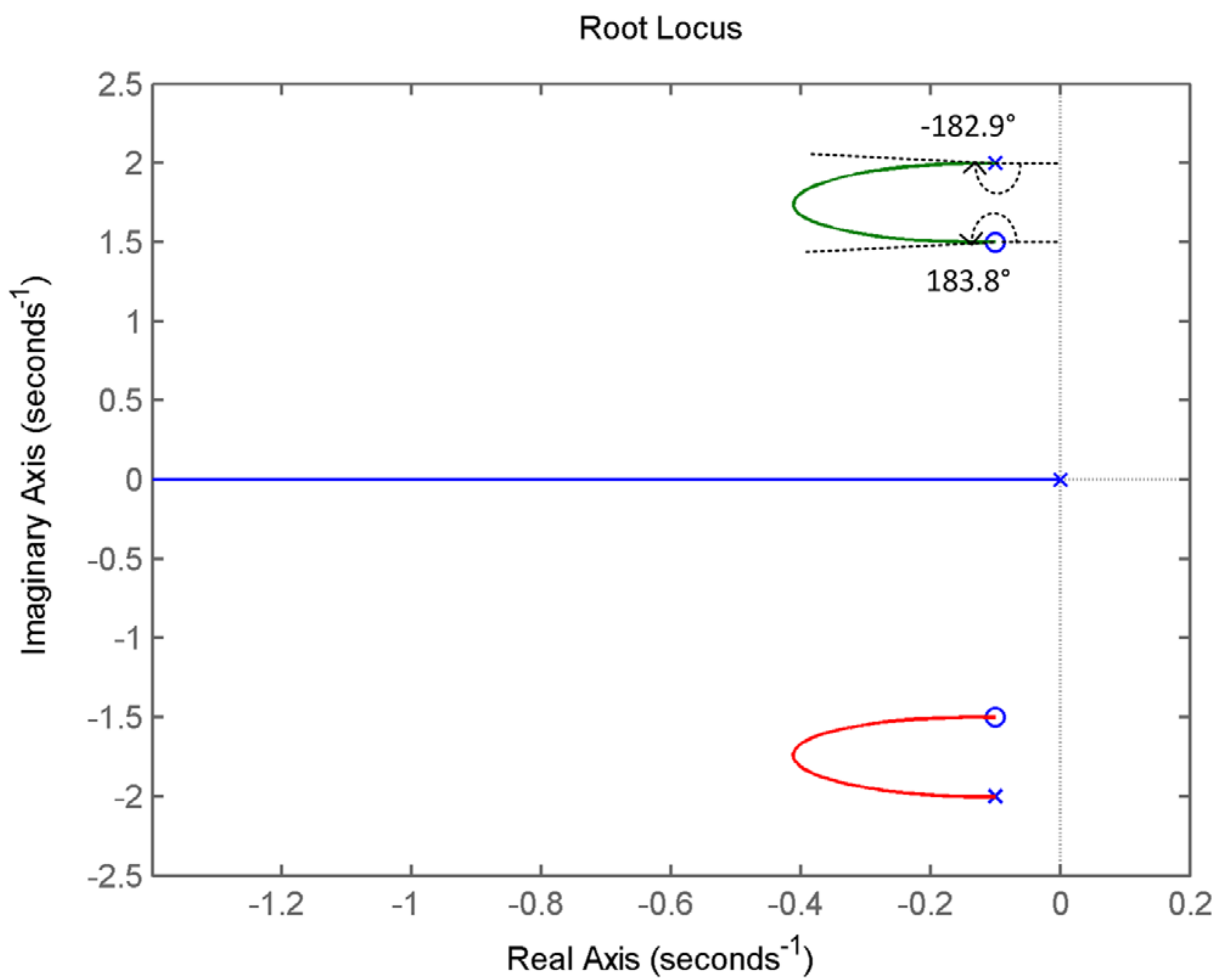
$$\psi_1 = \sum_{i=1}^{m} \phi_i - \sum_{i=2}^{n} \psi_i + 180°$$

$$\psi_1 = [-90° + 90° + 93.8°] - [90°] + 180°$$

$$\psi_1 = 183.8°, \qquad \psi_2 = -183.8°$$



Departure/Arrival Angles – Example



Departure/Arrival Angles – Example

- Next, consider the other system
- Angle of departure from p_1

$$\phi_1 = [-90° + 90°] - [90° + 93.8°] - 180°$$

$$\phi_1 = -363.8° \;\rightarrow\; -3.8°$$

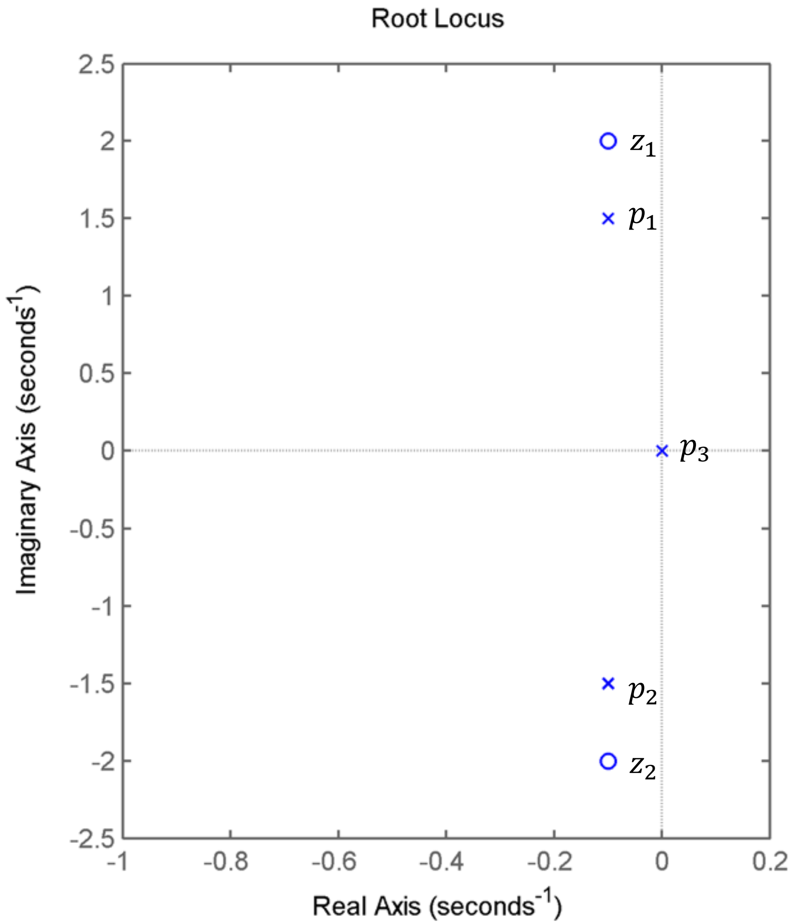
$$\phi_2 = 3.8°$$

- Angle of arrival at z_1

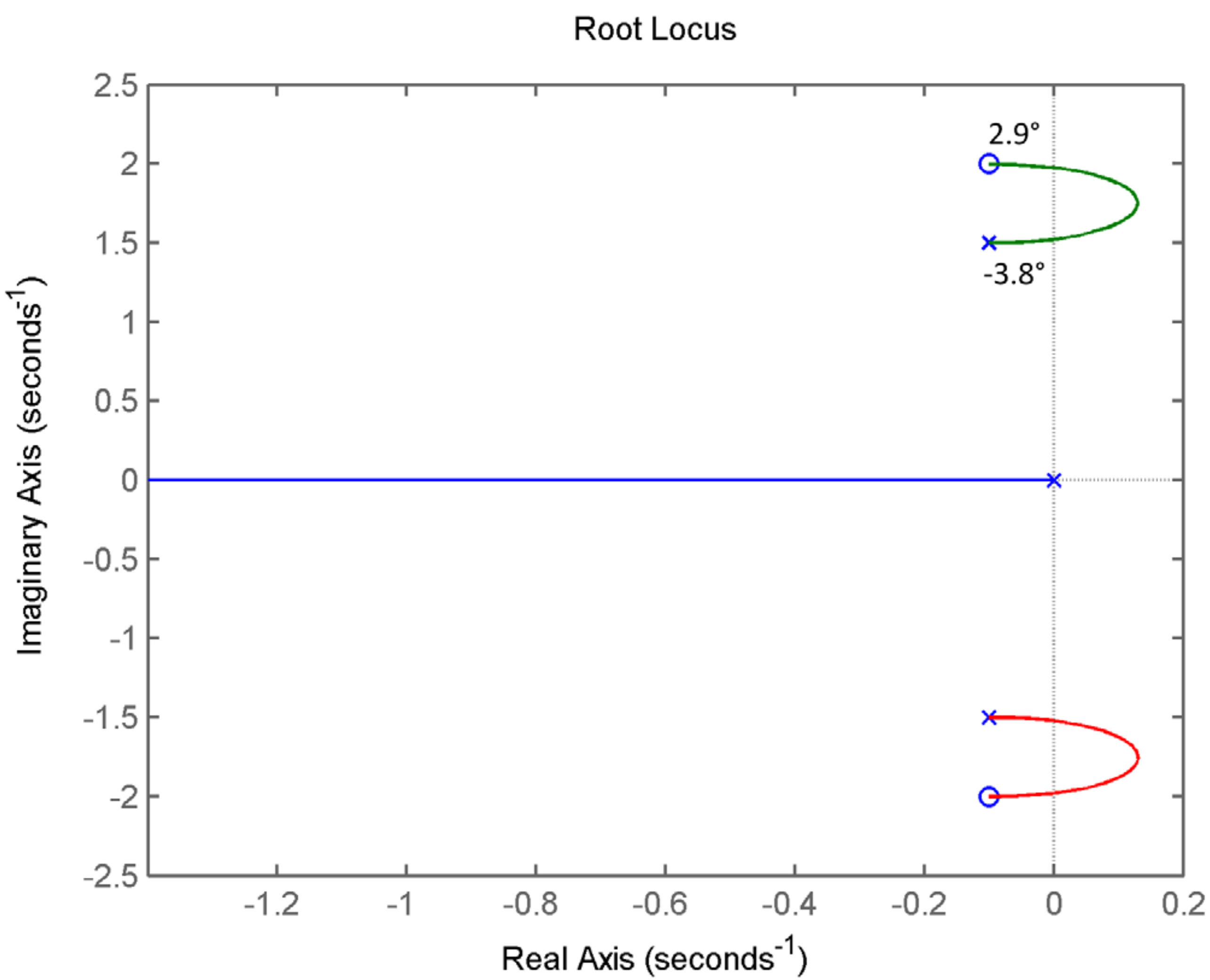
$$\psi_1 = [90° + 90° + 92.9°] - [90°] + 180°$$

$$\psi_1 = 362.9° \;\rightarrow\; 2.9°$$

$$\psi_2 = -2.9°$$



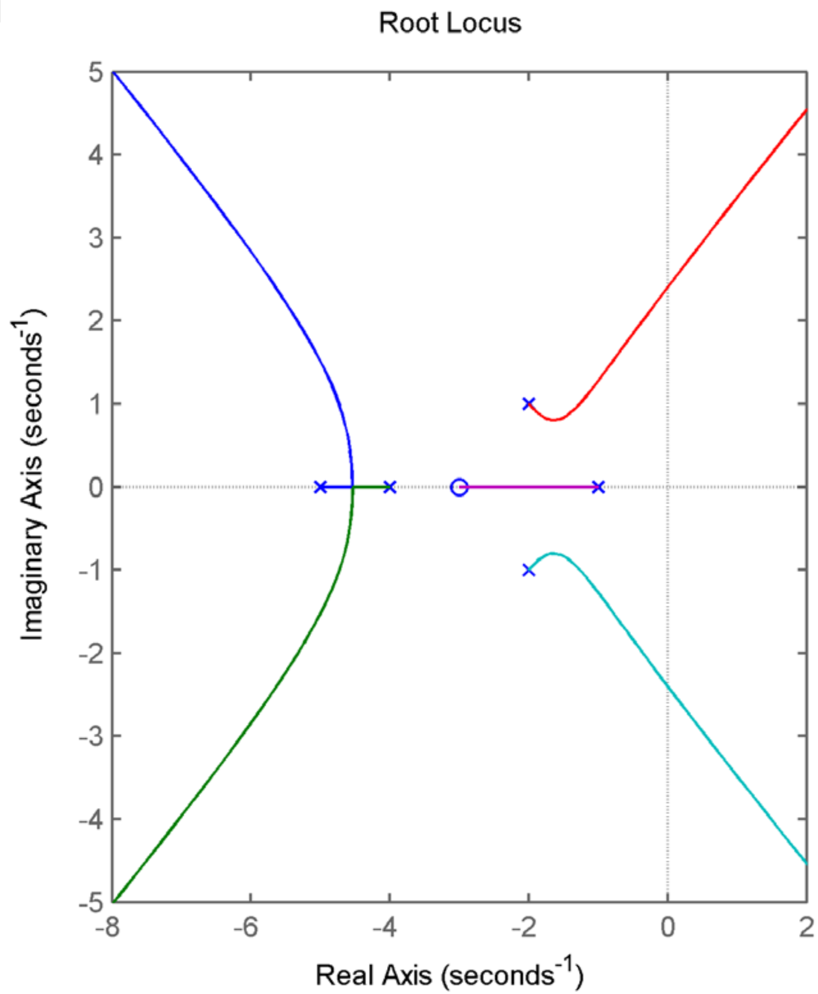
Departure/Arrival Angles – Example



$j\omega$-Axis Crossing Points

$j\omega$-Axis Crossing Points

- To determine the location of a $j\omega$-axis crossing
  - Apply Routh-Hurwitz
  - Find value of K that results in a row of zeros
    - Marginal stability
    - $j\omega$-axis poles
  - Roots of row preceding the zero row are $j\omega$-axis crossing points
- Or, plot in MATLAB
  - More on this later



Sketching the Root Locus - Summary

Root Locus Sketching Procedure – Summary

1. Plot open-loop ***poles*** and ***zeros*** in the s-plane
2. Plot locus segments on the ***real axis*** to the left of an odd number of poles and/or zeros
3. For the $(n-m)$ poles going to C^{∞}, sketch ***asymptotes*** at angles $\theta_{a,i}$, centered at σ_a, where

$$\theta_{a,i} = \frac{(2i+1)180°}{n-m}$$

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

Root Locus Sketching Procedure – Summary

4. Calculate ***departure angles*** from complex poles of multiplicity $q \geq 1$

$$\phi_i = \frac{\Sigma\angle(zeros) - \Sigma\angle(other\ poles) - (2i+1)180°}{q}$$

and ***arrival angles*** at complex zeros of multiplicity $q \geq 1$

$$\psi_i = \frac{\Sigma\angle(poles) - \Sigma\angle(other\ zeros) + (2i+1)180°}{q}$$

5. Determine real-axis breakaway/break-in points as the solutions to

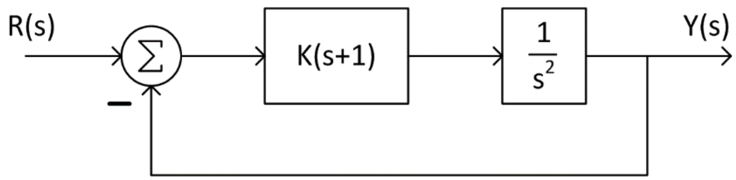
$$\frac{d}{d\sigma}\left(\frac{1}{G(\sigma)H(\sigma)}\right) = 0$$

Breakaway/break-in angles are $180°/n$ to the real axis

6. If desired, apply Routh-Hurwitz to determine $j\omega$-axis crossings

Sketching the Root Locus – Example 1

- Consider a satellite, controlled by a proportional-derivative (PD) controller



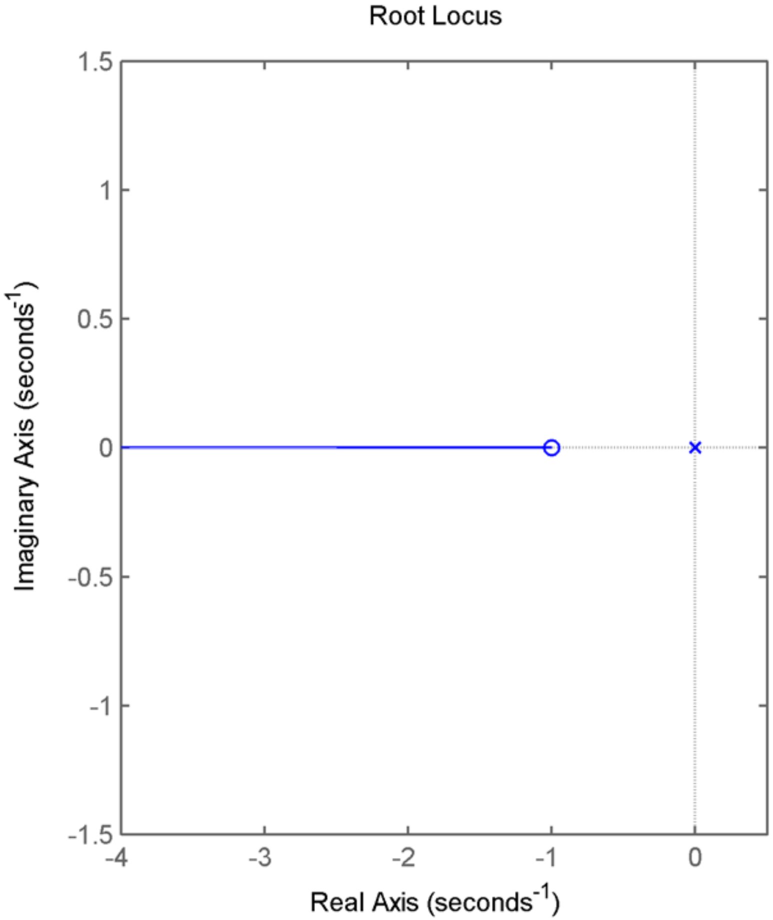
  - A example of a ***double-integrator*** plant
  - We'll learn about PD controllers in the next section
  - Closed-loop transfer function

$$T(s) = \frac{K(s+1)}{s^2 + Ks + K}$$

- Sketch the root locus
  - Two open-loop poles at the origin
  - One open-loop zero at $s = -1$

Sketching the Root Locus – Example 1

1. Plot open-loop poles and zeros
   - Two poles, one zero
2. Plot real-axis segments
   - To the left of the zero
3. Asymptotes to C^{∞}
   - One pole goes to the finite zero
   - One pole goes to ∞ at $180°$ - along the real axis



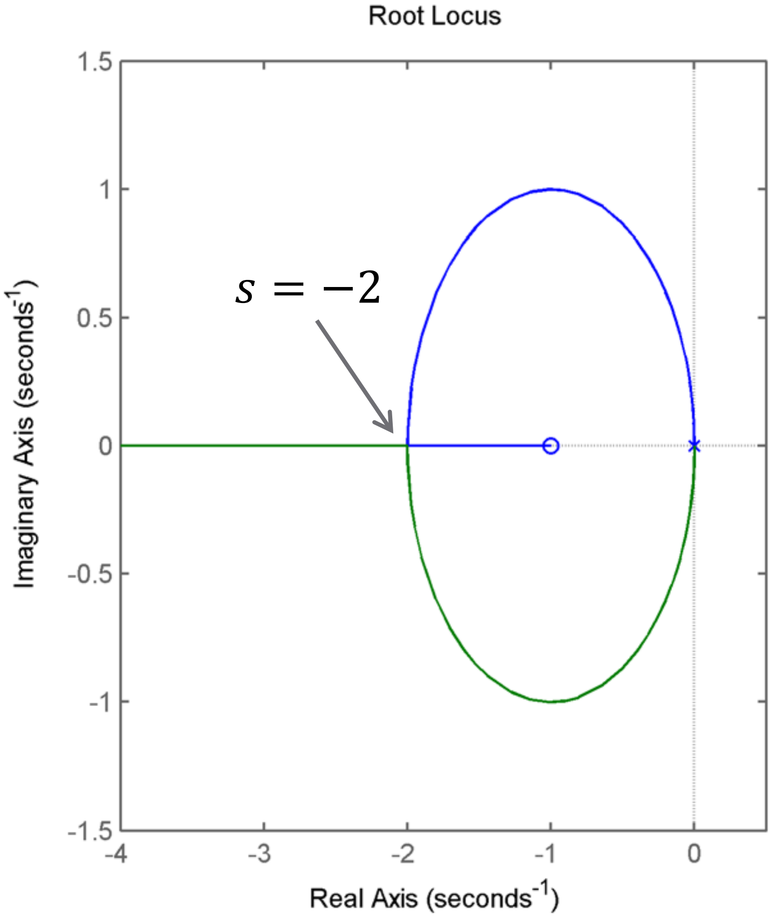
Sketching the Root Locus – Example 1

4. Departure/arrival angles
   - No complex poles or zeros
5. Breakaway/break-in points
   - Breakaway occurs at multiple roots – at $s = 0$
   - Break-in point:

$$\frac{d}{ds}\left(\frac{s^2}{(s+1)}\right) = 0$$

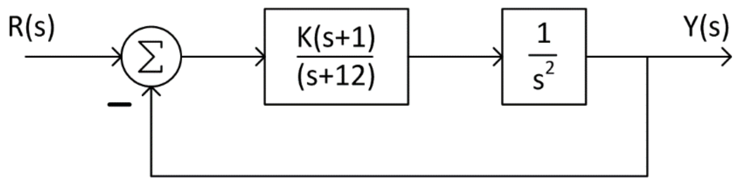
$$\frac{(s+1)2s - s^2}{(s+1)^2} = 0$$

$$s^2 + 2s = 0 \quad \rightarrow \quad s = -2, 0$$



Sketching the Root Locus – Example 2

- Now consider the same satellite with a different controller



  - A *lead compensator* – more in the next section
  - Closed-loop transfer function

$$T(s) = \frac{K(s+1)}{s^3 + 12s^2 + Ks + K}$$

- Sketch the root locus

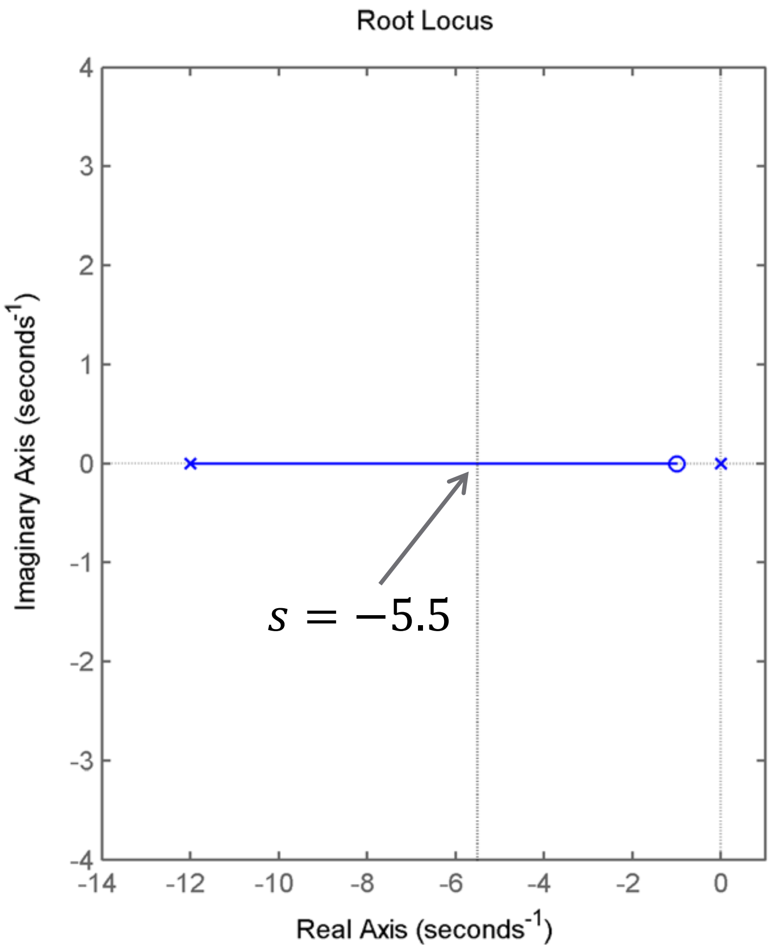
Sketching the Root Locus – Example 2

1. Plot open-loop poles and zeros
   - Now three open-loop poles and one zero
2. Plot real-axis segments
   - Between the zero and the pole at $s = -12$
3. Asymptotes to C^{∞}

$$\theta_{a,1} = \frac{180°}{2} = 90°$$

$$\theta_{a,2} = \frac{540°}{2} = 270°$$

$$\sigma_a = \frac{-12 - (-1)}{2} = -5.5$$



Sketching the Root Locus – Example 2

4. Departure/arrival angles
   - No complex open-loop poles or zeros

5. Breakaway/break-in points

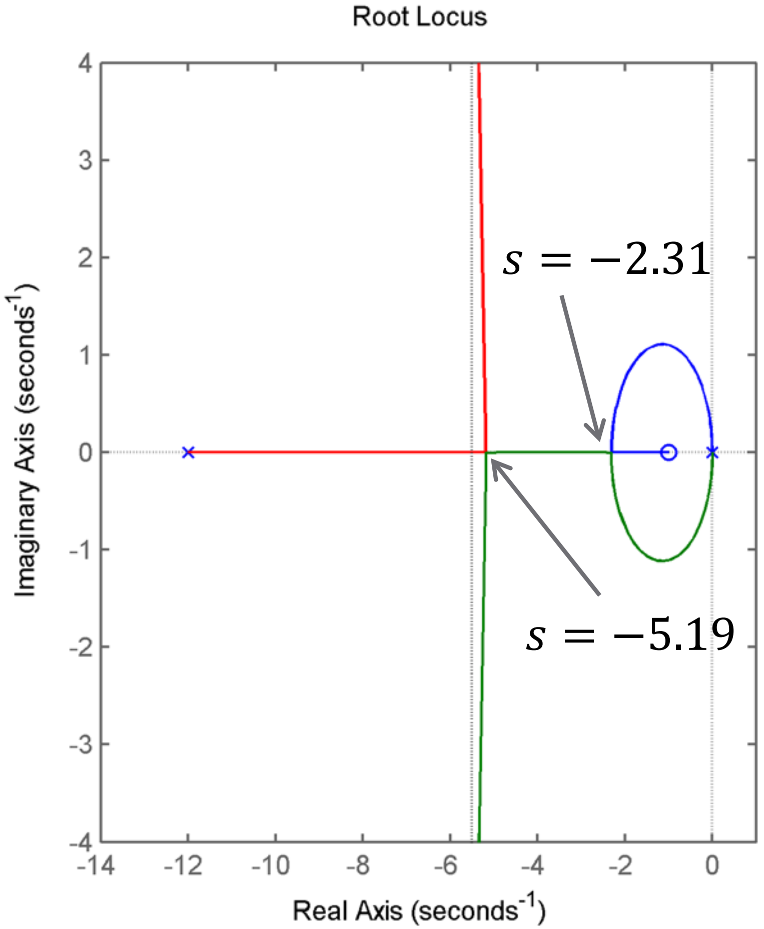
$$\frac{d}{ds}\left(\frac{s^2(s+12)}{(s+1)}\right)=0$$

$$\frac{(s+1)(3s^2+24s)-(s^3+12s^2)}{(s+1)^2}=0$$

$$2s^3+15s^2+24s=0$$

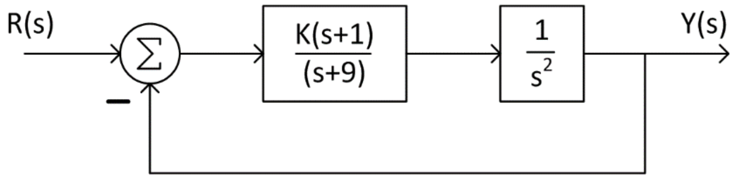
$$s=0,-2.31,-5.19$$

   - Breakaway: $s=0, s=-5.19$
   - Break-in: $s=-2.31$



Sketching the Root Locus – Example 3

- Now move the controller's pole to $s = -9$



  - Closed-loop transfer function

$$T(s) = \frac{K(s+1)}{s^3 + 9s^2 + Ks + K}$$

- Sketch the root locus

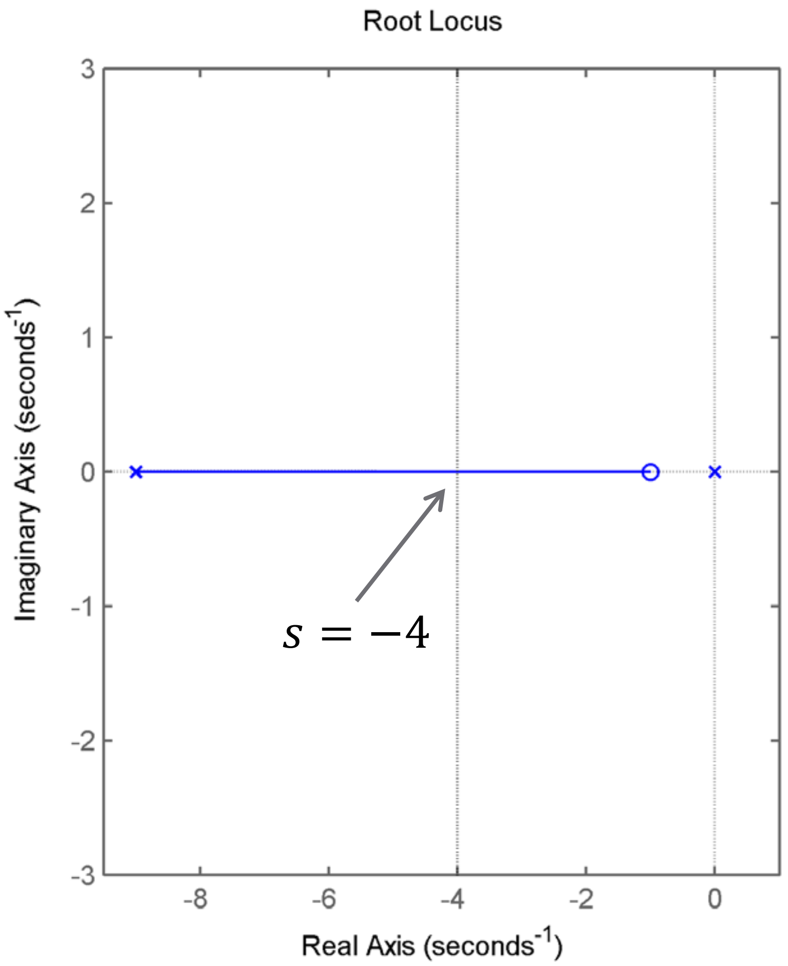
Sketching the Root Locus – Example 3

1. Plot open-loop poles and zeros
   - Again, three open-loop poles and one zero
2. Plot real-axis segments
   - Between the zero and the pole at $s = -9$
3. Asymptotes to C^{∞}

$$\theta_{a,1} = 90°$$

$$\theta_{a,2} = 270°$$

   - $\sigma_a = \frac{-9-(-1)}{2} = -4$
4. Departure/arrival angles
   - No complex open-loop poles or zeros



Sketching the Root Locus – Example 3

4. Breakaway/break-in points

$$\frac{d}{ds}\left(\frac{s^2(s+9)}{(s+1)}\right) = 0$$

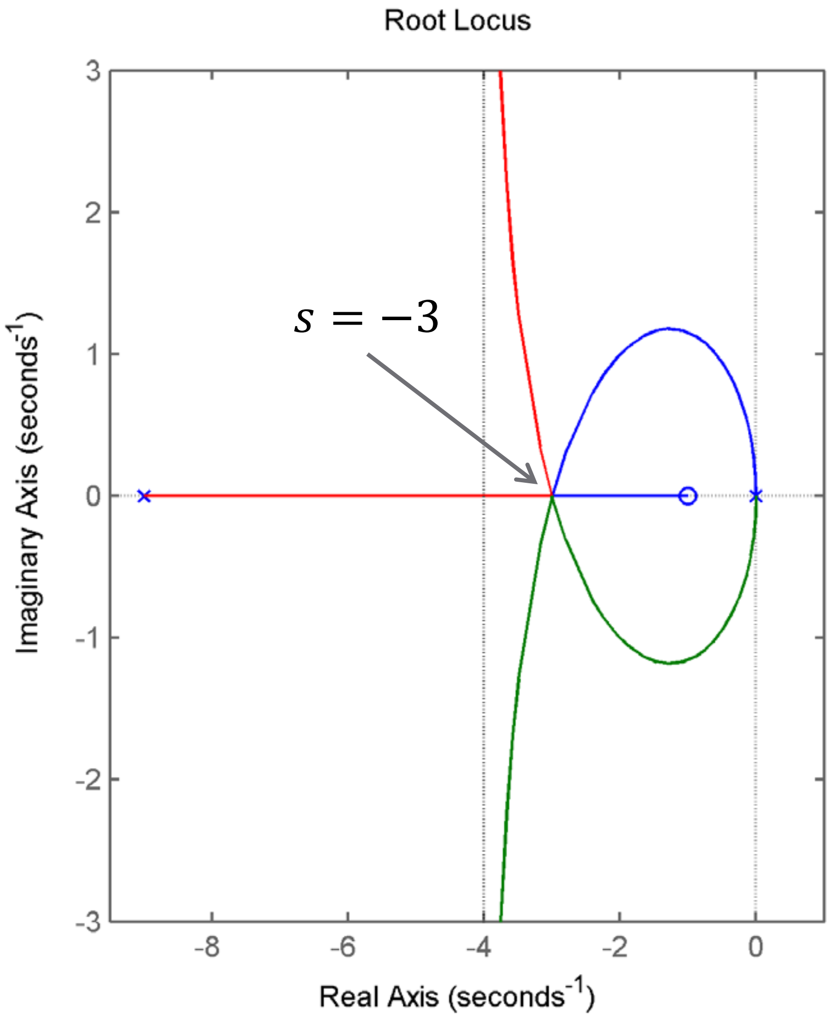
$$\frac{(s+1)(3s^2+18s)-(s^3+9s^2)}{(s+1)^2} = 0$$

$$2s^3+12s^2+18s=0$$

$$s = 0, -3, -3$$

- Breakaway: $s = 0, s = -3$
- Break-in: $s = -3$

- Three poles converge/diverge at $s = -3$
  - Breakaway angles: $0°, 120°, 240°$
  - Break-in angles: $60°, 180°, 300°$



Root Locus in MATLAB

feedback.m

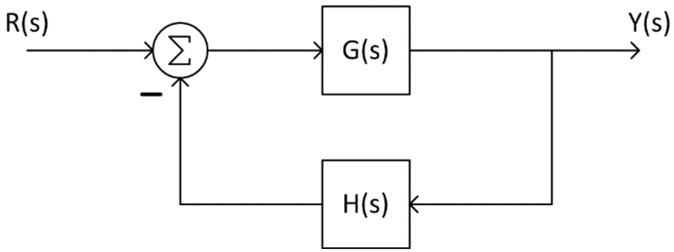

```
sys = feedback(G,H,sign)
```

- G: forward-path model – `tf`, `ss`, `zpk`, etc.
- H: feedback-path model
- `sign`: `-1` for neg. feedback, `+1` for pos. feedback – *optional* – default is `-1`
- `sys`: closed-loop system model object of the same type as G and H

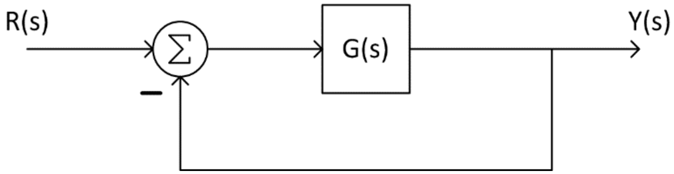
- Generates a closed-loop system model from forward-path and feedback-path models
- For unity feedback, `H=1`

feedback.m

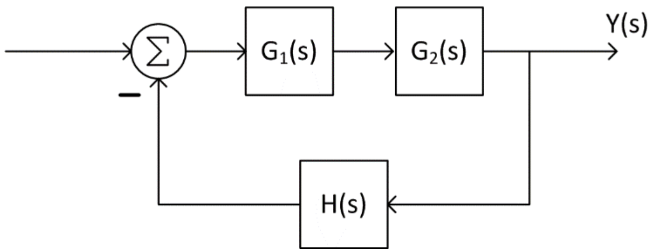
- For example:




```
T=feedback(G,H);
```




```
T=feedback(G,1);
```




```
T=feedback(G1*G2,H);
```

rlocus.m


```
[r,K] = rlocus(G,K)
```

- G: open-loop model – `tf`, `ss`, `zpk`, etc.
- K: vector of gains at which to calculate the locus – optional – MATLAB will choose gains by default
- `r`: vector of closed-loop pole locations
- K: gains corresponding to pole locations in `r`

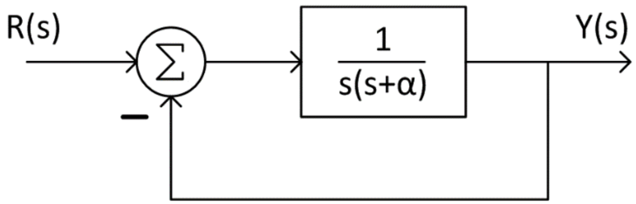
- If no outputs are specified a root locus is plotted in the current (or new) figure window
  - This is the most common use model, e.g.:

```
rlocus(G,K)
```

Generalized Root Locus

Generalized Root Locus

- We've seen that we can plot the root locus as a function of controller gain, K
- Can also plot the locus as a function of other parameters
  - For example, open-loop pole locations
- Consider the following system:



- Plot the root locus as a function of pole location, α
- Closed-loop transfer function is

$$T(s) = \frac{\frac{1}{s(s+\alpha)}}{1 + \frac{1}{s(s+\alpha)}} = \frac{1}{s^2 + \alpha s + 1}$$

Generalized Root Locus

$$T(s) = \frac{1}{s^2 + \alpha s + 1}$$

- Want the denominator to be in the root-locus form:

$$1 + \alpha G(s)H(s)$$

- First, isolate α in the denominator

$$T(s) = \frac{1}{(s^2 + 1) + \alpha s}$$

- Next, divide through by the remaining denominator terms

$$T(s) = \frac{\frac{1}{s^2 + 1}}{1 + \alpha \frac{s}{s^2 + 1}}$$

Generalized Root Locus

$$T(s) = \frac{\frac{1}{s^2+1}}{1 + \alpha \frac{s}{s^2+1}}$$

- The open-loop transfer function term in this form is

$$G(s)H(s) = \frac{s}{s^2+1}$$

- Sketch the root locus:
  1. Plot poles and zeros
     - A zero at the origin and poles at $s = \pm j$
  2. Plot real-axis segments
     - Entire negative real axis is left of a single zero

Generalized Root Locus

3. Asymptote to C^∞
   - Single asymptote along negative real axis
4. Departure angles

$$\phi_1 = 90° - 90° - 180°$$

$$\phi_1 = -180° = -\phi_2$$

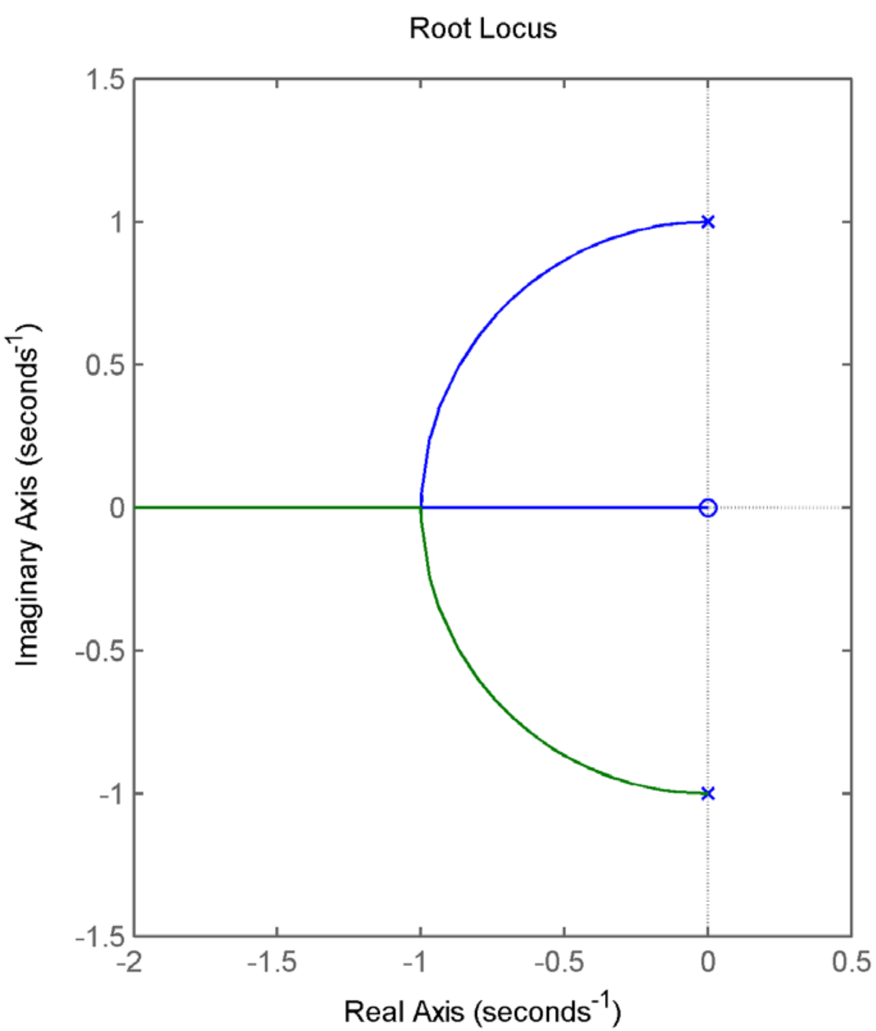
5. Break-in point

$$\frac{d}{d\sigma}\left(\frac{1}{G(\sigma)H(\sigma)}\right) = \frac{d}{d\sigma}\left(\frac{\sigma^2+1}{\sigma}\right) = 0$$

$$\frac{\sigma(2\sigma) - (\sigma^2+1)}{\sigma^2} = 0$$

$$\sigma^2 - 1 = 0 \quad \rightarrow \quad \sigma = +1, -1$$

   - $s = +1$ is not on the locus
   - Break-in point: $s = -1$



Design via Gain Adjustment

Design via Gain Adjustment

- Root locus provides a graphical representation of closed-loop pole locations vs. gain
- We have known relationships (some approx.) between pole locations and transient response
  - These apply to ***2nd-order systems with no zeros***
- Often, we don't have a 2nd-order system with no zeros
  - Would still like a link between pole locations and transient response
- Can sometimes approximate higher-order systems as 2nd-order
  - Valid only under certain conditions
  - Always verify response through simulation

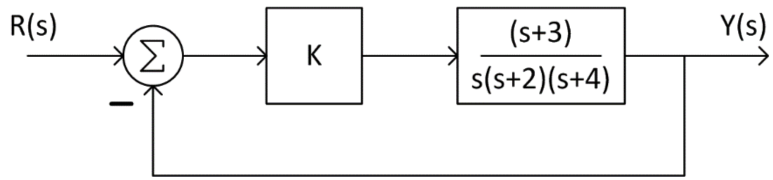
Second-Order Approximation

- A higher-order system with a pair of second-order poles can reasonably be approximated as second-order if:

1) Any higher-order closed-loop poles are either:
   a) at much higher frequency ($> \sim 5\times$) than the dominant 2nd-order pair of poles, or
   b) nearly canceled by closed-loop zeros

2) Closed-loop zeros are either:
   a) at much higher frequency ($> \sim 5\times$) than the dominant 2nd-order pair of poles, or
   b) nearly canceled by closed-loop poles

Design via Gain Adjustment – Example



- Determine K for 10% overshoot
- Assuming a 2nd-order approximation applies:

$$\zeta = \frac{-\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.59$$

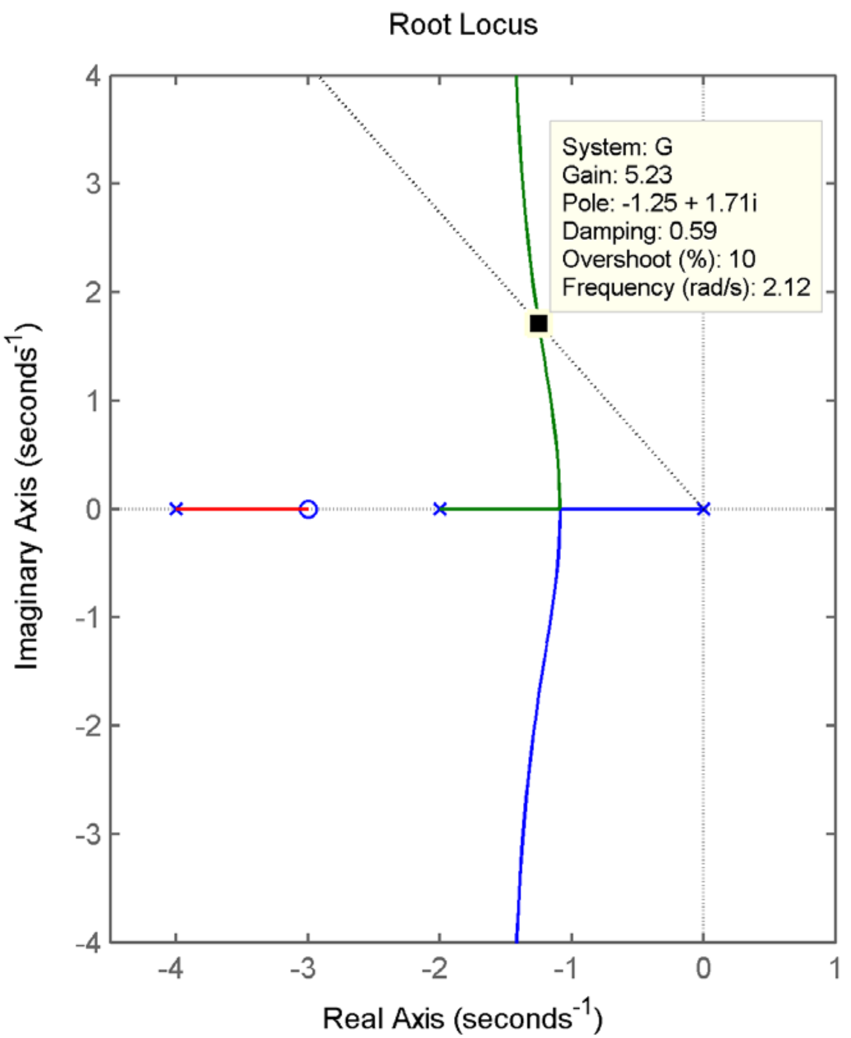
- Next, ***plot root locus*** in MATLAB
- ***Find gain*** corresponding to 2nd-order poles with $\zeta = 0.59$
  - *If possible* – often it is not
- Determine if a ***2nd-order approximation*** is justified
- Verify transient response through ***simulation***

Design via Gain Adjustment – Example

- Root locus shows that a pair of closed-loop poles with $\zeta = 0.59$ exist for $K = 5.23$:

$$s_{1,2} = -1.25 \pm j1.71$$

- Where is the third closed-loop pole?



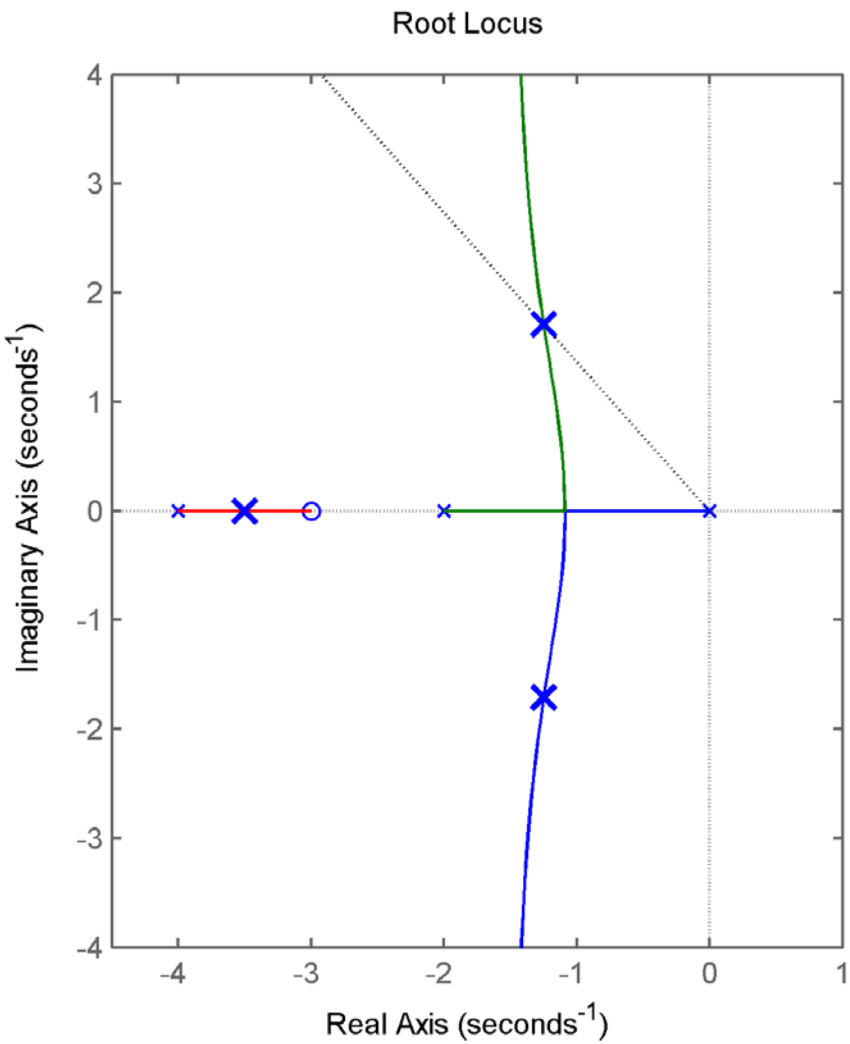
Design via Gain Adjustment – Example

- Third pole is at

$$s = -3.5$$

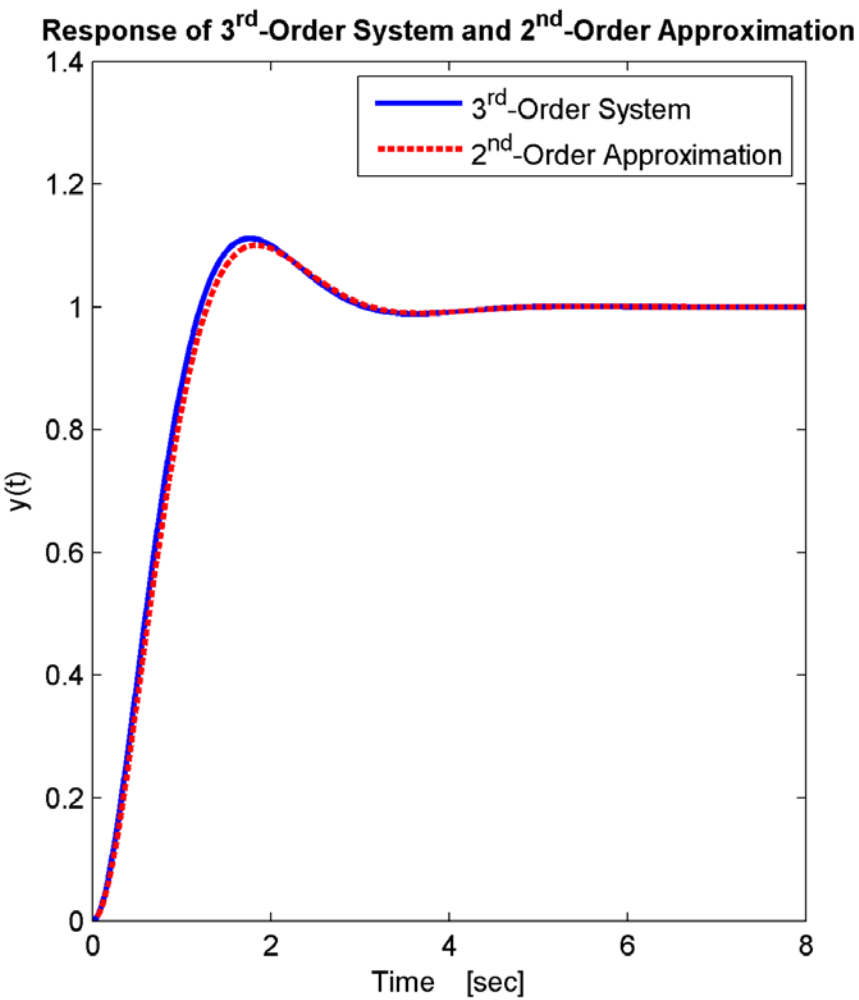
  - Not high enough in frequency for its effect to be negligible
  - But, it is in close proximity to a closed-loop zero

- Is a 2^{nd}-order approximation justified?
  - Simulate



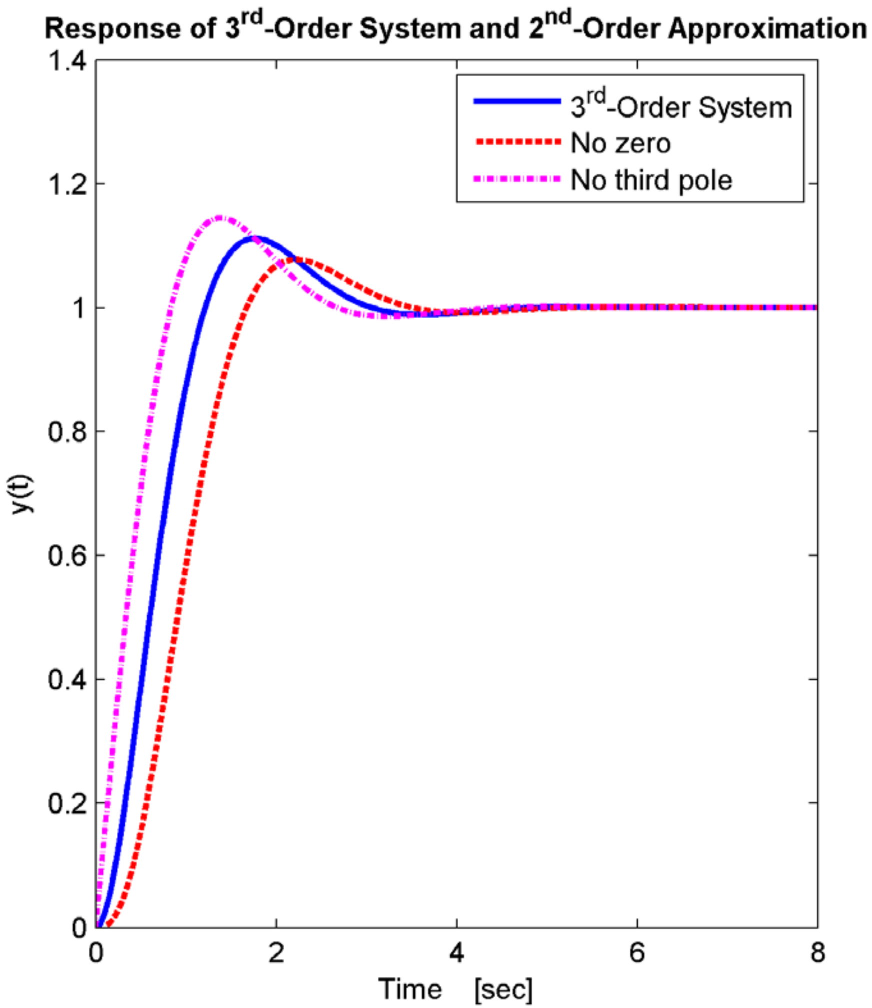
Design via Gain Adjustment – Example

- Step response compared to a true 2nd-order system
  - No third pole, no zero
- Very similar response
  - 11.14% overshoot
- 2nd-order approximation is valid
- Slight reduction in gain would yield 10% overshoot



Design via Gain Adjustment – Example

- Step response compared to systems with:
  - No zero
  - No third pole
- Quite different responses
- Partial pole/zero cancellation makes 2nd-order approximation valid, in this example



When Gain Adjustment Fails

- Root loci do not go through every point in the s-plane
  - Can't always satisfy a single performance specification, e.g. overshoot *or* settling time
  - Can satisfy two specifications, e.g. overshoot *and* settling time, even less often
- Also, gain adjustment affects steady-state error performance
  - In general, cannot simultaneously satisfy dynamic requirements and error requirements
- In those cases, we must ***add dynamics to the controller***
  - A ***compensator***