SECTION 7: FREQUENCY-RESPONSE ANALYSIS

ESE 430 – Feedback Control Systems

Introduction

Introduction

- We have seen how to design feedback control systems using the ***root locus***
- In the final two sections of the course, we'll learn how to do the same using the open-loop ***frequency response***

- **Objectives**:
  - Review the relationship between a system's frequency response and its transient response
  - Determine static error constants from the open-loop frequency response
  - Determine closed-loop stability from the open-loop frequency response

Relationship between Frequency Response and Transient Response

Transient/Frequency Response Relationship

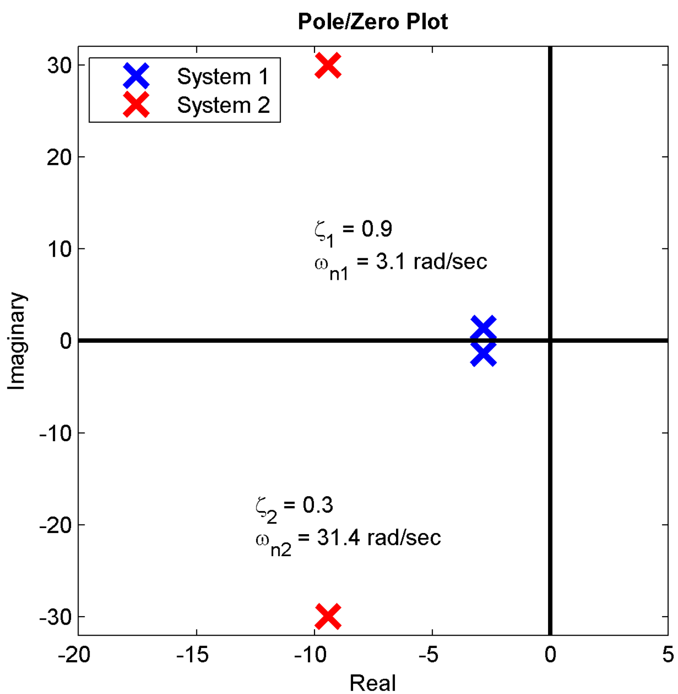
- We have relationships – some exact, some approximate – between closed-loop pole locations and closed-loop transient response
- Also have relationships between ***closed-loop frequency response*** and ***closed-loop transient responses***
- Applicable to ***second-order systems***:

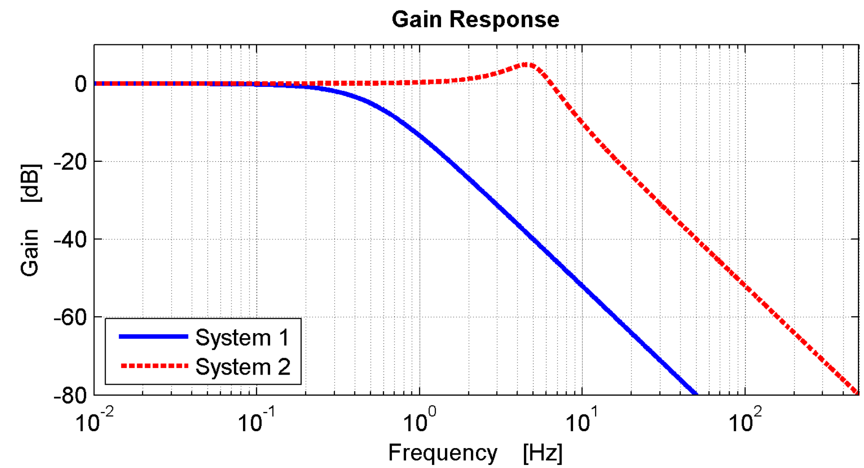
$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

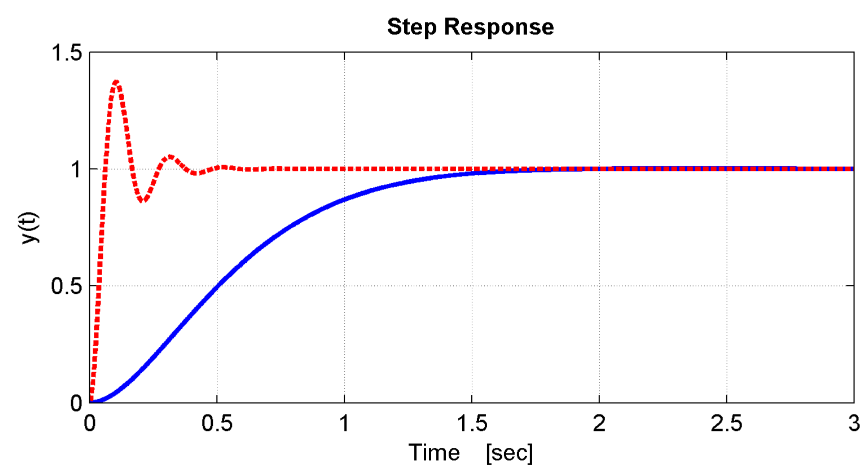
- Also applicable to higher-order systems that are reasonably ***approximated as second-order***
  - Systems with a pair of dominant second-order poles

Transient/Frequency Response Relationship

- Qualitative 2nd-order time/freq. response/pole relationships
  - Damping ratio vs. overshoot vs. peaking
  - Natural frequency vs. risetime vs. bandwidth







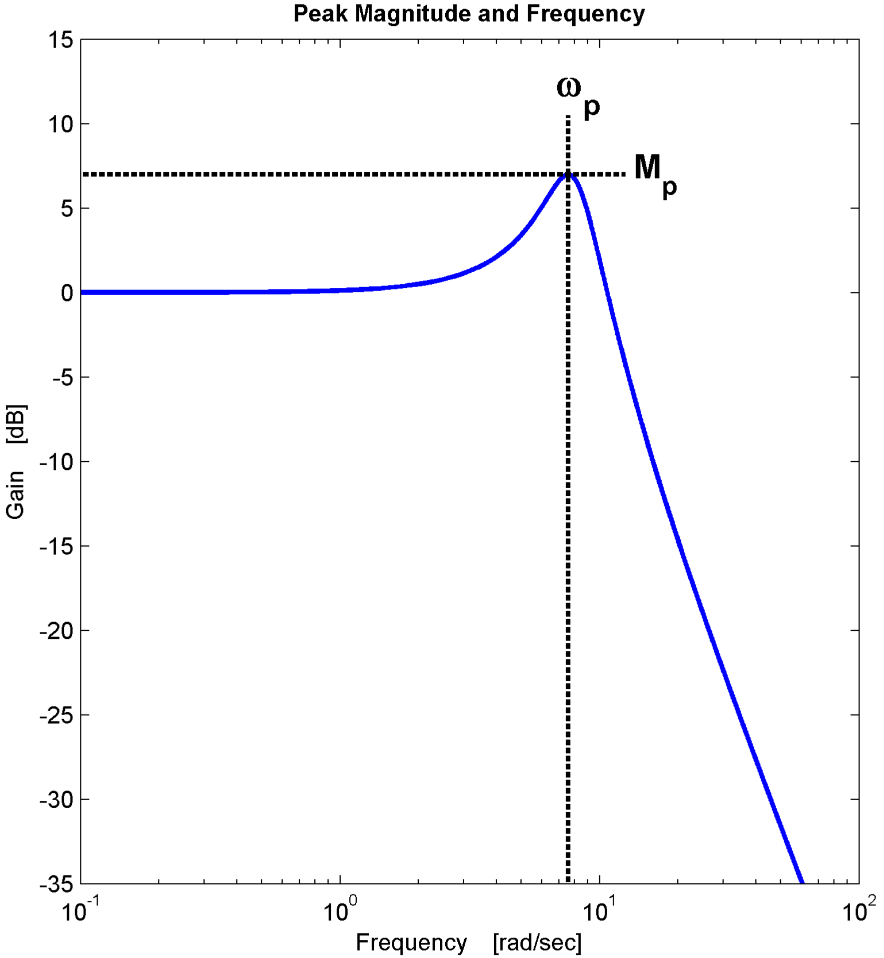
Frequency Response Peaking

- For systems with $\zeta < 0.707$, the gain response will exhibit ***peaking***
- Can relate ***peak magnitude*** to the damping ratio

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

  - Relative to low-frequency gain
- And the ***peak frequency*** to the damping ratio and natural frequency

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$



Transient/Frequency Response Relationship

- Can also relate a system's ***bandwidth*** (i.e., -3dB frequency, ω_{BW}) to the speed of its step response
- Bandwidth as a function of $\boldsymbol{\omega_n}$ and $\boldsymbol{\zeta}$

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

- Bandwidth as a function of ***1% settling time*** and $\boldsymbol{\zeta}$

$$\omega_{BW} = \frac{4.6}{t_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

- Bandwidth as a function of ***peak time*** and $\boldsymbol{\zeta}$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Steady-State Error from Bode Plots

Static Error Constants

- For unity-feedback systems, open-loop transfer function gives ***static error constants***
  - Use static error constants to calculate ***steady-state error***

$$K_p = \lim_{s \to 0} G(s)$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- We can also determine static error constants from a system's ***open-loop Bode plot***

Static Error Constant – Type 0

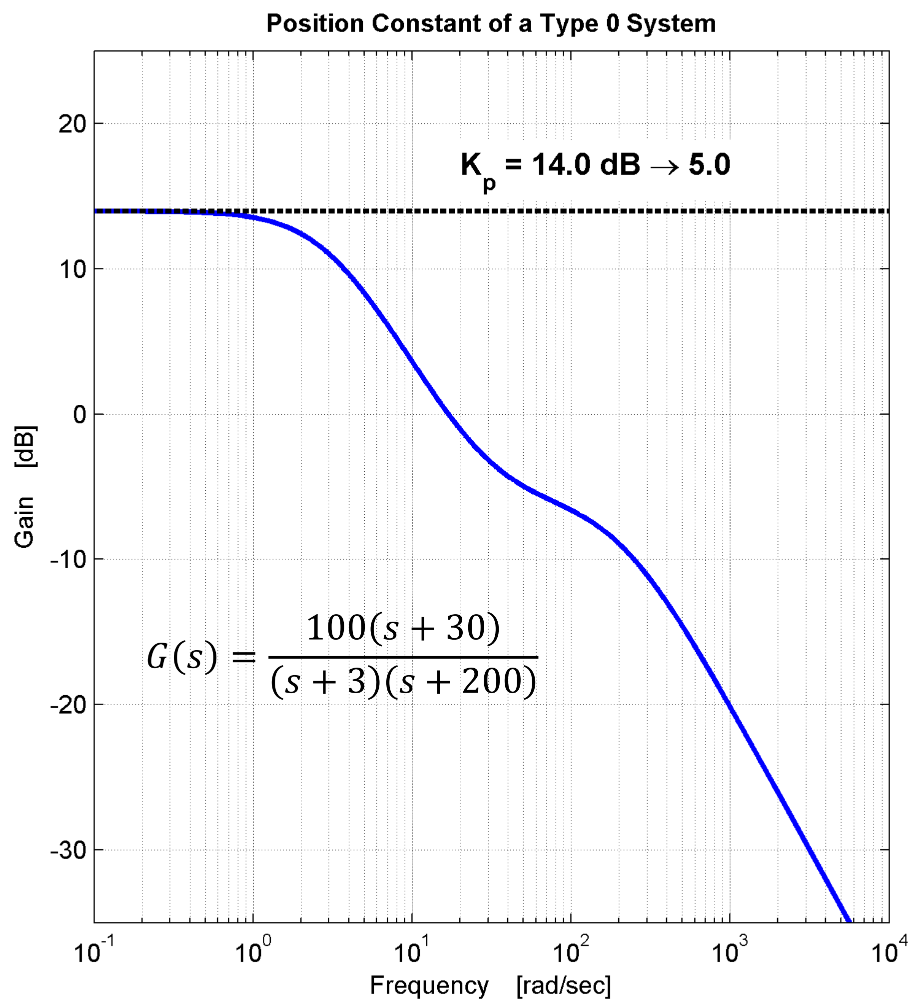
- For a type 0 system

$$K_p = \lim_{s \to 0} G(s)$$

- At low frequency, i.e. below any open-loop poles or zeros

$$G(s) \approx K_p$$

- Read K_p directly from the open-loop Bode plot
  - Low-frequency gain



Static Error Constant – Type 1

- For a type 1 system

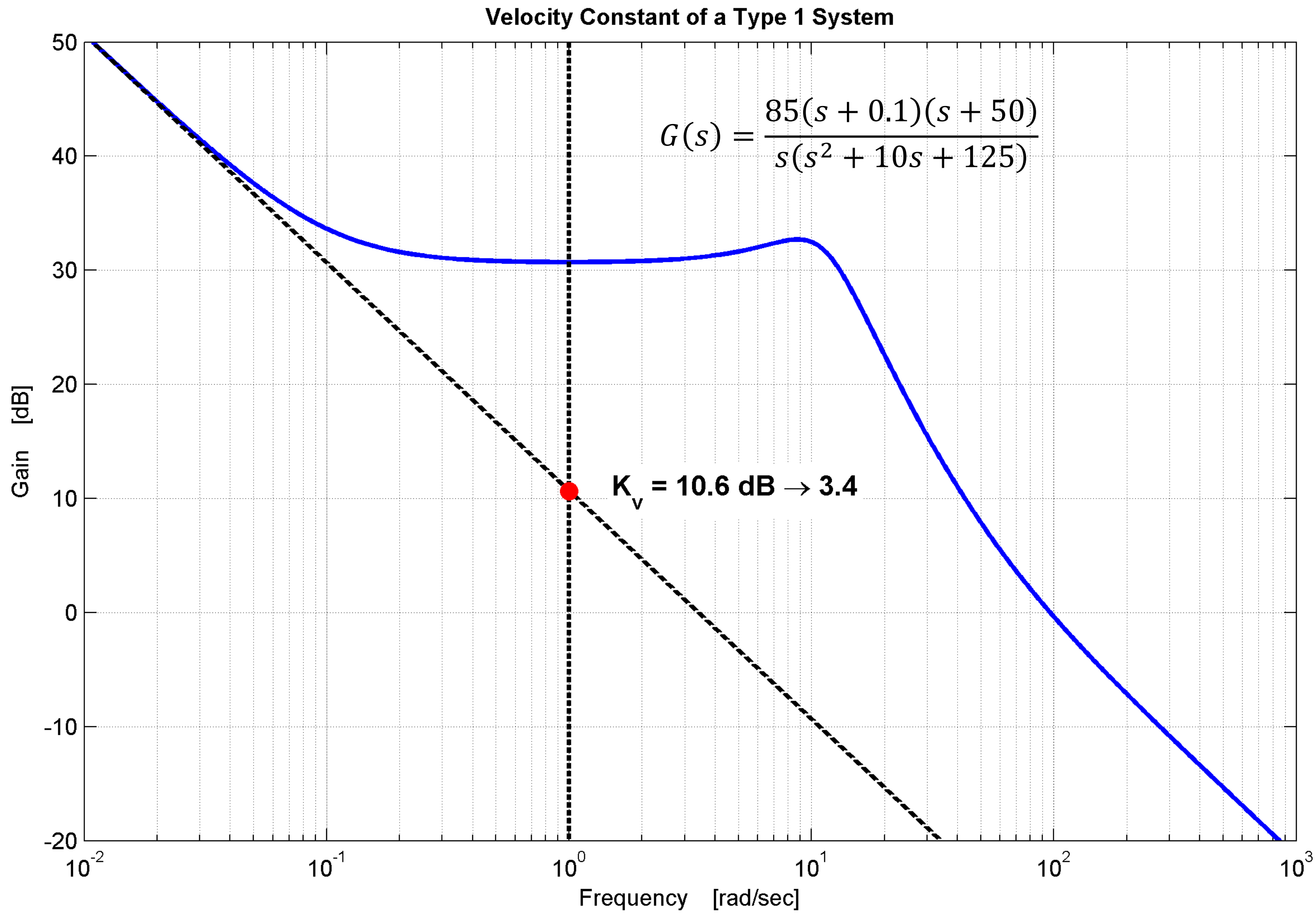
$$K_v = \lim_{s \to 0} sG(s)$$

- At low frequencies, i.e. below any other open-loop poles or zeros

$$G(s) \approx \frac{K_v}{s} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_v}{\omega}$$

- A straight line with a slope of $-20\ dB/dec$
- Evaluating this low-frequency asymptote at $\omega = 1$ yields the velocity constant, K_v
- On the Bode plot, extend the low-frequency asymptote to $\omega = 1$
  - Gain of this line at $\omega = 1$ is K_v

Static Error Constant – Type 1



Static Error Constant – Type 2

- For a type 2 system

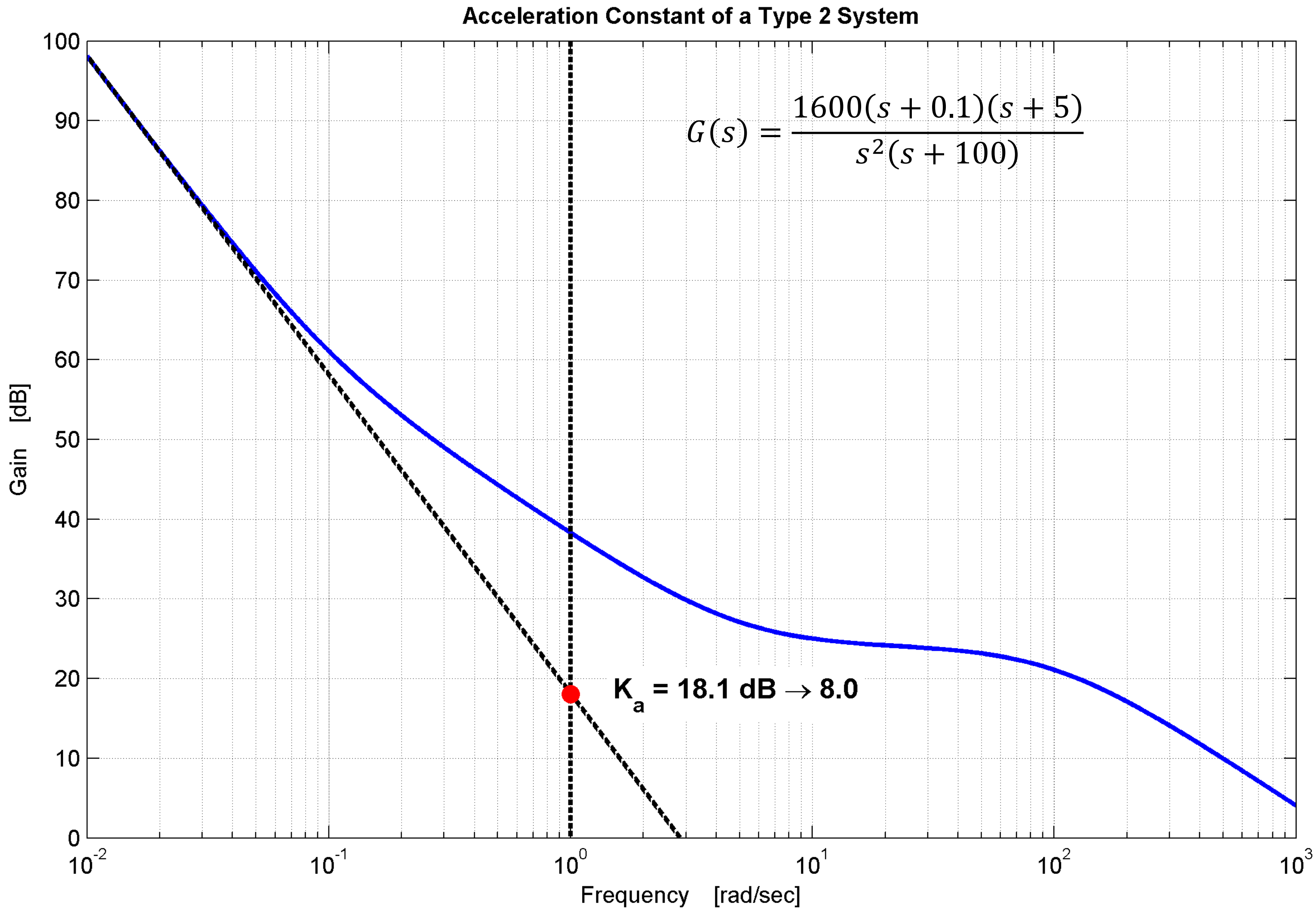
$$K_a = \lim_{s \to 0} s^2 G(s)$$

- At low frequencies, i.e. below any other open-loop poles or zeros

$$G(s) \approx \frac{K_a}{s^2} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_a}{\omega^2}$$

- A straight line with a slope of $-40\ dB/dec$
- Evaluating this low-frequency asymptote at $\omega = 1$ yields the acceleration constant, K_a
- On the Bode plot, extend the low-frequency asymptote to $\omega = 1$
  - Gain of this line at $\omega = 1$ is K_a

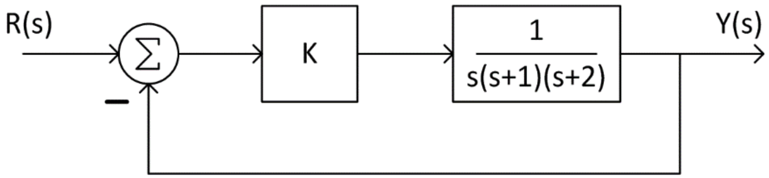
Static Error Constant – Type 2



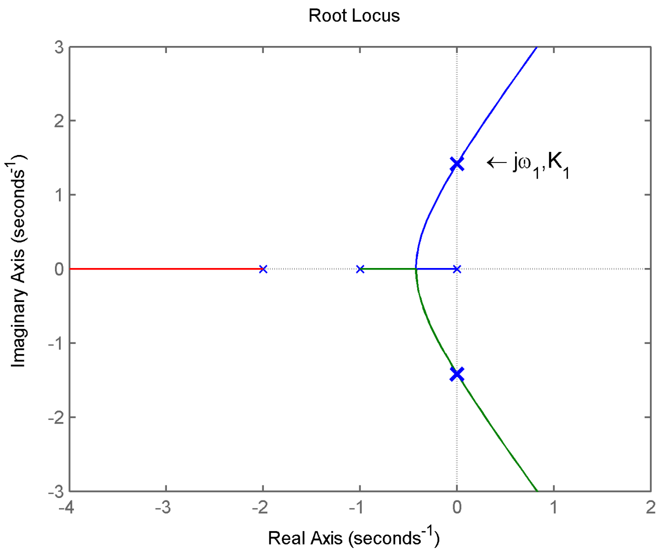
Frequency Response & Stability

Stability

- Consider the following system

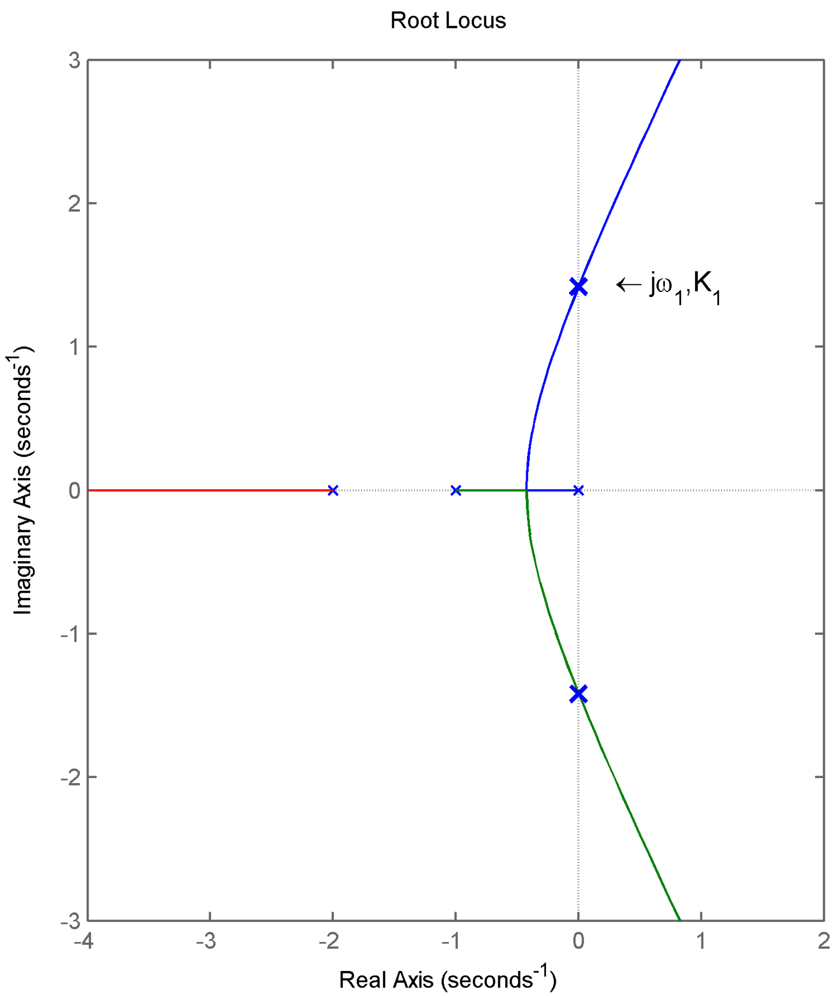


- We already have a couple of tools for assessing stability as a function of loop gain, K
  - Routh Hurwitz
  - Root locus
- Root locus:
  - Stable for some values of K
  - Unstable for others



Stability

- In this case gain is stable ***below*** some value
- Other systems may be stable for gain ***above*** some value
- Marginal stability point:
  - Closed-loop poles on the imaginary axis at $\pm j\omega_1$
  - For gain $K = K_1$



Open-Loop Frequency Response & Stability

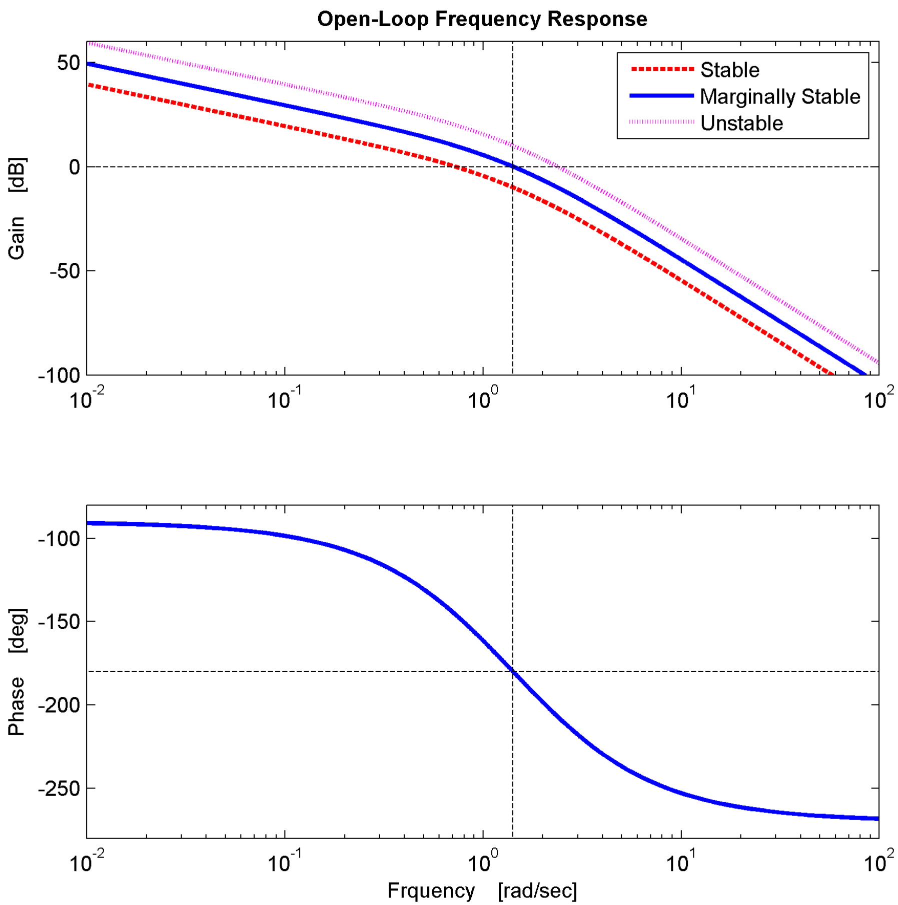
- Marginal stability point occurs when closed-loop poles are on the imaginary axis
  - Angle criterion satisfied at $\pm j\omega_1$

$$|KG(j\omega_1)| = 1 \quad \text{and} \quad \angle KG(j\omega_1) = -180°$$

  - Note that $-180° = 180°$
- $KG(j\omega)$ is the ***open-loop frequency response***
- ***Marginal stability*** occurs when:
  - Open-loop gain is: $KG(j\omega) = 0\ dB$
  - Open-loop phase is: $\angle KG(j\omega) = -180°$

Stability from Bode Plots

- Varying K simply shifts gain response up or down
- Here, stable for smaller gain values
  - $|KG(j\omega)| < 0\ dB$ when
    $\angle KG(j\omega) = -180°$
- Often, stable for larger gain values
  - $|KG(j\omega)| > 0\ dB$ when
    $\angle KG(j\omega) = -180°$
- Root locus provides this information
  - Bode plot does not



Open-Loop Frequency Response & Stability

- A method does exist for determining stability from the open-loop frequency response:
- ***Nyquist stability criterion***
  - Graphical technique
  - Uses open-loop frequency response
  - Determine system stability
  - Determine gain ranges for stability
- Before introducing the Nyquist criterion, we must first introduce the concept of ***complex functional mapping***

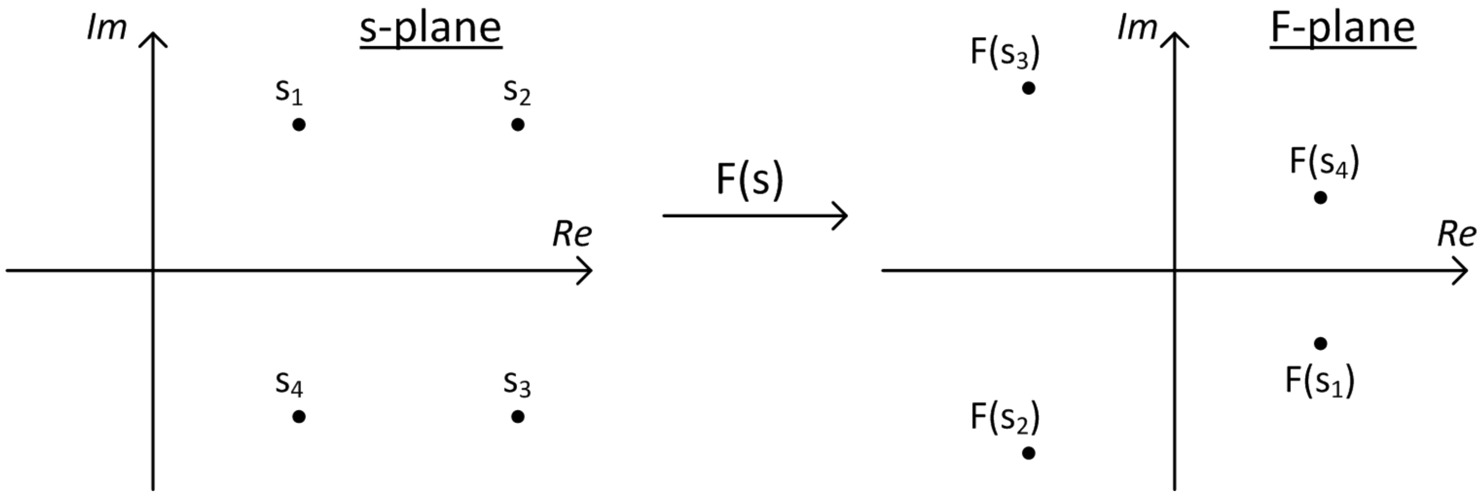
Complex Functional Mapping

Complex Functional Mapping

- Consider a complex function

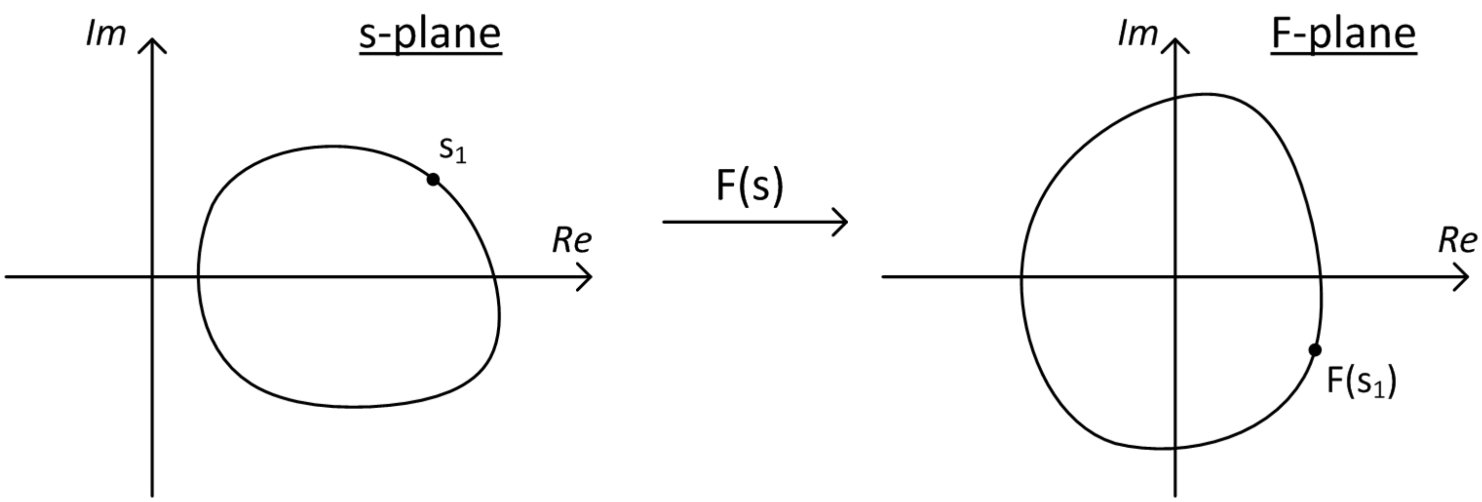
$$F(s) = \frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots}$$

- Takes one complex value, s, and yields a second complex value, $F(s)$
  - In other words, it ***maps*** s to $F(s)$



Mapping of Contours

- $F(s)$ provides a mapping of individual points in the s-plane to corresponding points in the F-plane
- Can also map all points around a ***contour*** in the s-plane to another contour in the F-plane



Mapping of Contours

- Recall how we approached the application of the angle criterion
  - Vector approach to the evaluation of a transfer function at a particular point in the s-plane

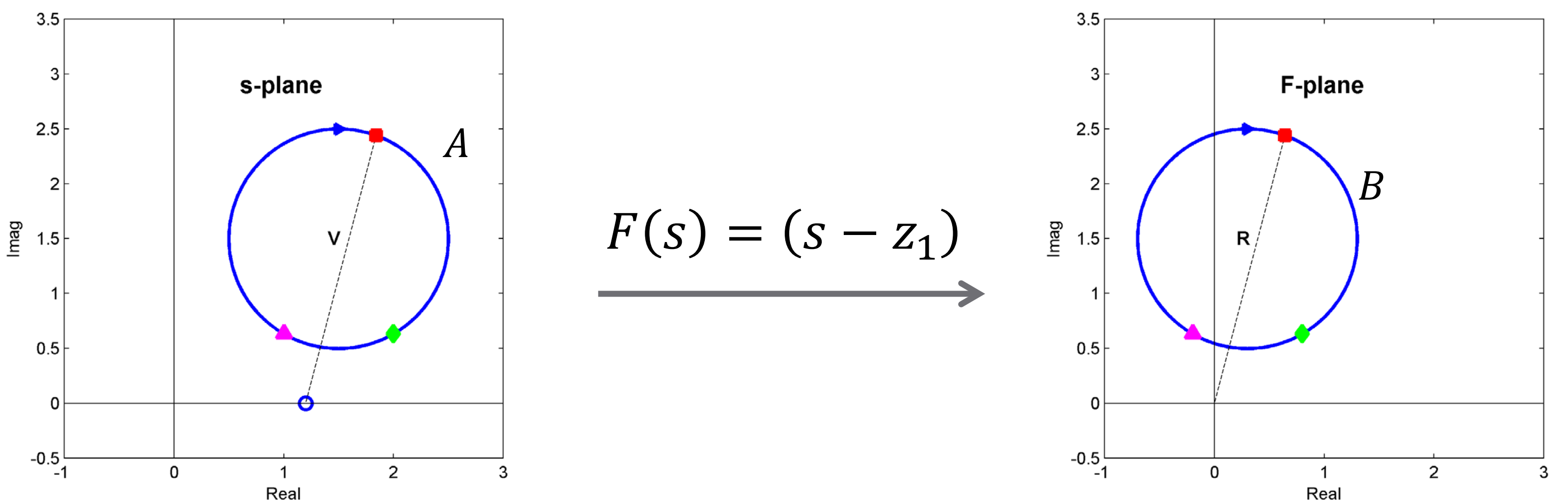
$$|G(s_1)| = \frac{\prod |vectors\ from\ zeros\ to\ s_1|}{\prod |vectors\ from\ poles\ to\ s_1|}$$

$$\angle G(s_1) = \Sigma\angle(\ from\ zeros\ to\ s_1) - \Sigma\angle(\ from\ poles\ to\ s_1)$$

- Can take the same approach to evaluating complex functions around ***contours*** in the s-plane

Mapping Contours – Example 1

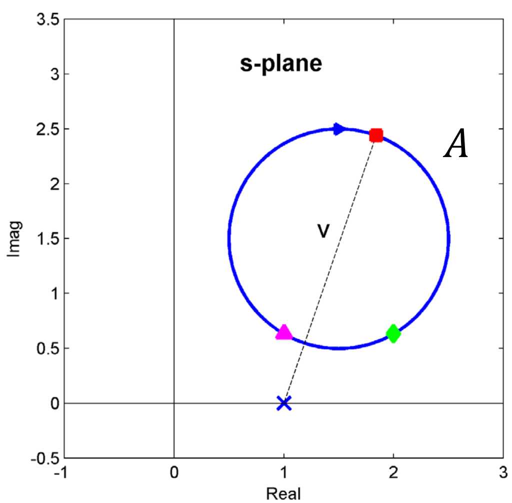
- Map contour A by $F(s) = (s - z_1)$ in a ***clockwise*** direction
  - Contour A does not enclose the zero
- Here, $R = V$, so $|R| = |V|$ and $\angle R = \angle V$



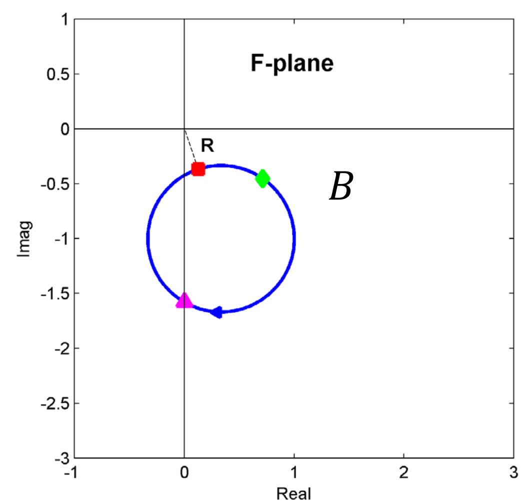
- As $F(s)$ is evaluated around A, $\angle V$ never exceeds $0°$ or $180°$
- R does the same:
  - Does not rotate through a full $360°$
  - ***Contour B does not encircle the origin***

Mapping Contours – Example 2

- Map contour A by $F(s)$ in a ***clockwise*** direction
  - Contour A does not enclose the pole
- Here, $R = 1/V$, so $|R| = 1/|V|$ and $\angle R = -\angle V$



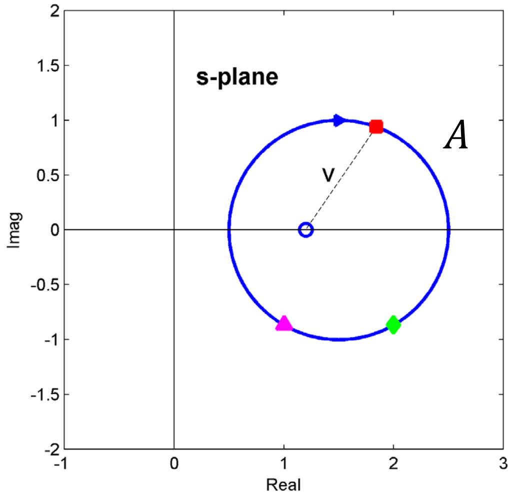
$$F(s) = \frac{1}{(s - p_1)}$$



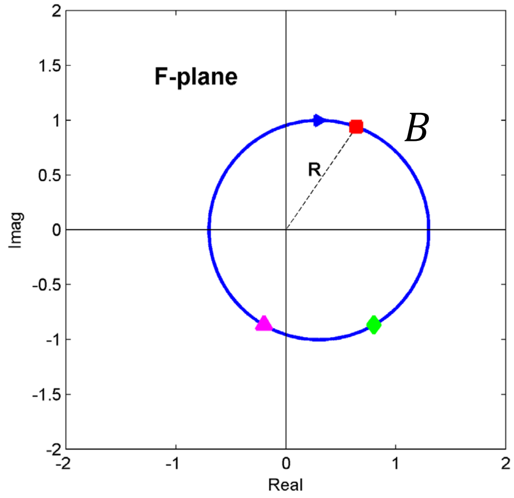
- $\angle V$ oscillates over some range well within $0°$ and $180°$
  - R rotates through the *negative* of the same range
  - ***Contour B does not encircle the origin***

Mapping Contours – Example 3

- Now, ***contour A encloses a single zero***
- $R = V$, so $|R| = |V|$ and $\angle R = \angle V$



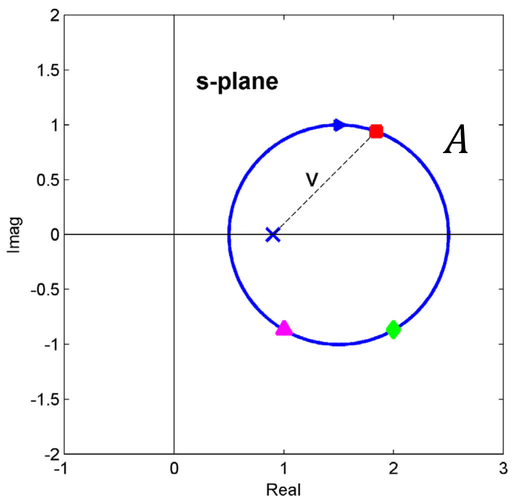
$$F(s) = (s - z_1)$$



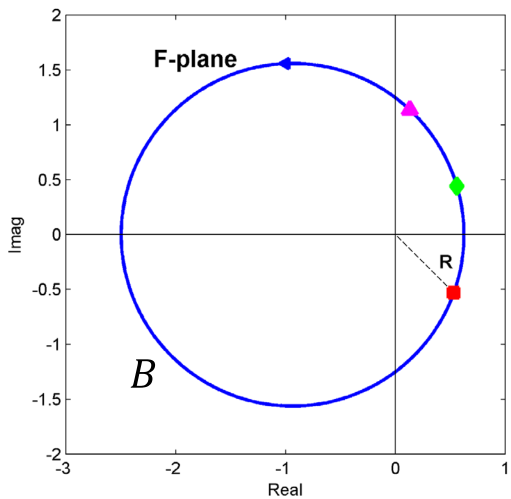
- V rotates through a full $360°$ in a clockwise direction
- R does the same:
  - ***Contour B encircles the origin in a <u>clockwise</u> direction***

Mapping Contours – Example 4

- Now, ***contour A encloses a single pole***
- $R = 1/V$, so $|R| = 1/|V|$ and $\angle R = -\angle V$



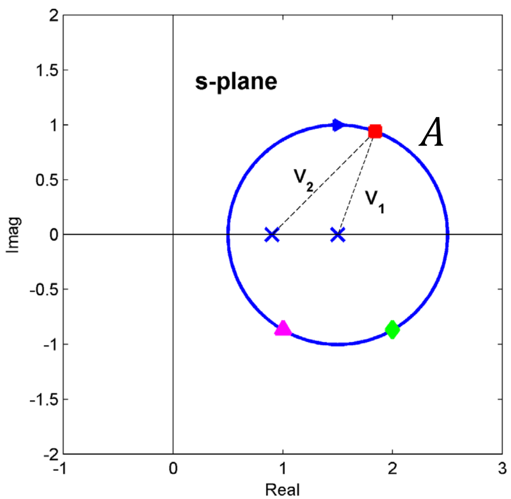
$$F(s) = \frac{1}{(s - p_1)}$$



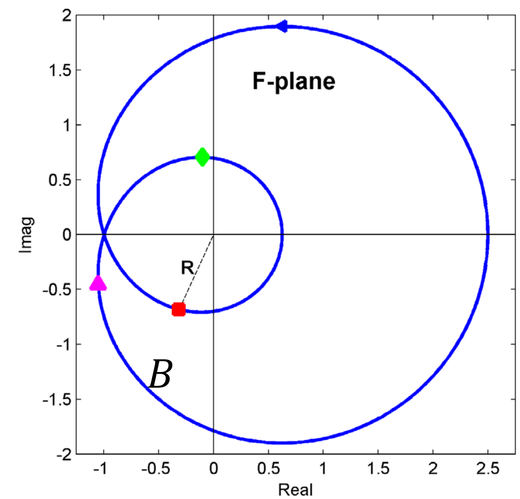
- V rotates through a full 360° in a clockwise direction
  - R rotates in the *opposite direction*
  - ***Contour B encircles the origin in a <u>CCW</u> direction***

Mapping Contours – Example 5

- Now, ***contour A encloses two poles***
- $R = \frac{1}{V_1 V_2}$, so $|R| = \frac{1}{|V_1||V_2|}$ and $\angle R = -(\angle V_1 + \angle V_2)$



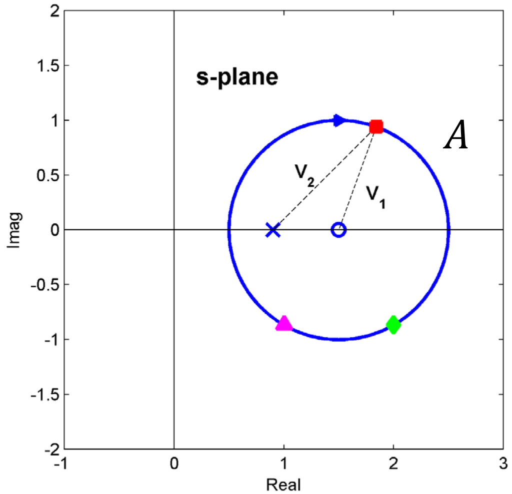
$$F(s) = \frac{1}{(s - p_1)(s - p_2)}$$



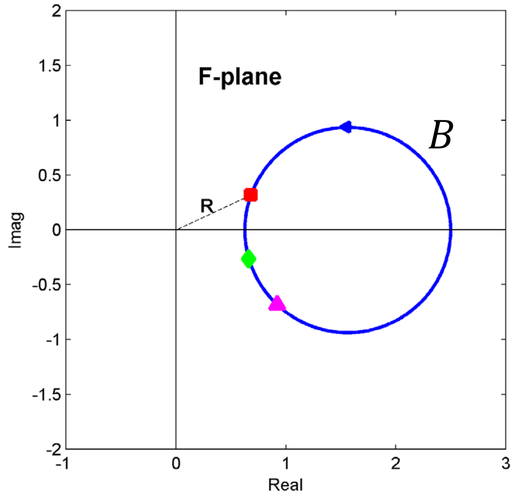
- V_1 and V_2 *each* rotate through a full 360° in a clockwise direction
  - R rotates in the *opposite direction*
  - ***Contour B encircles the origin <u>twice</u> in a <u>CCW</u> direction***

Mapping Contours – Example 6

- Now, ***contour A encloses one pole and one zero***
- $R = \frac{V_1}{V_2}$, so $|R| = \frac{|V_1|}{|V_2|}$ and $\angle R = \angle V_1 - \angle V_2$



$$F(s) = \frac{(s - z_1)}{(s - p_1)}$$



- $\angle V_1$ and$\angle V_2$ rotate through $360°$ in a CW direction
  - Their contributions rotate in *opposite* directions
  - $\angle R$ does not rotate through a full $360°$
  - ***Contour B does not encircle the origin***

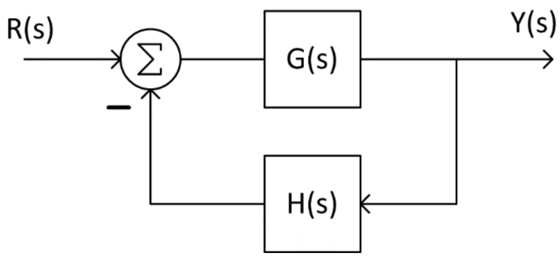
Complex Functional Mapping of Contours

- Some observations regarding complex mapping of contour A in a CW direction to contour B:
  - If A does not enclose any poles or zeros, B does not encircle the origin
  - If A encloses a single pole, B will encircle the origin once in a CCW direction
  - If A encloses two poles, B will make two CCW encirclements of the origin
  - If A encloses a pole and a zero, B will not encircle the origin
- Next, we'll use these observations to help derive the ***Nyquist stability criterion***

Nyquist Stability Criterion

Nyquist Stability Criterion

- Our goal is to assess closed-loop stability
  - Determine if there are any closed-loop poles in the RHP
- Consider a generic feedback system:



- Closed-loop transfer function

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

- Closed-loop poles are roots (zeros) of the closed-loop characteristic polynomial:

$$1 + G(s)H(s)$$

Nyquist Stability Criterion

- Can represent the individual transfer functions as

$$G(s) = \frac{N_G}{D_G} \quad \text{and} \quad H(s) = \frac{N_H}{D_H}$$

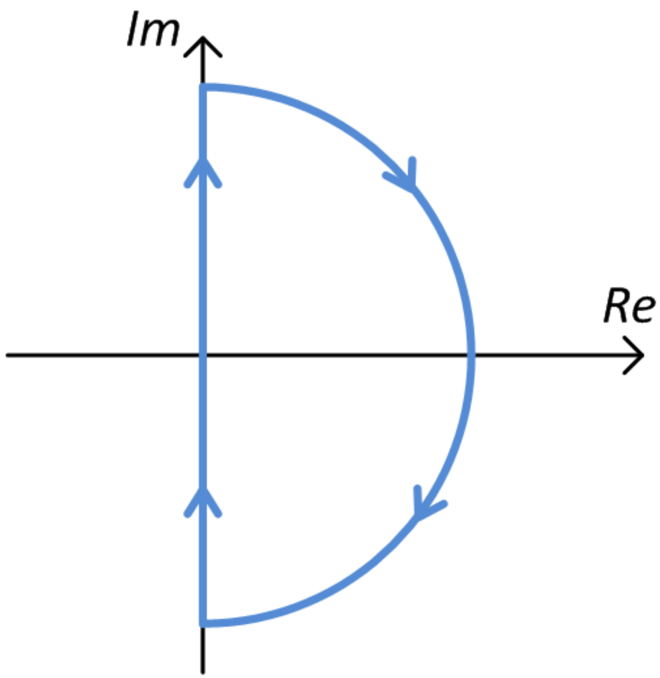
- The closed-loop characteristic polynomial becomes

$$1 + G(s)H(s) = 1 + \frac{N_G}{D_G}\frac{N_H}{D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

- From this, we can see that:
  - The ***poles*** of $1 + G(s)H(s)$ are the poles of $G(s)H(s)$, the ***open-loop poles***
  - The ***zeros*** of $1 + G(s)H(s)$ are the poles of $T(s)$, the ***closed-loop poles***

Nyquist Stability Criterion

- To determine stability, look for RHP closed-loop poles
- Evaluate $1 + G(s)H(s)$ CW around a contour that encircles the ***entire right half-plane***
  - Evaluate $1 + G(s)H(s)$ along ***entire $j\omega$-axis***
  - Encircle the entire RHP with an ***infinite-radius arc***
- If $1 + G(s)H(s)$ has ***one RHP pole***, resulting contour will ***encircle the origin once CCW***
- If $1 + G(s)H(s)$ has ***one RHP zero***, resulting contour will ***encircle the origin once CW***



Nyquist Stability Criterion

- Total number of CW encirclements of the origin, N, by the resulting contour will be

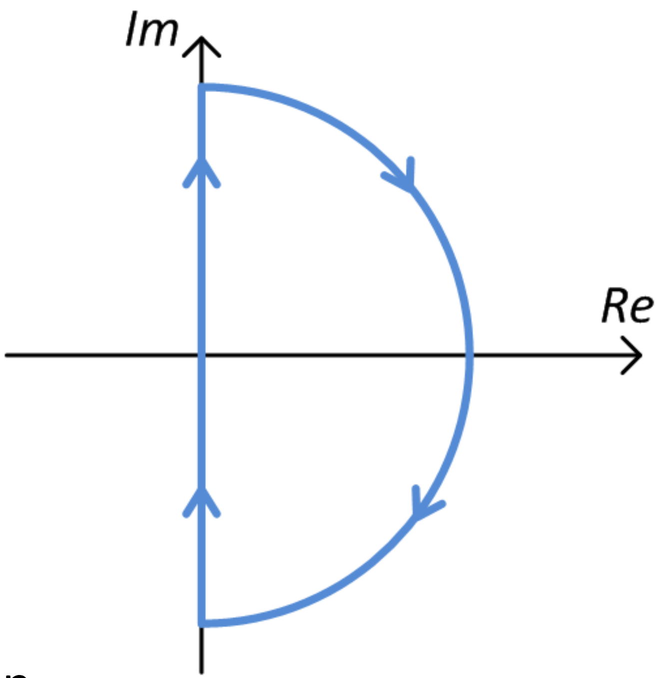
$$N = Z - P$$

  - $P =$ # of RHP poles of $1 + G(s)H(s)$
  - $Z =$ # of RHP zeros of $1 + G(s)H(s)$

- Want to detect RHP ***poles*** of $T(s)$, ***zeros*** of $1 + G(s)H(s)$, so

$$Z = N + P$$

  - $Z =$ # of closed-loop RHP poles
  - $P =$ # of open-loop RHP poles
  - $N =$ # of CW encirclements of the origin



Nyquist Stability Criterion

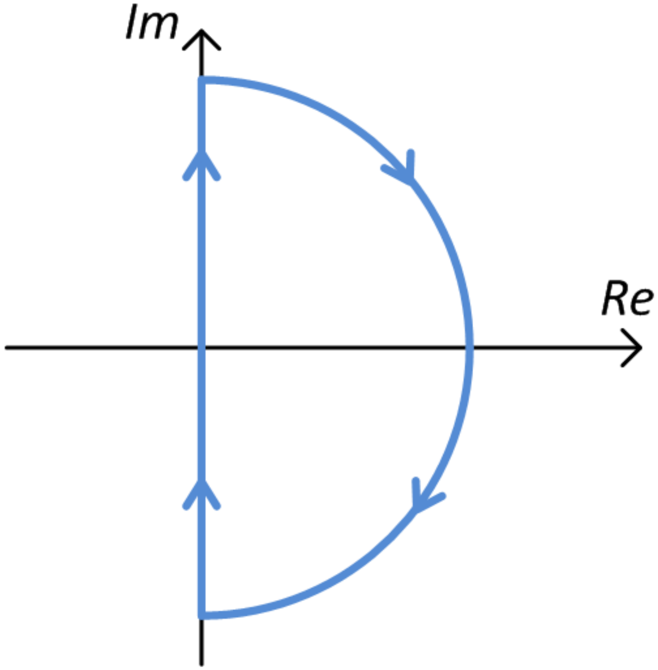
- Basis for detecting closed-loop RHP poles
  - Map contour encircling the entire RHP through closed-loop characteristic polynomial
  - Count number of CW encirclements of the origin by resulting contour
  - Calculate the number of closed-loop RHP poles:

$$Z = N + P$$

- Need to know:
  - Closed-loop characteristic polynomial
  - Number of RHP poles of closed-loop characteristic polynomial

Nyquist Stability Criterion

- Instead, map through $G(s)H(s)$
  - Open-loop transfer function
  - Easy to use for mapping – we know poles and zeros
  - Resulting contour shifts left by 1 – that's all
- Now, count encirclements of the point $s = -1$



Nyquist Stability Criterion

- ***Nyquist stability criterion***
  - *If a contour that encloses the entire RHP is mapped through the open-loop transfer function, $G(s)H(s)$, then the number of closed-loop RHP poles, Z, is given by*

$$Z = N + P$$

where

$N =$ # of CW encirclements of -1

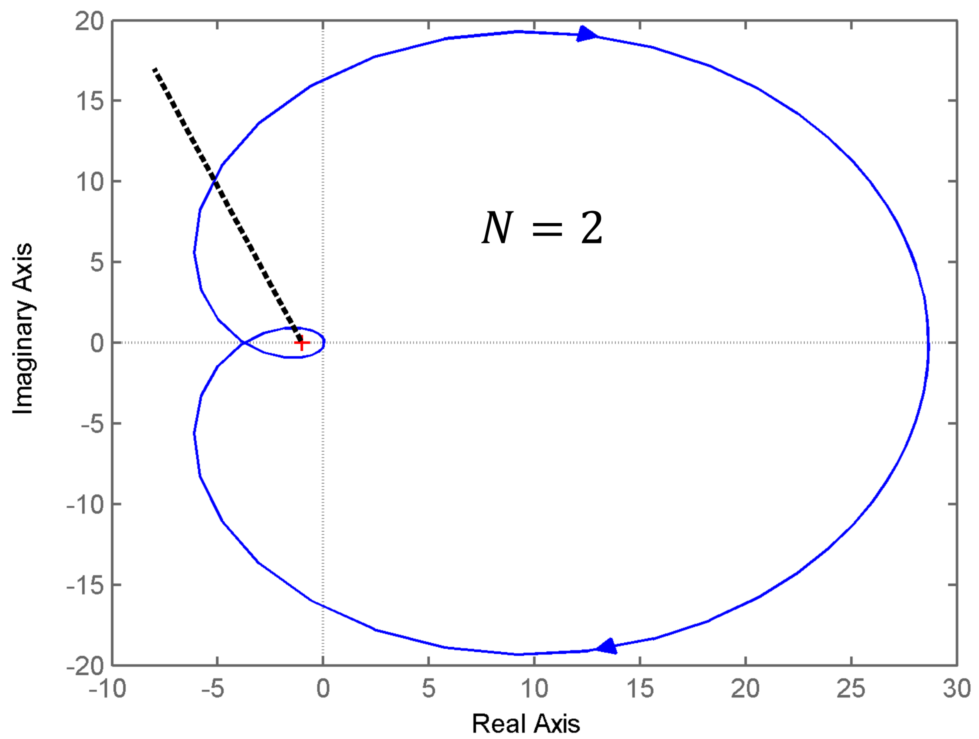
$P =$ # of open-loop RHP poles

Nyquist Stability Criterion

- Want to detect ***net clockwise encirclements***

$$N = \text{\# CW encirclements - \# CCW encirclements}$$

  - Draw a line from $s = -1$ in any direction
  - Count number of times contour crosses the line in each direction

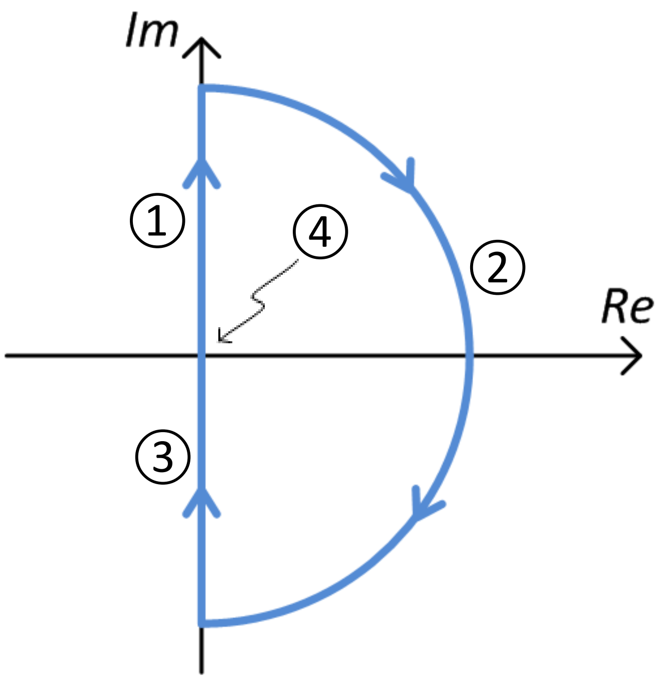


Nyquist Diagrams

Nyquist Diagram

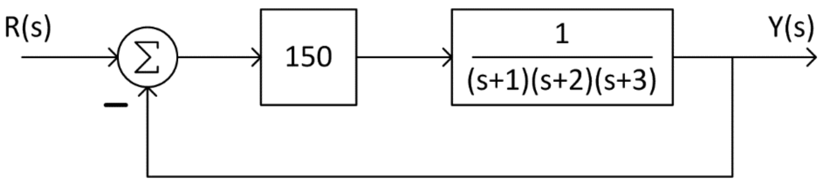
- The contour that results from mapping the perimeter of the entire RHP is a ***Nyquist diagram***
- Consider four segments of the contour:

1) Along positive $j\omega$-axis, we're evaluating $G(j\omega)H(j\omega)$
   - Open-loop frequency response
2) Here, $s \rightarrow C^{\infty}$
   - Maps to zero for any physical system
3) Here, evaluating $G(-j\omega)H(-j\omega)$
   - Complex conjugate of segment ①
   - Mirror ① about the real axis
4) The origin
   - Sometimes a special case – more later

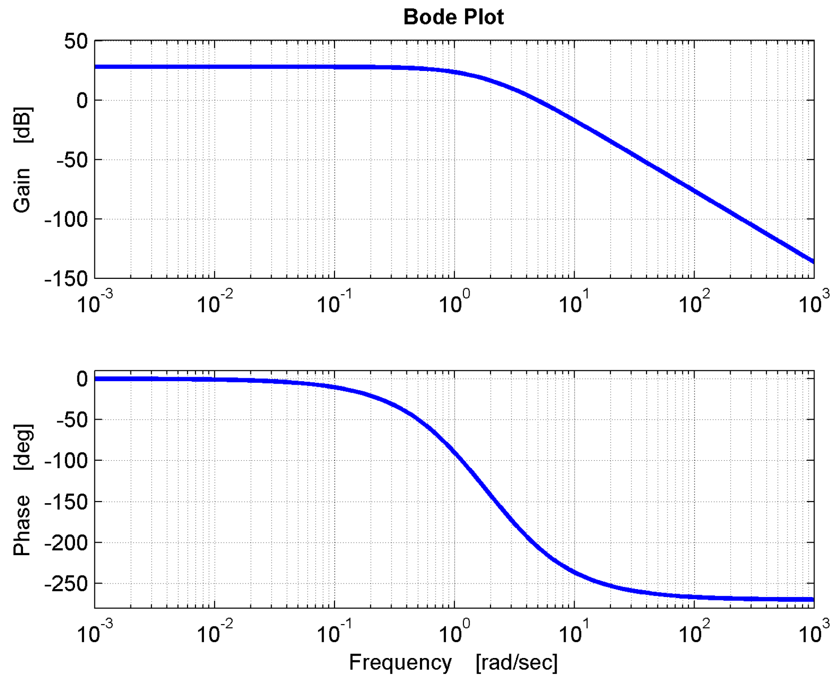


Nyquist Criterion – Example 1

- Apply the Nyquist criterion to determine stability for the following system

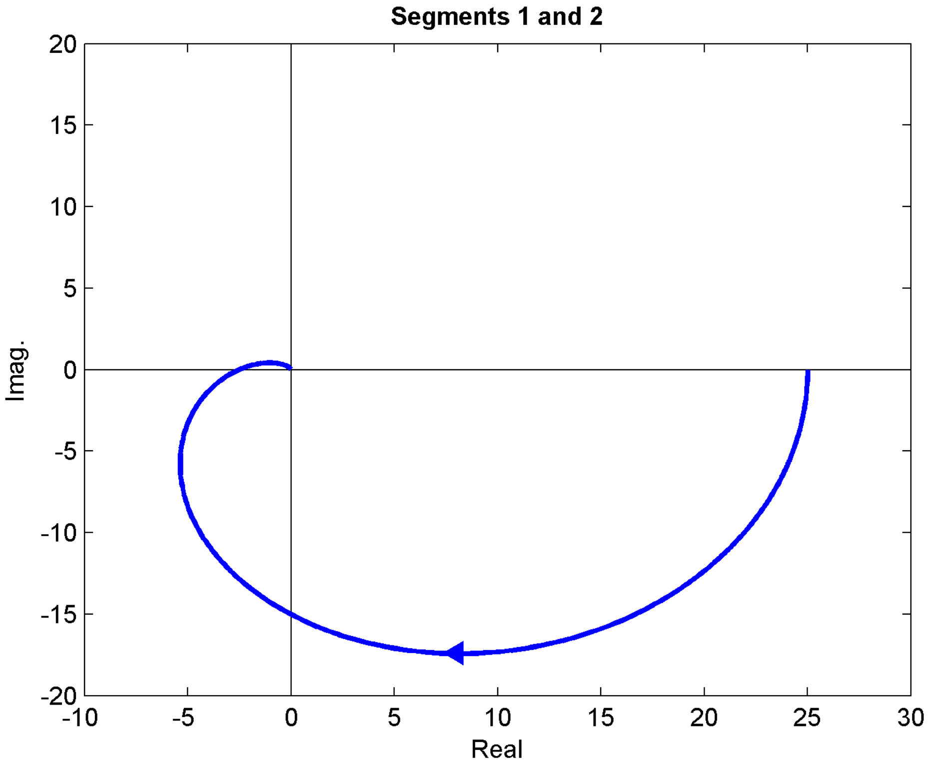


- First evaluate along segment ①, $+j\omega$-axis
  - This is the frequency response
  - Read values off of the Bode plot



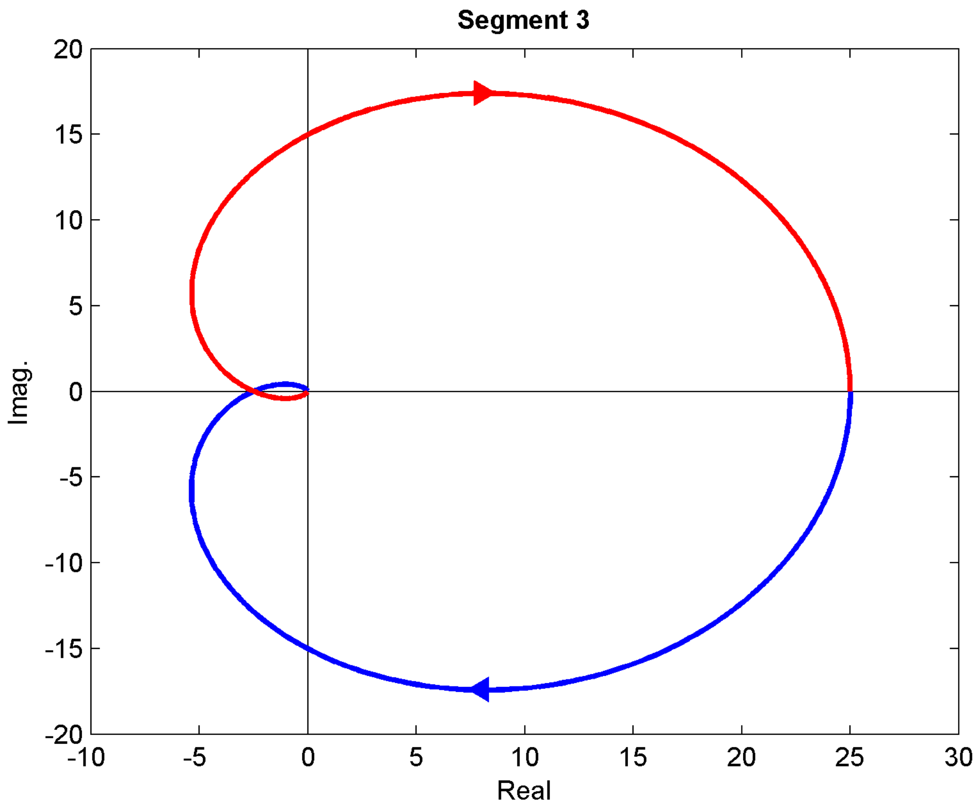
Nyquist Criterion – Example 1

- Segment ① is a polar plot of the frequency response
- All of segment ②, arc at C^{∞}, maps to the origin



Nyquist Criterion – Example 1

- Segment ③ is the complex conjugate of segment ①
  - Mirror about the real axis



Nyquist Criterion – Example 1

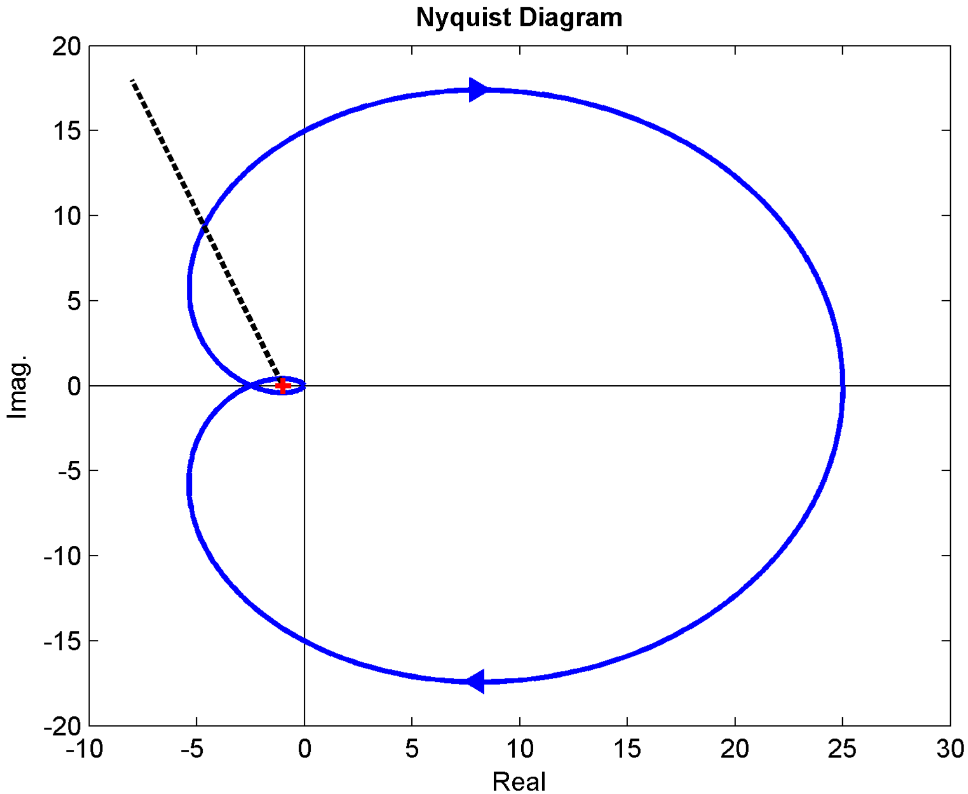
- Count CW encirclements of $s = -1$
  - Draw a line from $s = -1$ in any direction
- Here, $N = 2$
- Closed-loop RHP poles given by:

$$Z = N + P$$

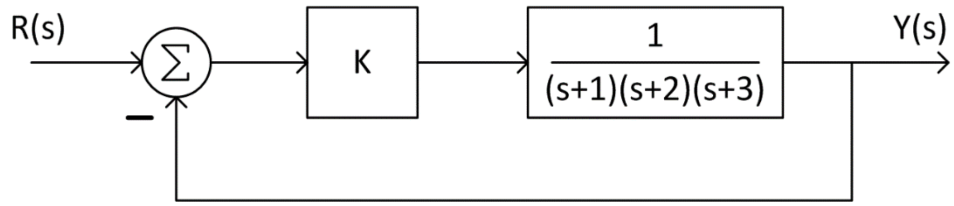
- No open-loop RHP poles, so $P = 0$

$$Z = 2 + 0 = 2$$

- Two RHP poles, so system is ***unstable***



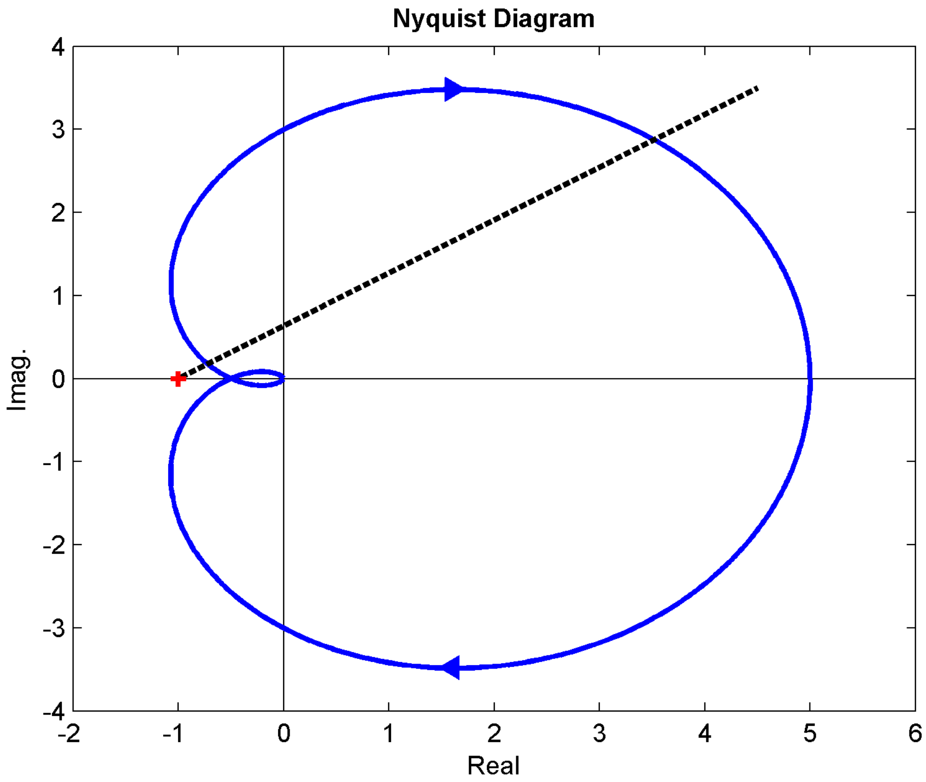
Nyquist Criterion – Example 2



- This system is open-loop stable
  - Stable for low enough K
  - Nyquist plot will not encircle $s = -1$
- Three poles and no zeros
  - Unstable for K above some value
  - Nyquist plot will encircle $s = -1$

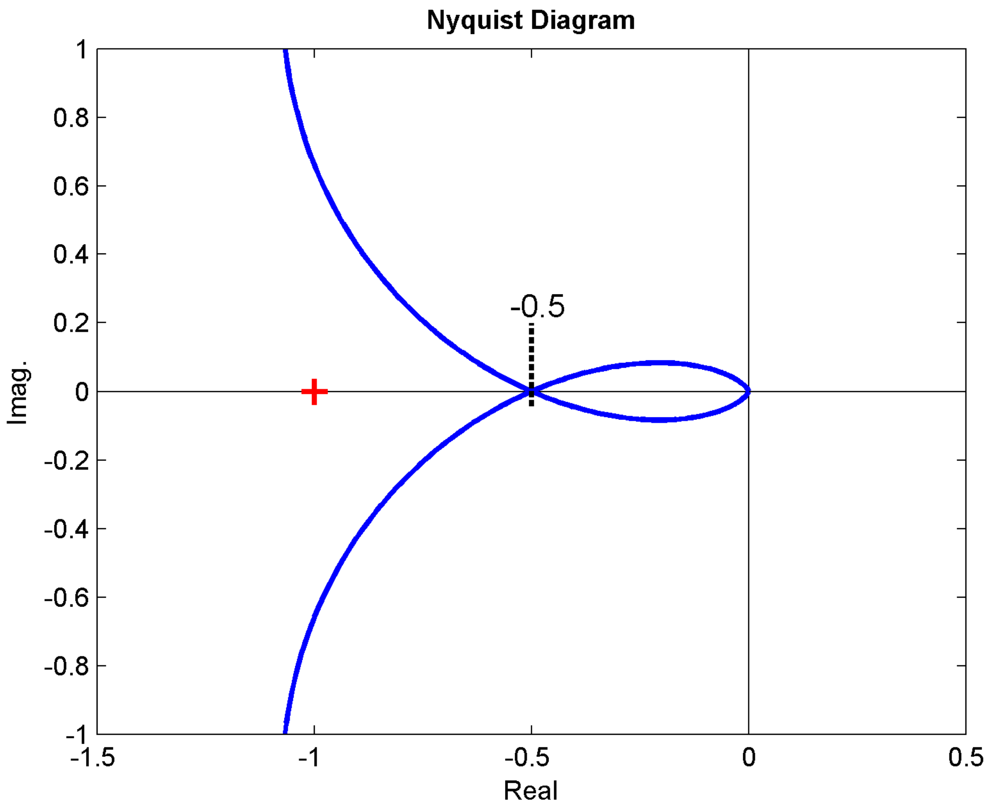
Nyquist Criterion – Example 2

- For $K = 30$, $N = 0$, and the system is stable
- Modifying K simply scales the magnitude of the Nyquist plot



Nyquist Criterion – Example 2

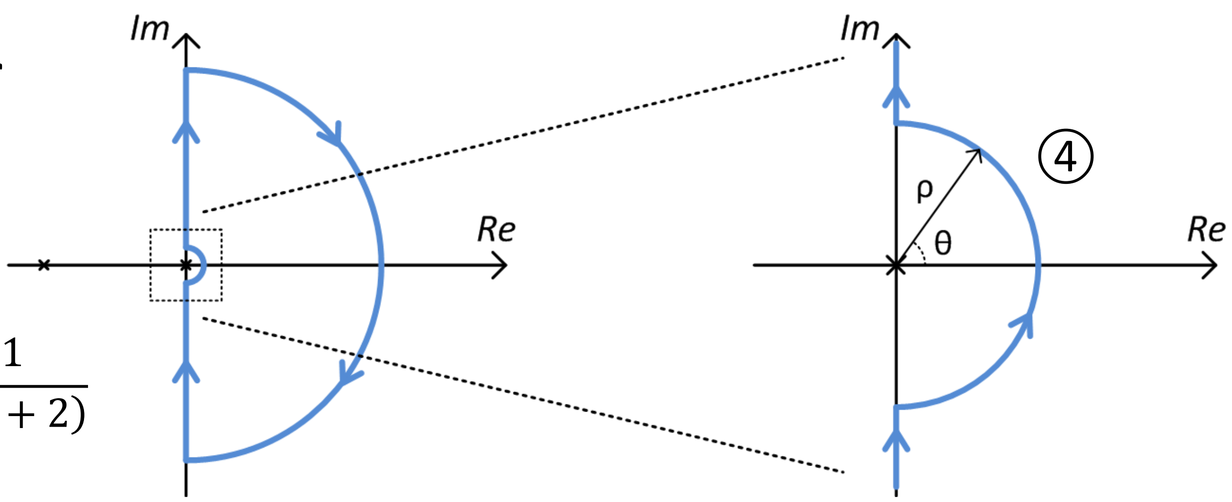
- Here, the Nyquist plot crosses the negative real axis at $s = -0.5$
- As gain increases real-axis crossing moves to the left
- Increasing K by 2x or more results in two encirclements of $s = -1$
  - Unstable for $K > 60$
  - More later …



Poles at the Origin

Nyquist Diagram – Poles at the Origin

- We evaluate the open-loop transfer function along a contour including the $j\omega$-axis
- $G(j\omega)$ is ***undefined*** at the pole
  - Must ***detour around the pole***
- Consider the common case of a pole at the origin
- This is the special case for segment ④



$$G(s) = \frac{1}{s(s+2)}$$

Nyquist Diagram – Poles at the Origin

- Segment ④ contour: $s = \rho e^{j\theta}$ for $0° \leq \theta \leq 90°$
- Evaluate $G(s)$ around segment ④ as $\rho \to 0$

$$G(s)\Big|_{s=\rho e^{j\theta}} = \frac{1}{\rho e^{j\theta}(\rho e^{j\theta} + 2)}$$

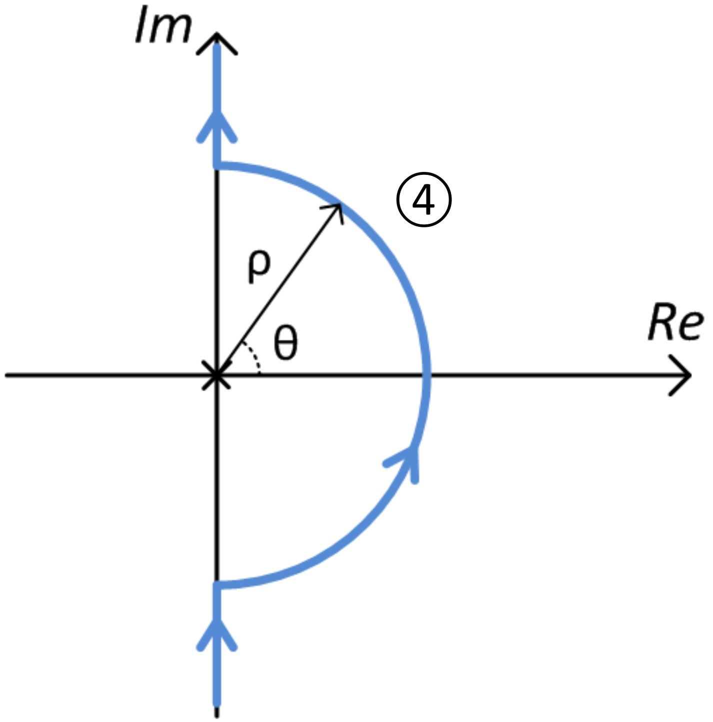
- ***Magnitude***:

$$\left|G\left(\rho e^{j\theta}\right)\right| = \frac{1}{\rho\left|\rho e^{j\theta} + 2\right|} = \frac{1}{2\rho}$$

- As $\rho \to 0$

$$\lim_{\rho\to 0}\left|G\left(\rho e^{j\theta}\right)\right| = \infty$$

- Maps to an arc at C^{∞}

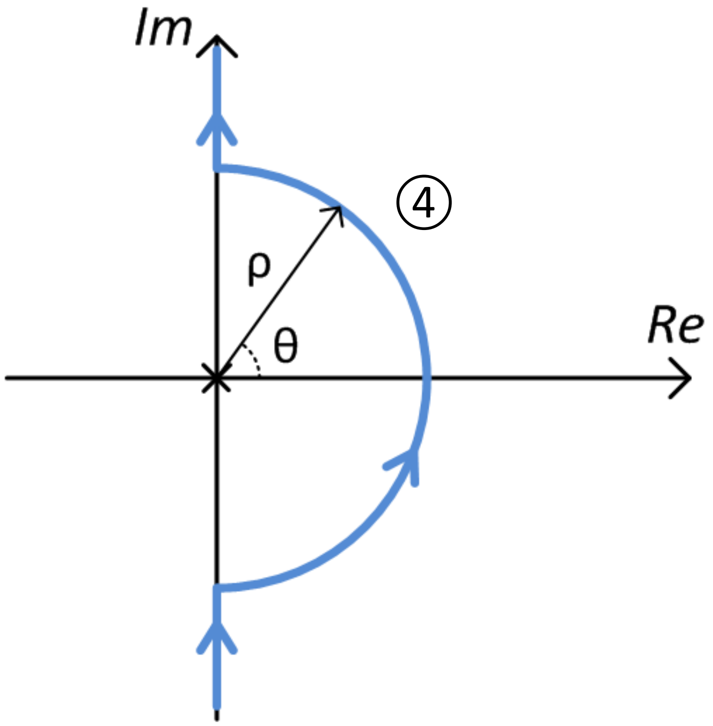


Nyquist Diagram – Poles at the Origin

- Segment ④ traversed in a CCW direction
  - θ varies from $0° \ldots + 90°$
- ***Phase*** of the resulting contour:

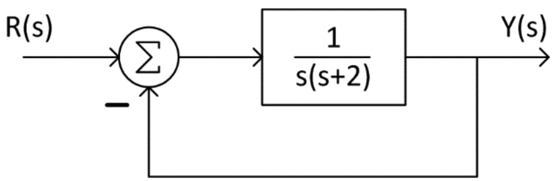
$$\angle G\left(\rho e^{j\theta}\right) = -\theta^{+}$$

  - Negative because it is angle from a pole
  - Extra phase from additional pole
- $G(s)$ maps segment ④ to:
  - An arc at C^{∞}
  - Rotating CW from $0°$ to $-90°^{+}$

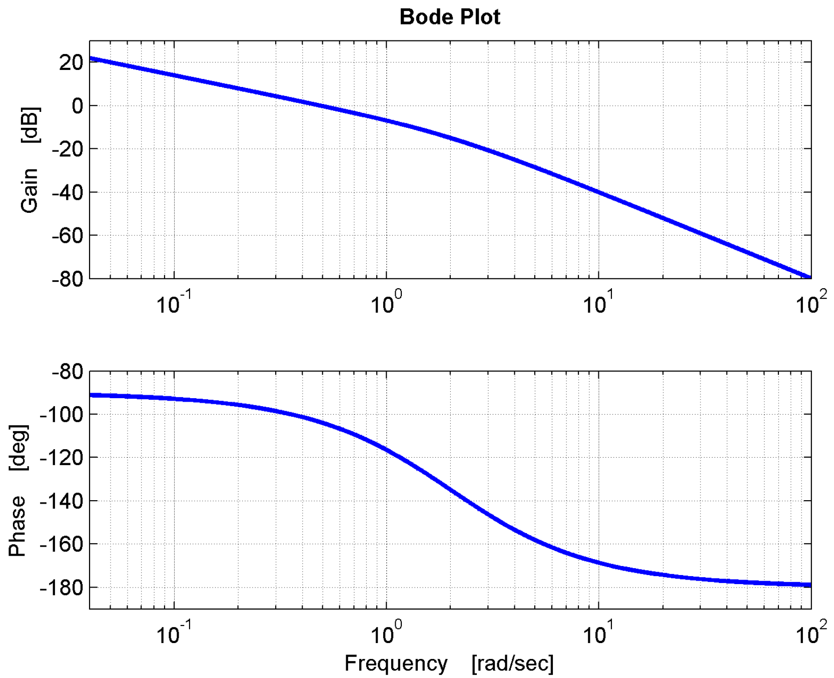


Nyquist Criterion – Example 3

- Apply the Nyquist criterion to determine stability for the following system

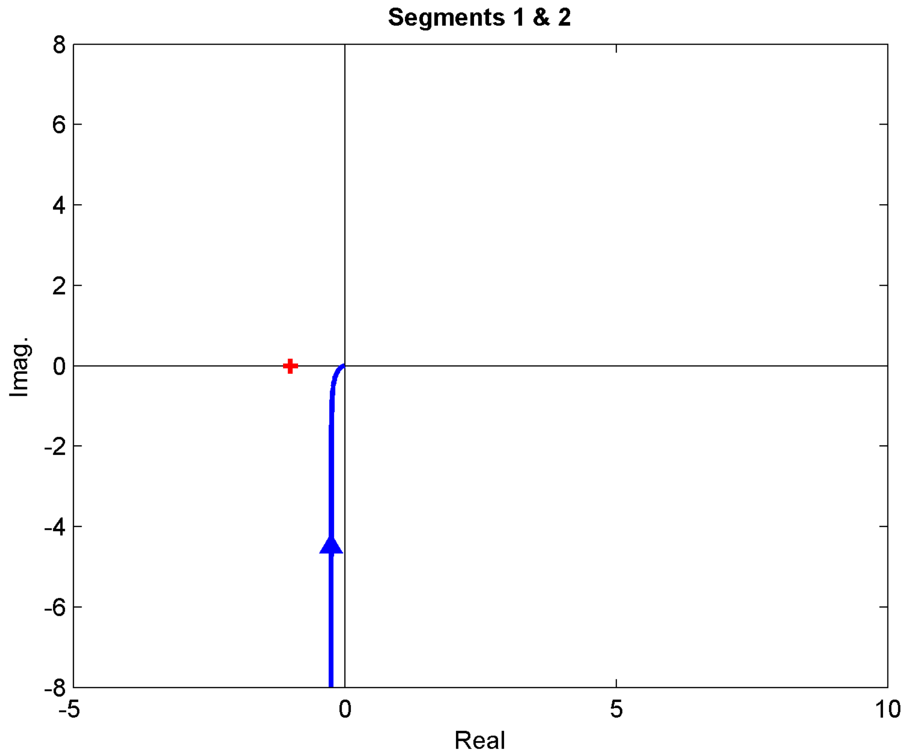


- Use Bode plot to map segment ①
  - Infinite DC gain
  - Starts at $-90°$ at C^{∞} for $\omega = 0$



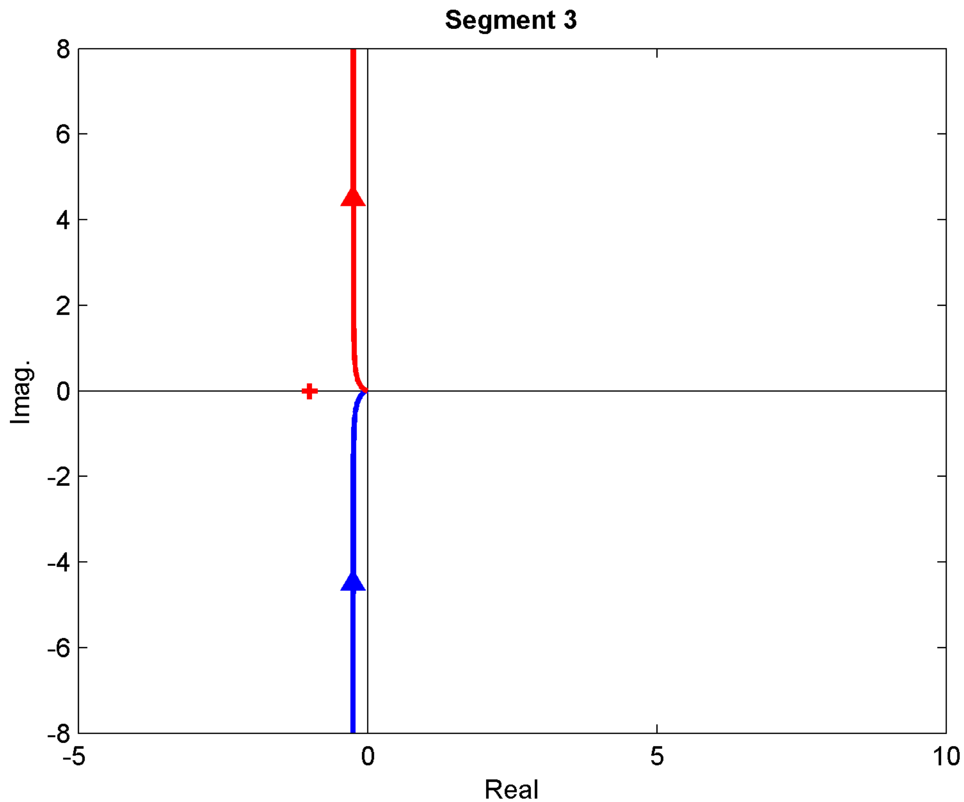
Nyquist Criterion – Example 3

- Segment ① starts at C^{∞} at $-90°$
- Heads to the origin at $-180°$
- All of segment ②, arc at C^{∞}, maps to the origin



Nyquist Criterion – Example 3

- Segment ③ is the complex conjugate of segment ①
  - Mirror about the real axis

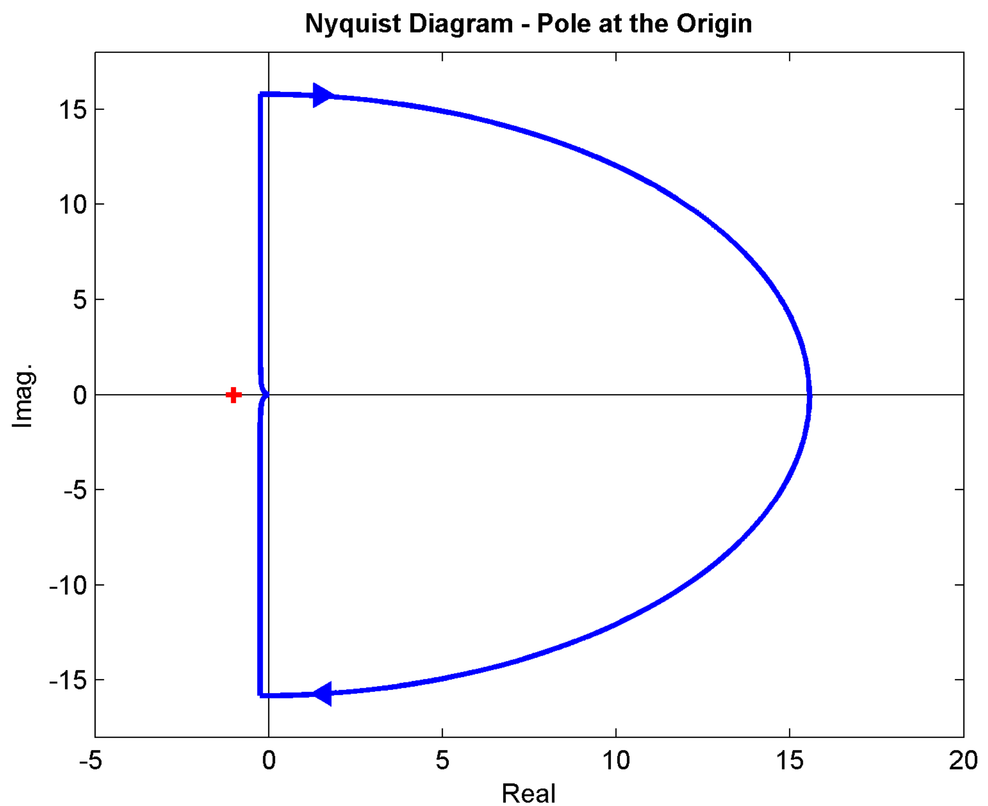


Nyquist Criterion – Example 3

- Segment 4 maps to a CW arc at C^{∞}
  - *CW*, so it does not encircle -1
  - Can't draw to scale
- Here, $N = 0$
- No open-loop RHP poles, so $P = 0$

$$Z = 0$$

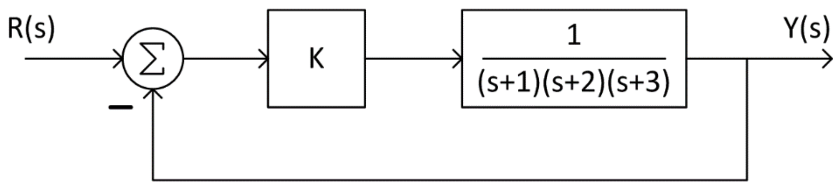
- No RHP poles, so system is ***stable***



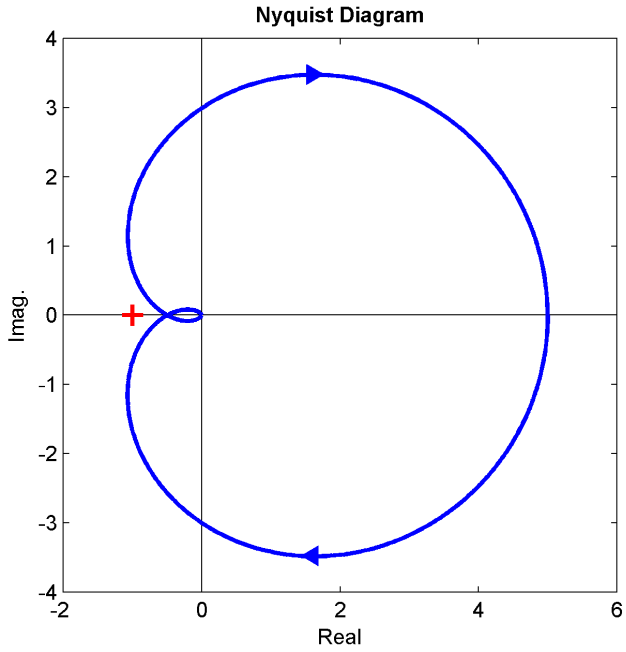
Stability Margins

Stability Margins

- Recall a previous example



- According to the Nyquist plot, the system is stable
  - How stable?



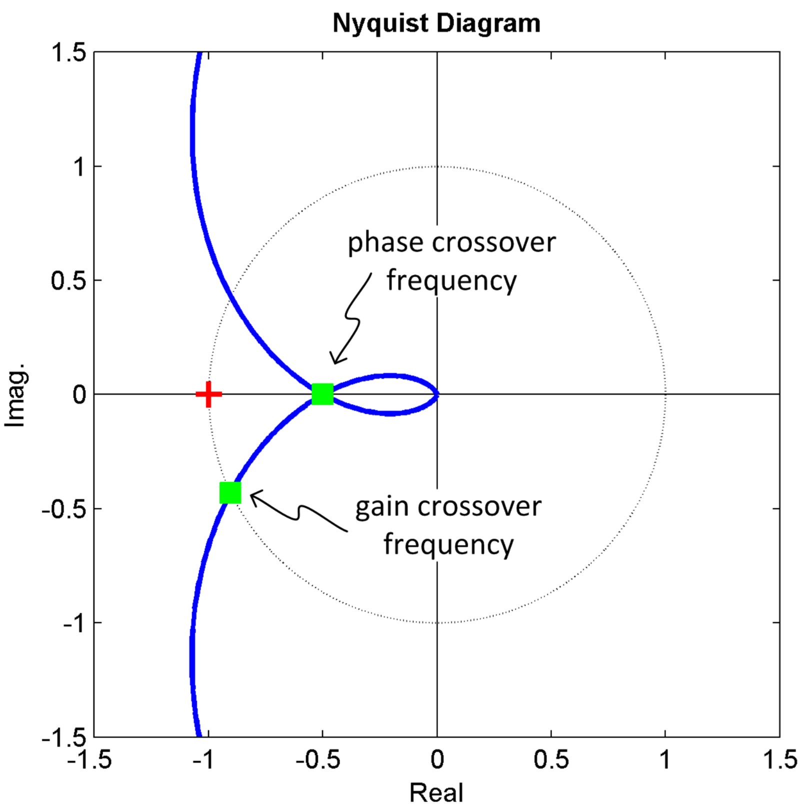
- Two stability metrics
  - Both are measures of how close the Nyquist plot is to encircling the point $s = -1$
  - ***Gain margin*** and ***phase margin***

Crossover Frequencies

- Two important frequencies when assessing stability:

- ***Gain crossover frequency***
  - The frequency at which the open-loop gain crosses $0\ dB$

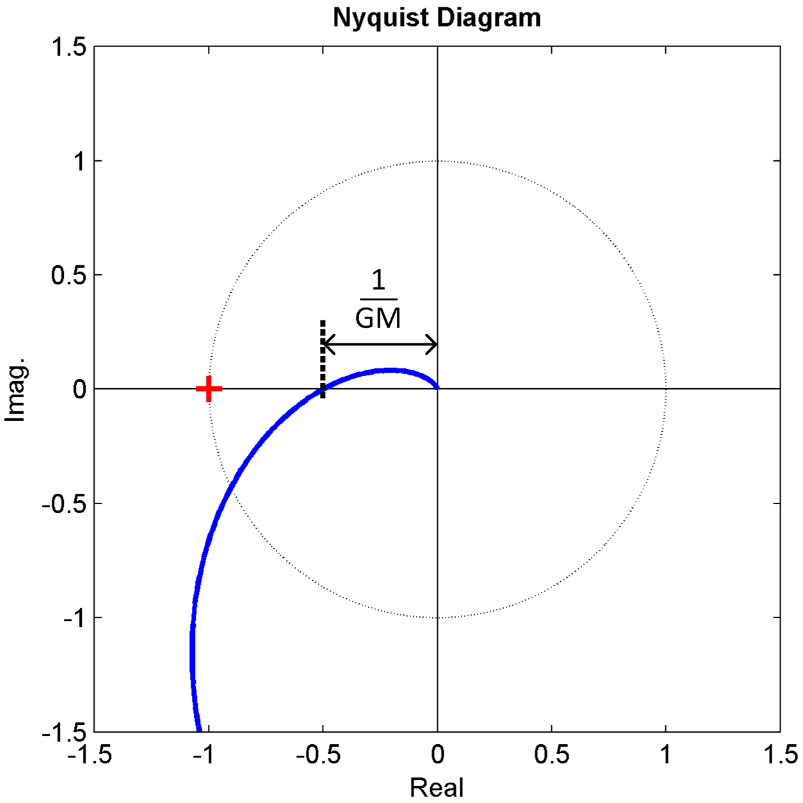
- ***Phase crossover frequency***
  - The frequency at which the open-loop phase crosses $-180°$



Gain Margin

- An open-loop-stable system will be closed-loop stable as long as its gain is less than unity at the phase crossover frequency

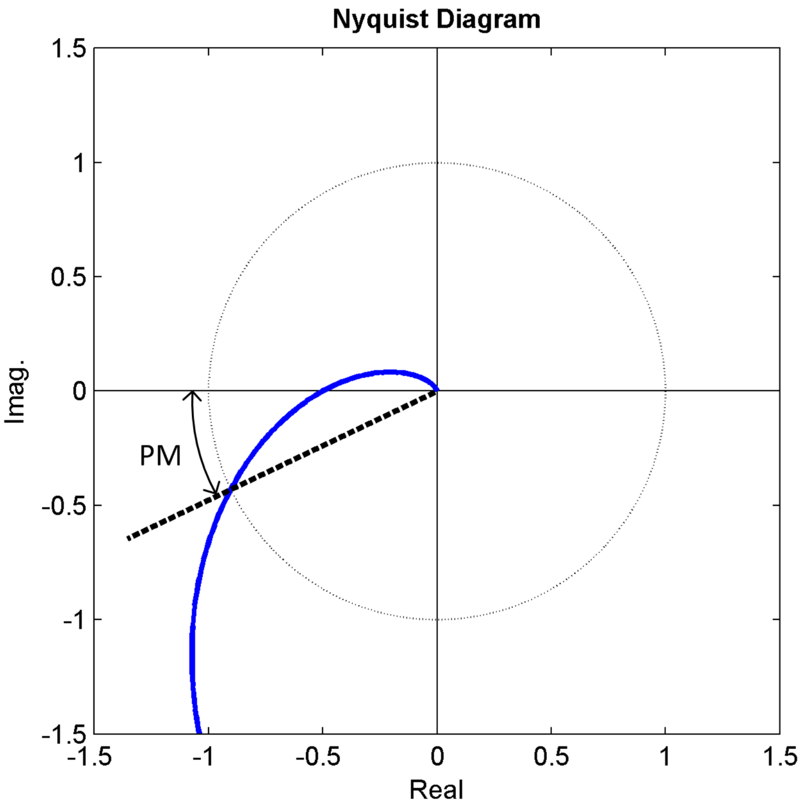
- **Gain margin, GM**
  - The change in open-loop gain at the phase crossover frequency required to make the closed-loop system unstable



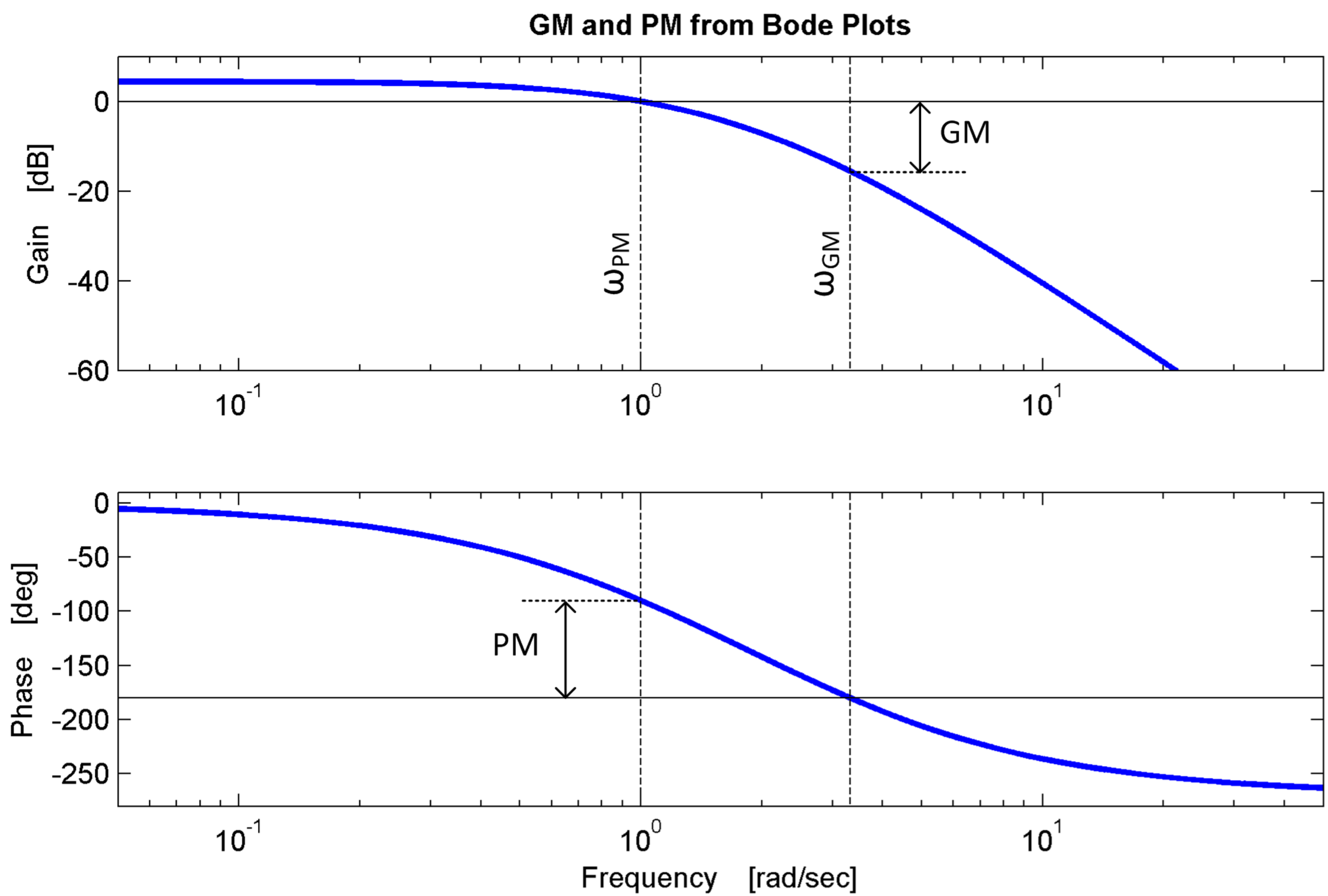
Phase Margin

- An open-loop-stable system will be closed-loop stable as long as its phase has not fallen below $-180°$ at the gain crossover frequency

- **<u>Phase margin, PM</u>**
  - The change in open-loop phase at the gain crossover frequency required to make the closed-loop system unstable



Gain and Phase Margins from Bode Plots



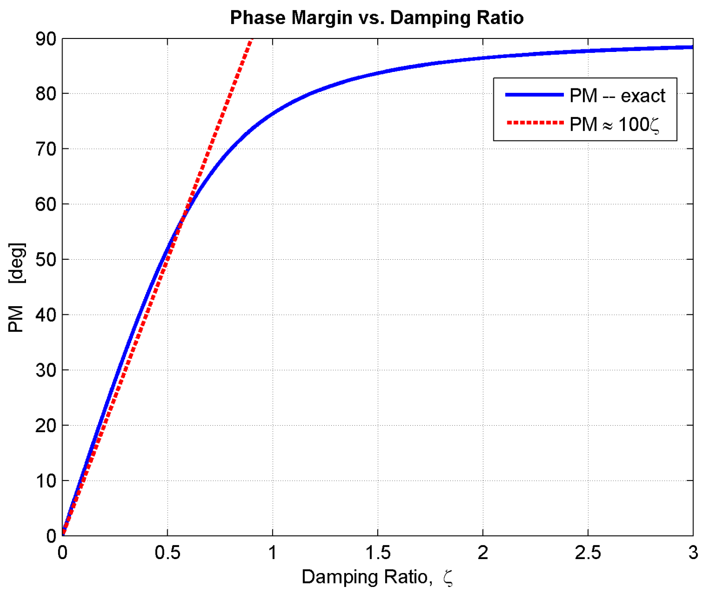
Phase Margin and Damping Ratio, ζ

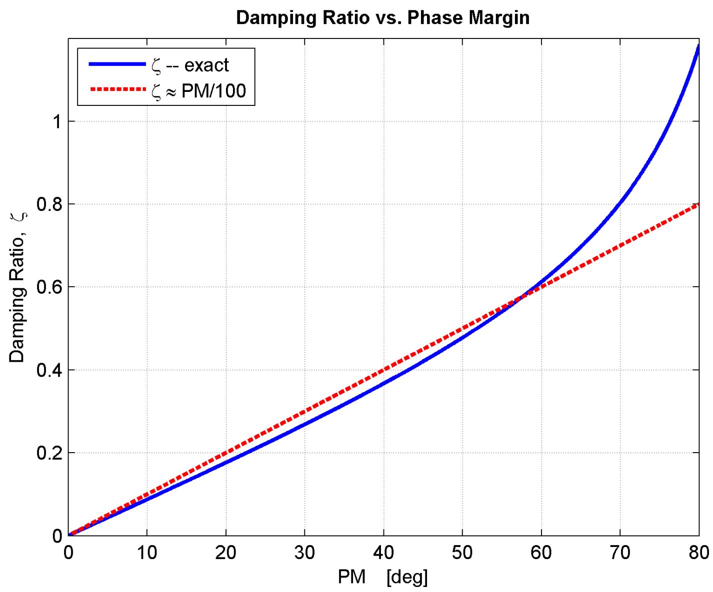
- PM can be expressed as a function of damping ratio, ζ, as

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}\right)$$

- For $PM \leq 65°$ or so, we can approximate:

$$PM \approx 100\zeta \quad \text{or} \quad \zeta \approx \frac{PM}{100}$$





Frequency Response Analysis in MATLAB

bode.m


```
[mag,phase] = bode(sys,w)
```

- `sys`: system model – state-space, transfer function, or other
- `w`: *optional* frequency vector – in rad/sec
- `mag`: system gain response vector
- `phase`: system phase response vector – in degrees

- If no outputs are specified, bode response is automatically plotted – preferable to plot yourself
- Frequency vector input is optional
  - If not specified, MATLAB will generate automatically

- May need to do: `squeeze(mag)` and `squeeze(phase)` to eliminate singleton dimensions of output matrices

nyquist.m


```
nyquist(sys,w)
```

- `sys`: system model – state-space, transfer function, or other
- `w`: *optional* frequency vector – in rad/sec

- MATLAB generates a Nyquist plot automatically
- Can also specify outputs, if desired:

```
[Re,Im] = nyquist(sys,w)
```

- Plot is not be generated in this case

margin.m


```
[GM,PM,wgm,wpm] = margin(sys)
```

- `sys`: system model – state-space, transfer function, or other
- `GM`: gain margin
- `PM`: phase margin – in degrees
- `wgm`: frequency at which GM is measured, the phase crossover frequency – in rad/sec
- `wpm`: frequency at which PM is measured, the gain crossover frequency

- If no outputs are specified, a Bode plot with GM and PM indicated is automatically generated