SECTION 3: LAPLACE TRANSFORMS & TRANSFER FUNCTIONS
This section of notes contains an introduction to Laplace transforms. This should mostly be a review of material covered in your differential equations course.
Transforms

- **What is a transform?**
  - A *mapping* of a mathematical function from one *domain* to another
  - A change in *perspective* not a change of the function

- **Why use transforms?**
  - Some mathematical problems are difficult to solve in their natural domain
    - Transform to and solve in a new domain, where the problem is simplified
    - Transform back to the original domain
  - Trade off the extra effort of transforming/inverse-transforming for simplification of the solution procedure
Transform Example – Slide Rules

- Slide rules make use of a logarithmic transform

- Multiplication/division of large numbers is difficult
  - Transform the numbers to the logarithmic domain
  - Add/subtract (easy) in the log domain to multiply/divide (difficult) in the linear domain
  - Apply the inverse transform to get back to the original domain

- Extra effort is required, but the problem is simplified
Laplace Transforms
Laplace Transforms

- An **integral transform** mapping functions from the **time domain** to the **Laplace domain** or **s-domain**

\[ g(t) \overset{\mathcal{L}}{\leftrightarrow} G(s) \]

- Time-domain functions are functions of time, \( t \)

\[ g(t) \]

- Laplace-domain functions are functions of \( s \)

\[ G(s) \]

- \( s \) is a complex variable

\[ s = \sigma + j\omega \]
Laplace Transforms – Motivation

- We’ll use Laplace transforms to solve differential equations
  
  - Differential equations in the time domain
    - difficult to solve
  
  - Apply the Laplace transform
    - Transform to the s-domain
  
  - Differential equations become algebraic equations
    - easy to solve
  
  - Transform the s-domain solution back to the time domain

- Transforming back and forth requires extra effort, but the solution is greatly simplified
Laplace Transform

- **Laplace Transform:**
  \[
  \mathcal{L}\{g(t)\} = G(s) = \int_0^\infty g(t)e^{-st} \, dt
  \]  

- **Unilateral or one-sided** transform
  - Lower limit of integration is \( t = 0 \)
  - Assumed that the time domain function is zero for all negative time, i.e.
    \[
    g(t) = 0, \quad t < 0
    \]
In the following subsection of notes, we’ll derive a few important properties of the Laplace transform.
Laplace Transform – Linearity

- Say we have two time-domain functions: \( g_1(t) \) and \( g_2(t) \)

- Applying the transform definition, (1)

\[
\mathcal{L}\{\alpha g_1(t) + \beta g_2(t)\} = \int_0^\infty (\alpha g_1(t) + \beta g_2(t))e^{-st} dt
\]

\[
= \int_0^\infty \alpha g_1(t)e^{-st} dt + \int_0^\infty \beta g_2(t)e^{-st} dt
\]

\[
= \alpha \int_0^\infty g_1(t)e^{-st} dt + \beta \int_0^\infty g_2(t)e^{-st} dt
\]

\[
= \alpha \mathcal{L}\{g_1(t)\} + \beta \mathcal{L}\{g_2(t)\}
\]

\[
\mathcal{L}\{\alpha g_1(t) + \beta g_2(t)\} = \alpha G_1(s) + \beta G_2(s) \hspace{1cm} (2)
\]

- The Laplace transform is a linear operation
Laplace Transform of a Derivative

- Of particular interest, given that we want to use Laplace transform to solve differential equations
  \[ \mathcal{L}\{\dot{g}(t)\} = \int_0^\infty \dot{g}(t)e^{-st} dt \]

- Use *integration by parts* to evaluate
  \[ \int u dv = uv - \int v du \]

- Let
  \[ u = e^{-st} \quad \text{and} \quad dv = \dot{g}(t) dt \]
  then
  \[ du = -se^{-st} dt \quad \text{and} \quad v = g(t) \]
The Laplace transform of the derivative of a function is the Laplace transform of that function multiplied by $s$ minus the initial value of that function.

\[
\mathcal{L}\{\dot{g}(t)\} = sG(s) - g(0)
\]
Higher-Order Derivatives

- The Laplace transform of a second derivative is

\[ \mathcal{L}\{\ddot{g}(t)\} = s^2 G(s) - sg(0) - \dot{g}(0) \]  

(4)

- In general, the Laplace transform of the \( n^{th} \) derivative of a function is given by

\[ \mathcal{L}\{g^{(n)}\} = s^n G(s) - s^{n-1} g(0) - s^{n-2} \dot{g}(0) - \cdots - g^{(n-1)}(0) \]  

(5)
Laplace Transform of an Integral

- The Laplace Transform of a **definite integral** of a function is given by

\[
\mathcal{L}\left\{ \int_0^t g(\tau) d\tau \right\} = \frac{1}{s} G(s)
\]  

(6)

- **Differentiation** in the time domain corresponds to **multiplication by** \( s \) in the Laplace domain
- **Integration** in the time domain corresponds to **division by** \( s \) in the Laplace domain
Next, we’ll derive the Laplace transform of some common mathematical functions
A useful and common way of characterizing a linear system is with its **step response**

- The system’s response (output) to a unit step input

The **unit step function** or **Heaviside step function**:

$$1(t) = \begin{cases} 
0, & t < 0 \\ 
1, & t \geq 0 
\end{cases}$$
Unit Step Function – Laplace Transform

- Using the definition of the Laplace transform

$$\mathcal{L}\{1(t)\} = \int_0^\infty 1(t)e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \bigg|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

- The Laplace transform of the unit step

$$\mathcal{L}\{1(t)\} = \frac{1}{s}$$ (7)

- Note that the unilateral Laplace transform assumes that the signal being transformed is zero for $t < 0$
  - Equivalent to multiplying any signal by a unit step
The unit ramp function is a useful input signal for evaluating how well a system tracks a constantly-increasing input.

The unit ramp function:

\[ g(t) = \begin{cases} 
0, & t < 0 \\
t, & t \geq 0 
\end{cases} \]
Could easily evaluate the transform integral
- Requires integration by parts
- Alternatively, recognize the relationship between the unit ramp and the unit step
  - *Unit ramp is the integral of the unit step*
- Apply the integration property, (6)

\[
\mathcal{L}\{t\} = \mathcal{L}\left\{ \int_0^t 1(\tau) d\tau \right\} = \frac{1}{s} \cdot \frac{1}{s}
\]

\[
\mathcal{L}\{t\} = \frac{1}{s^2}
\]

(8)
Exponential – Laplace Transform

\[ g(t) = e^{-at} \]

- Exponentials are common components of the responses of dynamic systems

\[
\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt
\]

\[
= - \frac{e^{-(s+a)t}}{s + a} \bigg|_0^\infty = 0 - \left( -\frac{1}{s + a} \right)
\]

\[
\mathcal{L}\{e^{-at}\} = \frac{1}{s + a}
\]
Another class of commonly occurring signals, when dealing with dynamic systems, is *sinusoidal signals* – both \( \sin(\omega t) \) and \( \cos(\omega t) \)

\[
g(t) = \sin(\omega t)
\]

Recall *Euler’s formula*

\[
e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)
\]

From which it follows that

\[
\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}
\]
Sinusoidal functions

\[ \mathcal{L}\{\sin(\omega t)\} = \frac{1}{2j} \int_0^\infty (e^{j\omega t} - e^{-j\omega t})e^{-st} dt \]

\[ = \frac{1}{2j} \int_0^\infty (e^{-(s-j\omega)t} - e^{-(s+j\omega)t}) dt \]

\[ = \frac{1}{2j} \int_0^\infty e^{-(s-j\omega)t} dt - \frac{1}{2j} \int_0^\infty e^{-(s+j\omega)t} dt \]

\[ = \frac{1}{2j} \left( e^{-(s-j\omega)t} \right)_0^\infty - \frac{1}{2j} \left( e^{-(s+j\omega)t} \right)_0^\infty \]

\[ = \frac{1}{2j} \left[ 0 + \frac{1}{s-j\omega} \right] - \frac{1}{2j} \left[ 0 + \frac{1}{s+j\omega} \right] = \frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2} \]

\[ \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2} \]  \hspace{1cm} (10)
Sinusoidal functions

- It can similarly be shown that
  \[ \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2} \]  
  (11)

- Note that for neither \(\sin(\omega t)\) nor \(\cos(\omega t)\) is the function equal to zero for \(t < 0\) as the Laplace transform assumes.

- Really, what we’ve derived is
  \[ \mathcal{L}\{1(t) \cdot \sin(\omega t)\} \quad \text{and} \quad \mathcal{L}\{1(t) \cdot \cos(\omega t)\} \]
More Properties and Theorems
Multiplication by an Exponential, $e^{-at}$

- We’ve seen that $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

- What if another function is multiplied by the decaying exponential term?

  $$\mathcal{L}\{g(t)e^{-at}\} = \int_0^\infty g(t)e^{-at}e^{-st} \, dt = \int_0^\infty g(t)e^{-(s+a)t} \, dt$$

- This is just the Laplace transform of $g(t)$ with $s$ replaced by $(s+a)$

  $$\mathcal{L}\{g(t)e^{-at}\} = G(s + a)$$  \hspace{1cm} (12)
Decaying Sinusoids

- The Laplace transform of a sinusoid is
  \[ \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2} \]

- And, multiplication by an decaying exponential, \( e^{-at} \), results in a substitution of \( (s + a) \) for \( s \), so
  \[ \mathcal{L}\{e^{-at} \sin(\omega t)\} = \frac{\omega}{(s + a)^2 + \omega^2} \]
  and
  \[ \mathcal{L}\{e^{-at} \cos(\omega t)\} = \frac{s + a}{(s + a)^2 + \omega^2} \]
Time Shifting

- Consider a time-domain function, \( g(t) \)

- To Laplace transform \( g(t) \) we’ve assumed \( g(t) = 0 \) for \( t < 0 \), or equivalently multiplied by \( 1(t) \)

- To shift \( g(t) \) by an amount, \( a \), in time, we must also multiply by a shifted step function, \( 1(t - a) \)
The transform of the shifted function is given by

\[ \mathcal{L}\{g(t \cdot a) \cdot 1(t \cdot a)\} = \int_a^\infty g(t \cdot a)e^{-st}dt \]

Performing a change of variables, let

\[ \tau = (t \cdot a) \text{ and } d\tau = dt \]

The transform becomes

\[ \mathcal{L}\{g(\tau) \cdot 1(\tau)\} = \int_0^\infty g(\tau)e^{-s(\tau+a)}d\tau = \int_0^\infty g(\tau)e^{-as}e^{-st}d\tau = e^{-as} \int_0^\infty g(\tau)e^{-s\tau}d\tau \]

A shift by \( a \) in the time domain corresponds to multiplication by \( e^{-as} \) in the Laplace domain

\[ \mathcal{L}\{g(t \cdot a) \cdot 1(t \cdot a)\} = e^{-as} G(s) \] (13)
Multiplication by time, $t$

- The Laplace transform of a function multiplied by time:
  \[ \mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds} F(s) \]  
  (14)

- Consider a unit ramp function:
  \[ \mathcal{L}\{t\} = \mathcal{L}\{t \cdot 1(t)\} = -\frac{d}{ds} \left(\frac{1}{s}\right) = \frac{1}{s^2} \]

- Or a parabola:
  \[ \mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = -\frac{d}{ds} \left(\frac{1}{s^2}\right) = \frac{2}{s^3} \]

- In general
  \[ \mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}} \]
Initial and Final Value Theorems

- **Initial Value Theorem**
  - Can determine the initial value of a time-domain signal or function from its Laplace transform

\[
g(0) = \lim_{s \to \infty} sG(s) \quad (15)
\]

- **Final Value Theorem**
  - Can determine the steady-state value of a time-domain signal or function from its Laplace transform

\[
g(\infty) = \lim_{s \to 0} sG(s) \quad (16)
\]
Convolution of two functions or signals is given by

\[ g(t) * x(t) = \int_{0}^{t} g(\tau)x(t - \tau)d\tau \]

Result is a function of time
- \( x(\tau) \) is \textit{flipped} in time and \textit{shifted} by \( t \)
- Multiply the flipped/shifted signal and the other signal
- Integrate the result from \( \tau = 0 \) ... \( t \)

May seem like an odd, arbitrary function now, but we’ll later see why it is very important

Convolution in the time domain corresponds to multiplication in the Laplace domain

\[ \mathcal{L}\{g(t) * x(t)\} = G(s)X(s) \]
Another common way to describe a dynamic system is with its **impulse response**

- System output in response to an impulse function input

**Impulse function** defined by

\[
\delta(t) = 0, \quad t \neq 0
\]

\[
\int_{-\infty}^{\infty} \delta(t) dt = 1
\]

- An infinitely tall, infinitely narrow pulse
To derive $\mathcal{L}\{\delta(t)\}$, consider the following function

$$g(t) = \begin{cases} \frac{1}{t_0}, & 0 \leq t \leq t_0 \\ 0, & t < 0 \text{ or } t > t_0 \end{cases}$$

Can think of $g(t)$ as the sum of two step functions:

$$g(t) = \frac{1}{t_0} 1(t) - \frac{1}{t_0} 1(t - t_0)$$

The transform of the first term is

$$\mathcal{L} \left\{ \frac{1}{t_0} 1(t) \right\} = \frac{1}{t_0 s}$$

Using the time-shifting property, the second term transforms to

$$\mathcal{L} \left\{ -\frac{1}{t_0} 1(t - t_0) \right\} = -\frac{e^{-t_0 s}}{t_0 s}$$
In the limit, as \( t_0 \to 0 \), \( g(t) \to \delta(t) \), so

\[
\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \mathcal{L}\{g(t)\}
\]

\[
\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \frac{1 - e^{-t_0 s}}{t_0 s}
\]

Apply l’Hôpital’s rule

\[
\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \frac{d}{dt_0} \left(1 - e^{-t_0 s}\right) = \lim_{t_0 \to 0} \frac{se^{-t_0 s}}{s} = \frac{s}{s}
\]

The Laplace transform of an impulse function is one

\[
\mathcal{L}\{\delta(t)\} = 1
\]
### Common Laplace Transforms

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<thead>
<tr>
<th>( g(t) )</th>
<th>( G(s) )</th>
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<tbody>
<tr>
<td>( \delta(t) )</td>
<td>1</td>
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<tr>
<td>( 1(t) )</td>
<td>( \frac{1}{s} )</td>
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<tr>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
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<tr>
<td>( t^m )</td>
<td>( \frac{m!}{s^{m+1}} )</td>
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<td>( e^{-at} )</td>
<td>( \frac{1}{s + a} )</td>
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<td>( te^{-at} )</td>
<td>( \frac{1}{(s + a)^2} )</td>
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<td>( \frac{\omega}{s^2 + \omega^2} )</td>
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<tr>
<td>( \cos(\omega t) )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
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### \( g(t) \) \( G(s) \) 

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</tr>
<tr>
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</tr>
<tr>
<td>( e^{-at}g(t) )</td>
<td>( G(s + a) )</td>
</tr>
<tr>
<td>( g(t - a) \cdot 1(t - a) )</td>
<td>( e^{-as}G(s) )</td>
</tr>
<tr>
<td>( t \cdot g(t) )</td>
<td>( -\frac{d}{ds}G(s) )</td>
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Inverse Laplace Transform

We’ve just seen how time-domain functions can be transformed to the Laplace domain. Next, we’ll look at how we can solve differential equations in the Laplace domain and transform back to the time domain.
Consider the simple spring/mass/damper system from the previous section of notes.

Applying Newton’s 2\textsuperscript{nd} law at the mass:

\[ F_{in}(t) - b \ddot{x} - kx = m\dddot{x} \]

Rearranging, gives the governing second-order differential equation

\[ \dddot{x} + \frac{b}{m} \ddot{x} + \frac{k}{m} x = \frac{1}{m} F_{in}(t) \]

This equation describes the dynamic response of the system to some applied input force, \( F_{in}(t) \).

That input may take many forms, e.g.:

- Step
- Sinusoidal
- Impulse, etc.
We’ll now use Laplace transforms to determine the *step response* of the system.

1N step force input

\[ F_{in}(t) = 1N \cdot 1(t) = \begin{cases} 0N, & t < 0 \\ 1N, & t \geq 0 \end{cases} \]  \hspace{1cm} (2)

For the step response, we assume *zero initial conditions*

\[ x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 0 \]  \hspace{1cm} (3)

Using the derivative property of the Laplace transform, (1) becomes

\[ s^2 X(s) - sx(0) - \dot{x}(0) + \frac{b}{m} sX(s) - \frac{b}{m} x(0) + \frac{k}{m} X(s) = \frac{1}{m} F_{in}(s) \]

\[ s^2 X(s) + \frac{b}{m} sX(s) + \frac{k}{m} X(s) = \frac{1}{m} F_{in}(s) \]  \hspace{1cm} (4)
The input is a step, so (7) becomes

\[ s^2X(s) + \frac{b}{m} sX(s) + \frac{k}{m} X(s) = \frac{1}{m} 1N \frac{1}{s} \]  \hspace{1cm} (5)

Solving (5) for \( X(s) \)

\[ X(s) \left( s^2 + \frac{b}{m} s + \frac{k}{m} \right) = \frac{1}{m} \frac{1}{s} \]

\[ X(s) = \frac{1/m}{s\left( s^2 + \frac{b}{m}s + \frac{k}{m} \right)} \] \hspace{1cm} (6)

Equation (6) is the solution to the differential equation of (1), given the step input and I.C.’s

- The system step response in the Laplace domain
- Next, we need to transform back to the time domain
Laplace Transforms – Differential Equations

\[ X(s) = \frac{1/m}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \]  

The form of (6) is typical of Laplace transforms when dealing with linear systems

- A rational polynomial in \( s \)
- Here, the numerator is 0\(^{\text{th}}\)-order

\[ X(s) = \frac{B(s)}{A(s)} \]

- Roots of the numerator polynomial, \( B(s) \), are called the zeros of the function
- Roots of the denominator polynomial, \( A(s) \), are called the poles of the function
Inverse Laplace Transforms

\[ X(s) = \frac{1/m}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \]  \hspace{1cm} (6)

- To get (6) back into the time domain, we need to perform an *inverse Laplace transform*
  - An integral inverse transform exists, but we don’t use it
  - Instead, we use *partial fraction expansion*

**Partial fraction expansion**
- Idea is to express the Laplace transform solution, (6), as a sum of Laplace transform terms that appear in the table
- Procedure depends on the type of roots of the denominator polynomial
  - Real and distinct
  - Repeated
  - Complex
Inverse Laplace Transforms – Example 1

- Consider the following system parameters
  
  \[ m = 1 \text{kg} \]
  
  \[ k = \frac{16N}{m} \]
  
  \[ b = 10 \frac{N \cdot s}{m} \]

- Laplace transform of the step response becomes
  
  \[ X(s) = \frac{1}{s(s^2+10s+16)} \]  
  \[ (7) \]

- Factoring the denominator
  
  \[ X(s) = \frac{1}{s(s+2)(s+8)} \]  
  \[ (8) \]

- In this case, the denominator polynomial has three \textit{real, distinct roots}
  
  \[ s_1 = 0, \quad s_2 = -2, \quad s_3 = -8 \]
Inverse Laplace Transforms – Example 1

- Partial fraction expansion of (8) has the form

\[ X(s) = \frac{1}{s(s+2)(s+8)} = \frac{r_1}{s} + \frac{r_2}{s+2} + \frac{r_3}{s+8} \]  

- The numerator coefficients, \( r_1, r_2, \) and \( r_3 \), are called **residues**

- Can already see the form of the time-domain function
  - Sum of a **constant** and **two decaying exponentials**

- To determine the residues, multiply both sides of (9) by the denominator of the left-hand side

\[ 1 = r_1(s + 2)(s + 8) + r_2 s(s + 8) + r_3 s(s + 2) \]
\[ 1 = r_1 s^2 + 10r_1 s + 16r_1 + r_2 s^2 + 8r_2 s + r_3 s^2 + 2r_3 s \]

- Collecting terms, we have

\[ 1 = s^2 (r_1 + r_2 + r_3) + s(10r_1 + 8r_2 + 2r_3) + 16r_1 \]
Inverse Laplace Transforms – Example 1

- Equating coefficients of powers of $s$ on both sides of (10) gives a system of three equations in three unknowns:
  
  \[ s^2: \quad r_1 + r_2 + r_3 = 0 \]
  \[ s^1: \quad 10r_1 + 8r_2 + 2r_3 = 0 \]
  \[ s^0: \quad 16r_1 = 1 \]

- Solving for the residues gives:
  
  \[ r_1 = 0.0625 \]
  \[ r_2 = -0.0833 \]
  \[ r_3 = 0.0208 \]

- The Laplace transform of the step response is
  
  \[ X(s) = \frac{0.0625}{s} - \frac{0.0833}{s+2} + \frac{0.0208}{s+8} \]  \hspace{1cm} (11)

- Equation (11) can now be transformed back to the time domain using the Laplace transform table.
The time-domain step response of the system is the **sum of a constant term and two decaying exponentials**:

\[ x(t) = 0.0625 - 0.0833e^{-2t} + 0.0208e^{-8t} \] (12)

- Step response plotted in MATLAB
- Characteristic of a signal having **only real poles**
  - No overshoot/ringing
- Steady-state displacement agrees with intuition
  - 1N force applied to a 16N/m spring
Inverse Laplace Transforms – Example 1

- Go back to (7) and apply the *initial value theorem*

\[ x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{1}{s(s^2 + 10s + 16)} = 0 \text{cm} \]

- Which is, in fact our assumed initial condition

- Next, apply the *final value theorem* to the Laplace transform step response, (7)

\[ x(\infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{1}{s(s^2 + 10s + 16)} \]

\[ x(\infty) = \frac{1}{16} = 0.0625m = 6.25 \text{cm} \]

- This final value agrees with both intuition and our numerical analysis
Reduce the damping and re-calculate the step response

\[ m = 1\text{kg} \]
\[ k = \frac{16N}{m} \]
\[ b = 8\frac{N \cdot s}{m} \]

Laplace transform of the step response becomes

\[ X(s) = \frac{1}{s(s^2+8s+16)} \]  \hspace{1cm} (13)

Factoring the denominator

\[ X(s) = \frac{1}{s(s+4)^2} \]  \hspace{1cm} (14)

In this case, the denominator polynomial has three real roots, two of which are identical

\[ s_1 = 0, \quad s_2 = -4, \quad s_3 = -4 \]
Inverse Laplace Transforms – Example 2

- Partial fraction expansion of (14) has the form
  
  \[ X(s) = \frac{1}{s(s+4)^2} = \frac{r_1}{s} + \frac{r_2}{s+4} + \frac{r_3}{(s+4)^2} \]  
  \[ (15) \]

- Again, find residues by multiplying both sides of (15) by the left-hand side denominator
  
  \[ 1 = r_1(s + 4)^2 + r_2s(s + 4) + r_3s \]
  
  \[ 1 = r_1s^2 + 8r_1s + 16r_1 + r_2s^2 + 4r_2s + r_3s \]

- Collecting terms, we have
  
  \[ 1 = s^2(r_1 + r_2) + s(8r_1 + 4r_2 + r_3) + 16r_1 \]  
  \[ (16) \]
Inverse Laplace Transforms – Example 2

- Equating coefficients of powers of $s$ on both sides of (16) gives a system of three equations in three unknowns
  
  \begin{align*}
  s^2: & \quad r_1 + r_2 = 0 \\
  s^1: & \quad 8r_1 + 4r_2 + r_3 = 0 \\
  s^0: & \quad 16r_1 = 1
  \end{align*}

- Solving for the residues gives
  
  \begin{align*}
  r_1 &= 0.0625 \\
  r_2 &= -0.0625 \\
  r_3 &= -0.2500
  \end{align*}

- The Laplace transform of the step response is
  
  $$X(s) = \frac{0.0625}{s} - \frac{0.0625}{s+4} - \frac{0.25}{(s+4)^2} \quad (17)$$

- Equation (17) can now be transformed back to the time domain using the Laplace transform table
The time-domain step response of the system is the **sum of a constant, a decaying exponential, and a decaying exponential scaled by time**:

\[ x(t) = 0.0625 - 0.0625e^{-4t} - 0.25te^{-4t} \]  

Step response plotted in MATLAB

Again, characteristic of a signal having **only real poles**

- Similar to the last case
- A bit faster – slow pole at \( s = -2 \) was eliminated
Reduce the damping even further and go through the process once again

\[ m = 1 \text{kg} \]
\[ k = \frac{16N}{m} \]
\[ b = 4 \frac{N \cdot s}{m} \]

Laplace transform of the step response becomes

\[ X(s) = \frac{1}{s(s^2 + 4s + 16)} \quad (19) \]

The second-order term in the denominator now has complex roots, so we won’t factor any further.

The denominator polynomial still has a root at zero and now has two roots which are a complex-conjugate pair

\[ s_1 = 0, \quad s_2 = -2 + j3.464, \quad s_3 = -2 - j3.464 \]
Inverse Laplace Transforms – Example 3

- Want to cast the partial fraction terms into forms that appear in the Laplace transform table
- Second-order terms should be of the form

\[ \frac{r_i(s+\sigma)+r_{i+1}\omega}{(s+\sigma)^2+\omega^2} \]  \hspace{1cm} (20)

- This will transform into the sum of damped sine and cosine terms

\[ \mathcal{L}^{-1} \left\{ r_i \frac{(s + \sigma)}{(s + \sigma)^2 + \omega^2} + r_{i+1} \frac{\omega}{(s + \sigma)^2 + \omega^2} \right\} = r_i e^{-\sigma t} \cos(\omega t) + r_{i+1} e^{-\sigma t} \sin(\omega t) \]

- To get the second-order term in the denominator of (19) into the form of (20), complete the square, to give the following partial fraction expansion

\[ X(s) = \frac{1}{s(s^2+4s+16)} = \frac{r_1}{s} + \frac{r_2(s+2)+r_3(3.464)}{(s+2)^2+(3.464)^2} \]  \hspace{1cm} (21)
Inverse Laplace Transforms – Example 3

- Note that the $\sigma$ and $\omega$ terms in (20) and (24) are the **real** and **imaginary parts** of the complex-conjugate denominator roots

$$s_{2,3} = -\sigma \pm j\omega = -2 \pm j3.464$$

- Multiplying both sides of (21) by the left-hand-side denominator, equate coefficients and solve for residues as before:

$$r_1 = 0.0625$$
$$r_2 = -0.0625$$
$$r_3 = -0.0361$$

- Laplace transform of the step response is

$$X(s) = \frac{0.0625}{s} - \frac{0.0625(s+2)}{(s+2)^2 + (3.464)^2} - \frac{0.0361(3.464)}{(s+2)^2 + (3.464)^2} \quad (22)$$
The time-domain step response of the system is the sum of a constant and two decaying sinusoids:

\[ x(t) = 0.0625 - 0.0625e^{-2t}\cos(3.464t) - 0.0361e^{-2t}\sin(3.464t) \]  

Step response and individual components plotted in MATLAB

Characteristic of a signal having complex poles

- Sinusoidal terms result in overshoot and (possibly) ringing
Laplace-Domain Signals with Complex Poles

- The Laplace transform of the step response in the last example had **complex poles**
  - A *complex-conjugate pair*: \( s = -\sigma \pm j\omega \)

- Results in sine and cosine terms in the time domain
  \[ Ae^{-\sigma t} \cos(\omega t) + Be^{-\sigma t} \sin(\omega t) \]

- **Imaginary part** of the roots, \( \omega \)
  - *Frequency of oscillation* of sinusoidal components of the signal

- **Real part** of the roots, \( \sigma \)
  - *Rate of decay* of the sinusoidal components

- Much more on this later
Natural and Forced Responses
So far, we’ve used Laplace transforms to determine the response of a system to a step input, given zero initial conditions

- The *driven response*

Now, consider the response of the same system to non-zero initial conditions only

- The *natural response*
Natural Response

- Same spring/mass/damper system
- Set the input to zero
- Second-order ODE for displacement of the mass:

\[ \ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0 \]  

(1)

- Use the derivative property to Laplace transform (1)
  - Allow for non-zero initial-conditions

\[ s^2 X(s) - sx(0) - \dot{x}(0) + \frac{b}{m} sX(s) - \frac{b}{m} x(0) + \frac{k}{m} X(s) = 0 \]  

(2)
Solving (2) for \( X(s) \) gives the Laplace transform of the output due solely to initial conditions.

The Laplace transform of the natural response is

\[
X(s) = \frac{s x(0) + \dot{x}(0) + \frac{b}{m}x(0)}{(s^2 + \frac{b}{m}s + \frac{k}{m})}
\]  

Consider the under-damped system with the following initial conditions:

- \( x(0) = 0.15 \, \text{m} \)
- \( \dot{x}(0) = 0.1 \, \text{m/s} \)

\( m = 1 \, \text{kg} \)
\( k = 16 \frac{\text{N}}{\text{m}} \)
\( b = 4 \frac{\text{N}\cdot\text{s}}{\text{m}} \)
Substituting component parameters and initial conditions into (3)

\[ X(s) = \frac{0.15s + 0.7}{s^2 + 4s + 16} \]  

(4)

Remember, it’s the roots of the denominator polynomial that dictate the form of the response

- Real roots – decaying exponentials
- Complex roots – decaying sinusoids

For the under-damped case, roots are complex

- Complete the square
- Partial fraction expansion has the form

\[ X(s) = \frac{0.15s + 0.7}{s^2 + 4s + 16} = \frac{r_1(s+2)+r_2(3.464)}{(s+2)^2+(3.464)^2} \]  

(5)
Natural Response

\[ X(s) = \frac{0.15s + 0.7}{s^2 + 4s + 16} = \frac{r_1(s+2) + r_2(3.464)}{(s+2)^2 + (3.464)^2} \]  

- Multiply both sides of (5) by the denominator of the left-hand side:

\[ 0.15s + 0.7 = r_1s + 2r_1 + 3.464r_2 \]

- Equating coefficients and solving for \( r_1 \) and \( r_2 \) gives:

\[ r_1 = 0.15, \quad r_2 = 0.115 \]

- The Laplace transform of the natural response:

\[ X(s) = \frac{0.15(s+2)}{(s+2)^2 + (3.464)^2} + \frac{0.115(3.464)}{(s+2)^2 + (3.464)^2} \]
Inverse Laplace transform is the natural response

\[ x(t) = 0.15e^{-2t} \cos(3.464 \cdot t) + 0.115e^{-2t} \sin(3.464 \cdot t) \]  

Under-damped response is the sum of decaying sine and cosine terms
Driven Response with Non-Zero I.C.’s

\[ \ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = \frac{1}{m} F_{in}(t) \]

- Now, Laplace transform, allowing for both **non-zero input and initial conditions**

\[ s^2 X(s) - sx(0) - \dot{x}(0) + \frac{b}{m} sX(s) - \frac{b}{m} x(0) + \frac{k}{m} X(s) = \frac{1}{m} F_{in}(s) \]

- Solving for \( X(s) \), gives the Laplace transform of the response to both the input and the initial conditions

\[ X(s) = \frac{s x(0) + \dot{x}(0) + \frac{b}{m} x(0) + \frac{1}{m} F_{in}(s)}{(s^2 + \frac{b}{m} s + \frac{k}{m})} \]  

\( (8) \)
Laplace transform of the response has two components:

\[
X(s) = \frac{s x(0) + \dot{x}(0) + \frac{b}{m} x(0)}{(s^2 + \frac{b}{m}s + \frac{k}{m})} + \frac{\frac{1}{m} F_{in}(s)}{(s^2 + \frac{b}{m}s + \frac{k}{m})}
\]

- **Natural response** - initial conditions
- **Driven response** - input

Total response is a superposition of the initial condition response and the driven response.

Both have the same denominator polynomial:

- Same roots, same type of response
  - Over-, under-, critically-damped
Driven Response with Non-Zero I.C.’s

- \( x(0) = 0.15 \, m \)
- \( \dot{x}(0) = 0.1 \, \frac{m}{s} \)
- \( F_{in}(t) = 1N \cdot 1(t) \)

Laplace transform of the total response

\[
X(s) = \frac{0.15s + 0.7 + \frac{1}{s}}{s^2 + 4s + 16} = \frac{0.15s^2 + 0.7s + 1}{s(s^2 + 4s + 16)}
\]

Transform back to time domain via partial fraction expansion

\[
X(s) = \frac{r_1}{s} + \frac{r_2(s + 2)}{(s + 2)^2 + (3.464)^2} + \frac{r_3(3.464)}{(s + 2)^2 + (3.464)^2}
\]

Solving for the residues gives

\( r_1 = 0.0625, \quad r_2 = 0.0875, \quad r_3 = 0.0794 \)

- \( m = 1 \, kg \)
- \( k = 16 \, \frac{N}{m} \)
- \( b = 4 \, \frac{N \cdot s}{m} \)
Driven Response with Non-Zero I.C.’s

- Total response:
  \[ x(t) = 0.0625 + 0.0875e^{-2t} \cos(3.464 \cdot t) + 0.0794e^{-2t} \sin(3.464 \cdot t) \]

- Superposition of two components
  - **Natural response** due to initial conditions
  - **Driven response** due to the input

![Graph showing driven response with non-zero initial conditions](image)
Transfer Functions
Transfer Functions

- Consider a system block diagram in the Laplace domain
  - $U(s)$ is the Laplace-domain input
  - $Y(s)$ is the Laplace-domain output
  - $G(s)$ is a Laplace-domain model for the plant

- To understand what $G(s)$ is, revisit the previous example

- We saw that if we assume zero initial conditions, the Laplace-domain output is

$$ Y(s) = \left[ \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}} \right] U(s) $$
**Transfer Functions**

\[ Y(s) = G(s) \cdot U(s) \]

- The Laplace transform of the output is the Laplace transform of the input multiplied by the stuff in square brackets

\[ Y(s) = \left[ \frac{1}{m} \right] \frac{1}{s^2 + \frac{b}{m} s + \frac{k}{m}} U(s) \]

- This is the transfer function

\[ G(s) = \frac{Y(s)}{U(s)} \]

- The ratio of the system’s output to input in the Laplace domain, assuming zero initial conditions

- A mathematical model for a dynamic system

- This is the model we will use for control system design in this course
The transfer function for our example system is

\[ G(s) = \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}} \]

Note that, as was the case for the Laplace transform of the output, the transfer function is a rational polynomial in \( s \)

\[ G(s) = \frac{B(s)}{A(s)} \]

Roots of \( B(s) \) are the system zeros

Roots of \( A(s) \) are the system poles

These poles and zeros dictate the nature of the system's response
SISO vs. MIMO Systems

- Note that we have assumed a system with one input and one output

\[ U(s) \longrightarrow G(s) \longrightarrow Y(s) \]

- A single-input single-output (SISO) system

- Often, systems have multiple inputs and multiple outputs

\[ U_1(s) \longrightarrow G(s) \longrightarrow Y_1(s) \]
\[ U_2(s) \longrightarrow G(s) \longrightarrow Y_2(s) \]
\[ \vdots \]
\[ U_m(s) \longrightarrow G(s) \longrightarrow Y_p(s) \]

- A multiple-input multiple-output (MIMO) system
MIMO Systems – Transfer Matrix

- For MIMO systems, the transfer function becomes a **transfer matrix**
  - For an m-input p-output system, a $p \times m$ **matrix of transfer functions**
    \[
    G(s) = \begin{bmatrix}
    G_{11}(s) & \cdots & G_{1m}(s) \\
    \vdots & \ddots & \vdots \\
    G_{p1}(s) & \cdots & G_{pm}(s)
    \end{bmatrix}
    \]
  - Transfer function $G_{ij}(s)$ relates the $i^{th}$ output to the $j^{th}$ input
    \[
    G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}
    \]
- In this course, we’ll restrict our focus entirely to SISO systems
Characteristic Polynomial

\[ G(s) = \frac{B(s)}{A(s)} \]

- The denominator of the transfer function is the *characteristic polynomial*, \( \Delta(s) \)

\[ G(s) = \frac{B(s)}{\Delta(s)} \]

- Poles of the transfer function are roots of \( \Delta(s) \)
  - *System poles* or *eigenvalues*
  - Poles determine the terms in the partial-fraction-expansion of the system’s natural response
  - Along with the input, system poles determine the nature of the time-domain response
Using the Transfer Function to Determine System Response
Using $G(s)$ to determine System Response

- System output in the Laplace domain can be expressed in terms of the transfer function as

$$Y(s) = U(s)G(s)$$  \hspace{1cm} (1)

- Laplace-domain output is the product of the Laplace-domain input and the transfer function

- Response to two specific types of inputs often used to characterize dynamic systems
  - Impulse response
  - Step response

- We’ll use the approach of (1) to determine these responses
Impulse response

- **Impulse function**
  \[ \delta(t) = 0, \; t \neq 0 \]
  \[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

- Laplace transform of the impulse function is
  \[ \mathcal{L}\{\delta(t)\} = 1 \]

- Impulse response in the Laplace domain is
  \[ Y(s) = 1 \cdot G(s) = G(s) \]

- The **transfer function is the Laplace transform of the impulse response**

- Impulse response in the time domain is the inverse transform of the transfer function
  \[ y(t) = g(t) = \mathcal{L}^{-1}\{G(s)\} \]
Step Response

- **Step function:**
  \[1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}\]

- Laplace transform of the step function
  \[\mathcal{L}\{1(t)\} = \frac{1}{s}\]

- Laplace-domain step response
  \[Y(s) = \frac{1}{s} \cdot G(s)\]

- Time-domain step response
  \[y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot G(s)\right\}\]

- Recall the integral property of the Laplace transform
  \[\mathcal{L}\left\{\int_{0}^{t} g(\tau)d\tau\right\} = \frac{1}{s} \cdot G(s), \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot G(s)\right\} = \int_{0}^{t} g(\tau)d\tau\]

- **The step response is the integral of the impulse response**
Transfer Function and Dynamic Response

- We just saw that the impulse response is
  \[ g(t) = \mathcal{L}^{-1}\{G(s)\} \]

- And the step response is
  \[ y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot G(s)\right\} \]

- Both are entirely determined by the system transfer function, \( G(s) \)
  - System poles and zeros determine the nature of these responses
  - \( G(s) \) is a complete mathematical model for the system