Introduction
Steady-State Error – Introduction

- Consider a simple *unity-feedback system*

\[ E(s) = R(s) - Y(s) \]

- The *error* is the difference between the reference and the output

- The input to the controller, \( D(s) \)

- Consider a case where:
  - Reference input is a step
  - Plant has no poles at the origin – finite DC gain
  - Controller is a simple gain block

- In *steady state*, the forward path reduces to a constant gain:
Steady-State Error – Introduction

- In steady state, we’d like:
  - Output to be equal to the input: \( y_{ss} = r_{ss} \)
  - Zero steady-state error: \( e_{ss} = 0 \)

- Is that the case here?
  \[
  e_{ss} = r_{ss} - y_{ss} = r_{ss} - e_{ss}K
  \]
  \[
  e_{ss} = r_{ss} \frac{1}{1 + K}
  \]
  - **No**, if \( r_{ss} \neq 0 \), then \( e_{ss} \neq 0 \)

- **Non-zero steady-state error to a step input for finite steady-state forward-path gain**
  - Finite DC gain implies **no poles at the origin** in \( D(s) \) or \( G(s) \)
Now, allow a single pole at the origin
- An integrator in the forward path

Now the error is

\[ E(s) = R(s) - E(s) \cdot \frac{K}{s} \]
\[ E(s) = R(s) \frac{s}{s + K} \]

For a step input

\[ E(s) = \frac{1}{s} \frac{s}{s + K} = \frac{1}{s + K} \]

Applying the final value theorem gives the steady-state error

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{s + K} = 0 \]

Zero steady-state error to a step input when there is an integrator in the forward path
Next, consider a ramp input to the same system

\[ r(t) = t \cdot 1(t) \quad \text{and} \quad R(s) = \frac{1}{s^2} \]

Now the error is

\[ E(s) = \frac{1}{s^2} \frac{s}{s + K} = \frac{1}{s(s + K)} \]

The steady-state error is

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{s(s + K)} = \frac{1}{K} \]

Non-zero, but finite, steady-state error to a ramp input when there is an integrator in the forward path
Steady-State Error – Introduction

- Two key observations from the preceding example involving *unity-feedback* systems:
  - *Steady-state error is related to the number of integrators in the open-loop transfer function*
  - *Steady-state error is related to the type of input*

- We’ll now explore both of these observations more thoroughly

- First, we’ll introduce the concept of *system type*
System Type and Steady-State Error
System Type

- **System Type**
  
  *The degree of the input polynomial for which the steady-state error is a finite, non-zero constant*

- **Type 0**: finite, non-zero error to a *step* input
- **Type 1**: finite, non-zero error to a *ramp* input
- **Type 2**: finite, non-zero error to a *parabolic* input

- For the remainder of this sub-section, and the one that follows, we’ll consider only the special case of *unity-feedback* systems
For **unity-feedback systems**, system type is determined by the **number of integrators in the forward path**

- **Type 0**: no integrators in the open-loop TF, e.g.:
  \[
  G(s) = \frac{s + 4}{(s + 6)(s^2 + 4s + 8)}
  \]

- **Type 1**: one integrator in the open-loop TF, e.g.:
  \[
  G(s) = \frac{15}{s(s^2 + 3s + 12)}
  \]

- **Type 2**: two integrators in the open-loop TF, e.g.:
  \[
  G(s) = \frac{s + 5}{s^2(s + 3)(s + 7)}
  \]
Types of Inputs

- When characterizing a control system’s error performance we focus on three main inputs:
  - Step
  - Ramp
  - Parabola

- We will derive expressions for the steady-state error due to each

- **Step:**
  - \( r(t) = 1(t) \leftrightarrow R(s) = \frac{1}{s} \)
  - For a positioning system, this represents a *constant position*
Types of Inputs

- **Ramp**:  
  - \[ r(t) = t \cdot 1(t) \iff R(s) = \frac{1}{s^2} \]  
  - For a positioning system, this represents a **constant velocity**

- **Parabola**:  
  - \[ r(t) = \frac{1}{2} t^2 \cdot 1(t) \iff R(s) = \frac{1}{s^3} \]  
  - For a positioning system, this represents a **constant acceleration**
For unity-feedback systems steady-state error can be expressed in terms of the open-loop transfer function, $G(s)$

$$E(s) = R(s) - Y(s) = R(s) - E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Steady-state error is found by applying the final value theorem

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

We’ll now consider this expression for each of the three inputs of interest
Steady-State Error – Step Input

- For a step input
  \[ r(t) = 1(t) \iff R(s) = \frac{1}{s} \]

- Steady-state error to a step input is
  \[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \frac{1}{s}}{1 + G(s)} \]
  \[ e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)} \]
  \[ e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)} \]
Steady-State Error – Step Input

\[ e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)} \]

- In order to have \( e_{ss} = 0 \), as we’d like, we must have
  \[ \lim_{s \to 0} G(s) = \infty \]

- That is, the **DC gain of the open-loop system must be infinite**

- If \( G(s) \) has the following form
  \[ G(s) = \frac{(s + z_1)(s + z_2)\cdots}{(s + p_1)(s + p_2)\cdots} \]
  then
  \[ \lim_{s \to 0} G(s) = \frac{z_1z_2\cdots}{p_1p_2\cdots} \neq \infty \]
  and we’ll have non-zero steady-state error
Steady-State Error – Step Input

- However, consider $G(s)$ of the following form

$$G(s) = \frac{(s + z_1)(s + z_2) \cdots}{s^n(s + p_1)(s + p_2) \cdots}$$

where $n \geq 1$

- That is, $G(s)$ includes $n$ integrators
  - It is a type $n$ system

$$\lim_{s \to 0} G(s) = \infty \text{ and } e_{ss} = 0$$

- A type 1 or greater system will exhibit zero steady-state error to a step input
Steady-State Error – Ramp Input

- For a ramp input

\[ r(t) = t \cdot 1(t) \iff R(s) = \frac{1}{s^2} \]

- Steady-state error to a ramp input is

\[
e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \frac{1}{s^2}}{1 + G(s)}
\]

\[
e_{ss} = \lim_{s \to 0} \frac{1}{s + sG(s)}
\]

\[
e_{ss} = \frac{1}{\lim_{s \to 0} sG(s)}
\]
Steady-State Error – Ramp Input

\[ e_{ss} = \frac{1}{\lim_{s \to 0} sG(s)} \]

- In order to have \( e_{ss} = 0 \), the following must be true
  \[ \lim_{s \to 0} sG(s) = \infty \]

- If there are no integrators in the forward path, then
  \[ \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{(s + z_1)(s + z_2) \cdots}{(s + p_1)(s + p_2) \cdots} = 0 \]

  and

  \[ e_{ss} = \infty \]

- **A type 0 system has infinite steady-state error to a ramp input**
If there is a single integrator in the forward path, i.e. a type 1 system

\[ G(s) = \frac{(s + z_1)(s + z_2) \cdots}{s(s + p_1)(s + p_2) \cdots} \]

then

\[ \lim_{s \to 0} sG(s) = \frac{z_1z_2 \cdots}{p_1p_2 \cdots} \]

and

\[ e_{ss} = \frac{p_1p_2 \cdots}{z_1z_2 \cdots} \]

A type 1 system has non-zero, but finite, steady-state error to a ramp input
Steady-State Error – Ramp Input

- If there are two or more integrators in the forward path, i.e. a type 2 or greater system

\[
G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}, \quad (n \geq 2)
\]

then

\[
\lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{(s + z_1)(s + z_2)\cdots}{s^n(s + p_1)(s + p_2)\cdots} = \infty
\]

and

\[e_{ss} = 0\]

- A type 2 or greater system has zero steady-state error to a ramp input
Steady-State Error – Parabolic Input

- For a Parabolic input
  \[ r(t) = \frac{t^2}{2} \cdot 1(t) \iff R(s) = \frac{1}{s^3} \]
- Steady-state error to a parabolic input is
  \[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \frac{1}{s^3}}{1 + G(s)} \]
  \[ e_{ss} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} \]
  \[ e_{ss} = \frac{1}{\lim_{s \to 0} s^2 G(s)} \]
Steady-State Error – Parabolic Input

\[ e_{ss} = \lim_{s \to 0} \frac{1}{s^2 G(s)} \]

- In order to have \( e_{ss} = 0 \), the following must be true
  \[ \lim_{s \to 0} s^2 G(s) = \infty \]

- If there are no integrators in the forward path, then
  \[ \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} s^2 \frac{(s + z_1)(s + z_2) \cdots}{(s + p_1)(s + p_2) \cdots} = 0 \]
  and
  \[ e_{ss} = \infty \]

- A type 0 system has infinite steady-state error to a parabolic input
If there is a single integrator in the forward path, i.e. a type 1 system

\[ G(s) = \frac{(s + z_1)(s + z_2)\cdots}{s(s + p_1)(s + p_2)\cdots} \]

then

\[ \lim_{{s \to 0}} s^2 G(s) = s \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} = 0 \]

and

\[ e_{ss} = \infty \]

A type 1 system has infinite steady-state error to a parabolic input
If there are two integrators in the forward path, i.e. a type 2 system

\[ G(s) = \frac{(s + z_1)(s + z_2)\cdots}{s^2(s + p_1)(s + p_2)\cdots} \]

then

\[ \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{(s + z_1)(s + z_2)\cdots}{(s + p_1)(s + p_2)\cdots} = \frac{z_1z_2\cdots}{p_1p_2\cdots} \]

and

\[ e_{ss} = \frac{p_1p_2\cdots}{z_1z_2\cdots} \]

A type 2 system has non-zero, but finite, steady-state error to a parabolic input
Steady-State Error — Parabolic Input

- If there are three or more integrators in the forward path, i.e. a type 3 or greater system

\[
G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}, \quad (n \geq 3)
\]

then

\[
\lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} s^2 \frac{(s + z_1)(s + z_2)\cdots}{s^n(s + p_1)(s + p_2)\cdots} = \infty
\]

and

\[
e_{ss} = 0
\]

- A type 3 or greater system has zero steady-state error to a parabolic input
Static Error Constants
We’ve seen that the steady-state error to each of the inputs considered is

- **Step**: 
  \[ e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)} \]

- **Ramp**: 
  \[ e_{ss} = \frac{1}{\lim_{s \to 0} sG(s)} \]

- **Parabola**: 
  \[ e_{ss} = \frac{1}{\lim_{s \to 0} s^2G(s)} \]

The limit term in each expression is the **static error constant** associated with that particular input:

- **Position constant**: 
  \[ K_p = \lim_{s \to 0} G(s) \]

- **Velocity constant**: 
  \[ K_v = \lim_{s \to 0} sG(s) \]

- **Acceleration constant**: 
  \[ K_a = \lim_{s \to 0} s^2G(s) \]
Steady-State Error vs. System Type

- Steady-state error vs. input and system type

<table>
<thead>
<tr>
<th>System Type</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Step</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{1 + K_p}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

- Note that the given steady-state error is for inputs of unit magnitude
  - Actual error is scaled by the magnitude of the reference input
Non-Unity-Feedback Systems
Non-Unity-Feedback Systems

- So far, we’ve focused on the special case of *unity-feedback* systems

- **System type** determined by *# of integrators* in the forward path – i.e., *# of open-loop* poles at the origin

- Steady-state error determined using *static error constants*

- Static error constants determined from the *open-loop transfer function*
Non-Unity-Feedback Systems

- More general approach to determining steady-state error is to use the **closed-loop transfer function**
  - Applicable to non-unity-feedback systems, e.g.:
    \[ T(s) = \frac{G_1(s)G_2(s)}{1 + G_2(s)H(s)} \]

- The error is
  \[ E(s) = R(s) - Y(s) = R(s) - R(s)T(s) \]
  \[ E(s) = R(s)[1 - T(s)] \]
Non-Unity-Feedback Systems

- Apply the **final value theorem** to determine the steady-state error:

  \[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)[1 - T(s)] \]

- Here, system type is determined by using the more general definition:

  **System type** is the degree of the input polynomial for which the steady-state error is a finite, non-zero constant
Non-Unity-Feedback Systems

- Alternatively, find steady-state error by converting to a unity-feedback configuration, e.g.:

- Add and subtract unity-feedback paths:
Non-Unity-Feedback Systems

- Combine the two upper parallel feedback paths:

- Collapsing the inner feedback form leaves a unity-feedback system
  - Can now apply unity-feedback error analysis techniques
Steady-State Error – Examples
What is the steady-state error to a constant reference input, \( r(t) = 3 \text{ cm} \cdot 1(t) \), for the following feedback positioning system?

A type 0 system
- Non-zero error to a constant reference

Position constant:

\[
K_p = \lim_{s \to 0} G(s) = 10
\]

Steady-state error:

\[
e_{ss} = r_{ss} \frac{1}{1 + K_p} = 3 \text{ cm} \frac{1}{1 + 10}
\]

\[
e_{ss} = 0.27 \text{ cm}
\]
Steady-State Error – Example 1

Closed-Loop Response to $r(t) = 3\text{cm} \cdot 1(t)$

$e_{ss} = 0.27 \text{ cm}$
Steady-State Error – Example 1

- What is the same system’s steady-state error to a unit ramp input, \( r(t) = t \cdot 1(t) \)?
  - A type 0 system, so error to a ramp reference will be \textit{infinite}

- Verify using closed-loop transfer function
  
  \[
  T(s) = \frac{G(s)}{1 + G(s)} = \frac{10}{s + 11}
  \]

- Steady-state error is
  
  \[
  e_{ss} = \lim_{s \to 0} sR(s)[1 - T(s)] = \lim_{s \to 0} s \left( \frac{1}{s^2} \left[ 1 - \frac{10}{s + 11} \right] \right)
  \]
  
  \[
  e_{ss} = \lim_{s \to 0} \frac{1}{s} \left[ \frac{s + 1}{s + 11} \right] = \infty
  \]
Steady-State Error – Example 1

Closed-Loop Response to $r(t) = t \cdot 1(t)$

$e_{ss} = \infty$
Steady-State Error – Example 2

- Design the controller, $D(s)$, for error of 0.05 to a unit ramp input

![Block Diagram]

- Plant is type 0
  - Forward path must be type 1 for finite error to a ramp input
  - $D(s)$ must be type 1, so one very simple option is:
    \[ D(s) = \frac{K}{s} \]
  - Forward-path transfer function is
    \[ D(s)G(s) = \frac{K(s + 2)}{s(s + 1)(s + 5)} \]
The velocity constant is

\[ K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{K(s + 2)}{s(s + 1)(s + 5)} = \frac{2K}{5} \]

Steady-state error is

\[ e_{ss} = \frac{1}{K_v} = \frac{5}{2K} \]

For error of 0.05:

\[ e_{ss} = 0.05 = \frac{5}{2K} \]

\[ K = 50 \]
Steady-State Error – Example 2

Closed-Loop Response to \( r(t) = t \, 1(t) \)

\[ e_{ss} = 0.05 \]
Next, consider a non-unity-feedback system:

- Determine controller gain, $K$, to provide a 2% steady-state error to a constant reference input

- First, convert to a unity-feedback system
  - Combine forward-path blocks
  - Simultaneously add and subtract unity-feedback paths
Steady-State Error – Example 3

- Combine the top two parallel feedback paths
  
  ![Diagram 1](image1)

- Simplifying the inner feedback form leaves a unity-feedback system
  
  ![Diagram 2](image2)
Steady-State Error – Example 3

- Steady-state error for this type 0 system is
  \[ e_{ss} = \frac{1}{1 + K_p} \]
  where
  \[ K_p = \lim_{s \to 0} G(s) = \frac{20 \cdot K}{10} = 2 \cdot K \]

- For 2% steady-state error
  \[ e_{ss} = 0.02 = \frac{1}{1 + 2 \cdot K} \]

- The controller gain is
  \[ K = 24.5 \]
Note that the controller gain has been set to satisfy a steady-state error requirement only.

- Closed loop poles are very lightly-damped.
- Dynamic response is likely unacceptable.
Now, consider a unity-feedback system with a disturbance input.

\[ D(s) = K \quad \text{and} \quad G(s) = \frac{1}{s+5} \]

Determine the controller gain, \( K \), such that error due to a constant disturbance is 1% of \( W(s) \).

For this value of \( K \), what is the steady-state error to a constant reference input?
The total error is given by

\[ E(s) = R(s) - Y(s) = R(s) - [E(s)D(s)G(s) + W(s)G(s)] \]

\[ E(s)[1 + D(s)G(s)] = R(s) - W(s)G(s) \]

\[ E(s) = R(s) \frac{1}{1 + D(s)G(s)} - W(s) \frac{G(s)}{1 + D(s)G(s)} \]

Substituting in controller and plant transfer functions gives

\[ E(s) = R(s) \frac{s + 5}{s + 5 + K} - W(s) \frac{1}{s + 5 + K} \]
Steady-State Error – Example 4

- Error due to a constant disturbance can be found by applying the final value theorem

\[
e_{ss,w} = \lim_{s \to 0} s \left( -W(s) \frac{1}{s + 5 + K} \right)
\]

\[
e_{ss,w} = \lim_{s \to 0} \left( -s \frac{1}{s + 5 + K} \right) = -\frac{1}{5 + K}
\]

- We can calculate the required gain for 1% error

\[
|e_{ss,w}| = 0.01 = \frac{1}{5 + K} \quad \rightarrow \quad K = 95
\]

- At this gain value, the error due to a constant reference is

\[
e_{ss,r} = \lim_{s \to 0} \left( s \frac{1}{s} \frac{s + 5}{s + 5 + K} \right) = \frac{5}{100} \quad \rightarrow \quad 5\%
\]