SECTION 7: FREQUENCY-RESPONSE ANALYSIS
Introduction
Introduction

- We have seen how to design feedback control systems using the *root locus*
- In this section of the course, we’ll learn how to do the same using the open-loop *frequency response*

**Objectives:**
- Review frequency-response fundamentals
- Relate a system’s frequency response to its transient response
- Determine static error constants from the open-loop frequency response
- Determine closed-loop stability from the open-loop frequency response
Consider an $n^{th}$-order system

- $n$ poles: $p_1, p_2, \ldots p_n$
  - Real or complex
  - Assume all are distinct

Transfer function is:

$$G(s) = \frac{Num(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$  \hspace{1cm} (1)

Apply a sinusoidal input to the system

$$u(t) = A \sin(\omega t) \quad \mathcal{L} \quad U(s) = A \frac{\omega}{s^2+\omega^2}$$

Output is given by

$$Y(s) = G(s)U(s) = \frac{Num(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \cdot A \frac{\omega}{s^2+\omega^2}$$  \hspace{1cm} (2)
System Response to a Sinusoidal Input

- Partial fraction expansion of (2) gives
  \[ Y(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \ldots + \frac{r_n}{s-p_n} + \frac{r_{n+1}s}{s^2+\omega^2} + \frac{r_{n+2}\omega}{s^2+\omega^2} \]  

- Inverse transform of (3) gives the time-domain output
  \[ y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots + r_n e^{p_n t} + r_{n+1} \cos(\omega t) + r_{n+2} \sin(\omega t) \]  

- Two portions of the response:
  - Transient
    - Decaying exponentials or sinusoids – goes to zero in steady state
    - Natural response to initial conditions
  - Steady state
    - Due to the input – sinusoidal in steady state
We are interested in the **steady-state response**

\[ y_{ss}(t) = r_{n+1} \cos(\omega t) + r_{n+2} \sin(\omega t) \]  

(5)

A trig. identity provides insight into \( y_{ss}(t) \):

\[ \alpha \cos(\omega t) + \beta \sin(\omega t) = \sqrt{\alpha^2 + \beta^2} \sin(\omega t + \phi) \]

where

\[ \phi = \tan^{-1} \left( \frac{\alpha}{\beta} \right) \]

Steady-state response to a sinusoidal input

\[ u(t) = A \sin(\omega t) \]

is a sinusoid of the same frequency, but, in general different amplitude and phase

\[ y_{ss}(t) = B \sin(\omega t + \phi) \]

Where

\[ B = \sqrt{r_{n+1}^2 + r_{n+2}^2} \quad \text{and} \quad \phi = \tan^{-1} \left( \frac{r_{n+1}}{r_{n+2}} \right) \]  

(6)
Steady-State Sinusoidal Response

\[ u(t) = A \sin(\omega t) \rightarrow y_{ss}(t) = B \sin(\omega t + \phi) \]

- Steady-state sinusoidal response is a **scaled** and **phase-shifted** sinusoid of the same frequency
  - Equal frequency is a property of linear systems

- Note the \( \omega \) term in the numerator of (3)
  - \( \omega \) will affect the residues
  - Residues determine amplitude and phase of the output
  - **Output amplitude and phase are frequency-dependent**

\[ y_{ss}(t) = B(\omega) \sin(\omega t + \phi(\omega)) \]
Steady-State Sinusoidal Response

\[ u(t) = A \sin(\omega t + \theta) \quad \text{Linear System} \quad G(s) \quad y_{ss}(t) = B \sin(\omega t + \phi) \]

- **Gain** – the ratio of amplitudes of the output and input of the system
  \[ Gain = \frac{B}{A} \]

- **Phase** – phase difference between system input and output
  \[ Phase = \phi - \theta \]

- Systems will, in general, exhibit **frequency-dependent** gain and phase

- We’d like to be able to determine these functions of frequency
  - The system’s **frequency response**
Frequency Response

A system’s frequency response, or sinusoidal transfer function, describes its gain and phase shift for sinusoidal inputs as a function of frequency.
Frequency Response

- System output in the Laplace domain is
  \[ Y(s) = U(s) \cdot G(s) \]

- Multiplication in the Laplace domain corresponds to convolution in the time domain
  \[ y(t) = u(t) \ast g(t) = \int_0^t g(\tau)u(t - \tau)d\tau \]

- Consider an exponential input of the form
  \[ u(t) = e^{st} \]
  where \( s \) is the complex Laplace variable: \( s = \sigma + j\omega \)

- Now the output is
  \[ y(t) = u(t) \ast g(t) = \int_0^t g(\tau)e^{s(t-\tau)}d\tau = \int_0^t g(\tau)e^{st}e^{-st}d\tau \]
  \[ y(t) = \int_0^t g(\tau)e^{-st}d\tau \cdot e^{st} \] (1)
Frequency Response

\[ y(t) = \int_{0}^{t} g(\tau)e^{-s\tau}d\tau \cdot e^{st} \]  

- We’re interested in the steady-state response, so let the upper limit of integration go to infinity

\[ y(t) = \int_{0}^{\infty} g(\tau)e^{-s\tau}d\tau \cdot e^{st} \]

\[ y(t) = G(s) \cdot e^{st} \]  

- Time-domain response to an exponential input is the time-domain input multiplied by the system transfer function

- What is this input?

\[ u(t) = e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} \]  

- If we let \( \sigma \to 0 \), i.e. let \( s \to j\omega \), then we have

\[ y(t) = G(j\omega) \cdot e^{j\omega t} \]
Recall **Euler’s formula**:

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$  \hspace{1cm} (5)

From which it follows that

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$  \hspace{1cm} (6)

and

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$  \hspace{1cm} (7)
Frequency Response

- We’re interested in the sinusoidal steady-state system response, so let the input be

\[ u(t) = A \cos(\omega t) = A \frac{e^{j\omega t} + e^{-j\omega t}}{2} \]

- A sum of complex exponentials in the form of (3)
  - We’ve let \( s \rightarrow j\omega \) in the first term and \( s \rightarrow -j\omega \) in the second

\[ u(t) = \frac{A}{2} e^{j\omega t} + \frac{A}{2} e^{-j\omega t} \] \hspace{1cm} (8)

- According to (4) the output in response to (8) will be

\[ y(t) = \frac{A}{2} G(j\omega) \cdot e^{j\omega t} + \frac{A}{2} G(-j\omega) \cdot e^{-j\omega t} \] \hspace{1cm} (9)
Frequency Response

\[ y(t) = \frac{A}{2} G(j\omega) \cdot e^{j\omega t} + \frac{A}{2} G(-j\omega) \cdot e^{-j\omega t} \]  

- \( G(j\omega) \) is a complex function of frequency
  - Evaluates to a complex number at each value of \( \omega \)
  - Has both magnitude and phase
  - Can be expressed in polar form as
    \[ G(j\omega) = Me^{j\phi} \]  
    where
    \[ M = |G(j\omega)| \text{ and } \phi = \angle G(j\omega) \]
- It follows that
  \[ G(-j\omega) = Me^{-j\phi} \]
Using (11), the output given by (9) becomes

\[ y(t) = \frac{A}{2} M [ e^{j\omega t} e^{j\phi} + e^{-j\omega t} e^{-j\phi} ] \]

\[ y(t) = \frac{A}{2} M [ e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)} ] \] (12)

\[ y(t) = M \cdot A \cos(\omega t + \phi) \] (13)

where, again

\[ M = |G(j\omega)| \text{ and } \phi = \angle G(j\omega) \] (14)
Frequency response Function – \( G(j\omega) \)

- \( G(j\omega) \) is the system’s **frequency response function**
  - Transfer function, where \( s \to j\omega \)
    \[
    G(j\omega) = G(s)|_{s\to j\omega}\quad (15)
    \]
  - A complex-valued function of frequency

- \( |G(j\omega)| \) at each \( \omega \) is the **gain** at that frequency
  - Ratio of output amplitude to input amplitude

- \( \angle G(j\omega) \) at each \( \omega \) is the **phase** at that frequency
  - Phase shift between input and output sinusoids

- Another representation of system behavior
  - Along with state-space model, impulse/step responses, transfer function, etc.
  - Typically represented graphically
Plotting the Frequency Response Function

- $G(j\omega)$ is a complex-valued function of frequency
  - Has both magnitude and phase
  - Plot gain and phase separately
- Frequency response plots formatted as **Bode plots**
  - Two sets of axes: gain on top, phase below
  - Identical, logarithmic frequency axes
  - Gain axis is logarithmic – either explicitly or as units of decibels (dB)
  - Phase axis is linear with units of degrees
Bode Plots

Units of magnitude are dB

Units of phase are degrees

Magnitude plot on top

Logarithmic frequency axes

Phase plot below
Interpreting Bode Plots

Bode plots tell you the gain and phase shift at all frequencies:
choose a frequency, read gain and phase values from the plot

For a 10KHz sinusoidal input, the gain is 0dB (1) and the phase shift is 0°.

For a 10MHz sinusoidal input, the gain is -32dB (0.025), and the phase shift is -176°. 
Interpreting Bode Plots
Decibels - dB

- Frequency response gain most often expressed and plotted with units of decibels (dB)
  - A logarithmic scale
  - Provides detail of very large and very small values on the same plot
  - Commonly used for ratios of powers or amplitudes

- Conversion from a linear scale to dB:
  \[ |G(j\omega)|_{dB} = 20 \cdot \log_{10}(|G(j\omega)|) \]

- Conversion from dB to a linear scale:
  \[ |G(j\omega)| = 10^{\frac{|G(j\omega)|_{dB}}{20}} \]
Decibels – dB

- Multiplying two gain values corresponds to adding their values in dB
  - E.g., the overall gain of cascaded systems
    \[ |G_1(j\omega) \cdot G_2(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} \]

- Negative dB values corresponds to sub-unity gain
- Positive dB values are gains greater than one

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Value of Logarithmic Axes - dB

- Gain axis is linear in dB
  - A logarithmic scale
  - Allows for displaying detail at very large and very small levels on the same plot

- Gain plotted in dB
  - Two resonant peaks clearly visible

- Linear gain scale
  - Smaller peak has disappeared
Value of Logarithmic Axes - dB

- Frequency axis is logarithmic
  - Allows for displaying detail at very low and very high frequencies on the same plot

- Log frequency axis
  - Can resolve frequency of both resonant peaks

- Linear frequency axis
  - Lower resonant frequency is unclear
Gain Response – Terminology

- **Corner frequency, cut off frequency, -3dB frequency:**
  - Frequency at which gain is 3dB below its low-frequency value
    \[ f_c = \frac{\omega_c}{2\pi} \]
  - This is the *bandwidth* of the system

- **Peaking**
  - Any increase in gain above the low frequency gain
    \[ \omega_c = 1.45 \frac{\text{rad}}{\text{sec}} \]
    \[ f_c = \frac{\omega_c}{2\pi} = 0.23 \text{Hz} \]
    \(~5dB\) of peaking
Frequency-Response Factors
Transfer Function Factors

- Numerator and denominator of a transfer function can be factored into first- and second-order terms

\[ G(s) = \frac{(s - z_1)(s - z_2) \cdots (s^2 + 2\zeta_a\omega_{na}s + \omega_{na}^2)(s^2 + 2\zeta_2\omega_{nb}s + \omega_{nb}^2) \cdots}{(s - p_1)(s - p_2) \cdots (s^2 + 2\zeta_1\omega_{n1}s + \omega_{n1}^2)(s^2 + 2\zeta_2\omega_{n2}s + \omega_{n2}^2) \cdots} \]

- Can think of the transfer function as a product of the individual factors

- For example, consider the following system

\[ G(s) = \frac{(s - z_1)}{(s - p_1)(s^2 + 2\zeta_1\omega_{n1}s + \omega_{n1}^2)} \]

- Can rewrite as

\[ G(s) = (s - z_1) \cdot \frac{1}{(s - p_1)} \cdot \frac{1}{(s^2 + 2\zeta_1\omega_{n1}s + \omega_{n1}^2)} \]
Transfer Function Factors

\[ G(s) = (s - z_1) \cdot \frac{1}{s - p_1} \cdot \frac{1}{s^2 + 2\zeta_1 \omega_n s + \omega_n^2} \]

Think of this as three cascaded transfer functions

\[ G_1(s) = (s - z_1), \quad G_2(s) = \frac{1}{s - p_1}, \quad G_3(s) = \frac{1}{s^2 + 2\zeta_1 \omega_n s + \omega_n^2} \]

or

\[ \frac{1}{s - p_1} \]

\[ \frac{1}{s^2 + 2\zeta_1 \omega_n s + \omega_n^2} \]
In the **Laplace domain**, transfer function of a *cascade* of systems is the **product** of the individual transfer functions.

- In the time domain, overall impulse response is the convolution of the individual impulse responses.

- Same holds true in the **frequency domain**.
  - *Frequency response* of a *cascade* is the **product** of the individual frequency responses.
  - Or, the **product of individual factors**.
Consider the following system

\[ G(s) = \frac{20(s + 20)}{(s + 1)(s + 100)} \]

The system’s frequency response function is

\[ G(j\omega) = \frac{20(j\omega + 20)}{(j\omega + 1)(j\omega + 100)} \]

As we’ve seen we can consider this a product of individual frequency response factors

\[ G(j\omega) = 20 \cdot (j\omega + 20) \cdot \frac{1}{(j\omega + 1)} \cdot \frac{1}{(j\omega + 100)} \]

Overall response is the composite of the individual responses

- Product of individual gain responses – sum in dB
- Sum of individual phase responses
Frequency Response Components - Example

Gain response

Pole/Zero Plot

Gain response

- $G_1(j\omega) = 20$
- $G_2(j\omega) = (j\omega + 20)$
- $G_3(j\omega) = 1/(j\omega + 1)$
- $G_4(j\omega) = 1/(j\omega + 100)$

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Frequency Response Components - Example

- Phase response

**Pole/Zero Plot**

- **Graph**
  - **Red** line: $G_1(j\omega) = 20$
  - **Pink** line: $G_2(j\omega) = (j\omega + 20)$
  - **Green** line: $G_3(j\omega) = 1/(j\omega + 1)$
  - **Light Blue** line: $G_4(j\omega) = 1/(j\omega + 100)$

**Axes**
- Real
- Imaginary

**Frequency Response**

- **X-axis**: $\omega$ [rad/sec]
- **Y-axis**: Phase [deg]

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In this section, we’ll look at a method for sketching, by hand, a straight-line, asymptotic approximation for a Bode plot.
We’ve just seen that a system’s frequency response function can be factored into first- and second-order terms. Each factor contributes a component to the overall gain and phase responses.

Now, we’ll look at a technique for manually sketching a system’s Bode plot. In practice, you’ll almost always plot with a computer. But, learning to do it by hand provides valuable insight.

We’ll look at how to approximate Bode plots for each of the different factors.
Consider the general transfer function form:

$$G(s) = K \frac{(s - z_1)(s - z_2) \cdots (s^2 + 2\zeta_\omega \omega_n s + \omega_n^2) \cdots}{(s - p_1)(s - p_2) \cdots (s^2 + 2\zeta_\omega \omega_n s + \omega_n^2) \cdots}$$

We first want to put this into Bode form:

$$G(s) = K_0 \frac{\left( \frac{s}{\omega_{ca}} + 1 \right) \left( \frac{s}{\omega_{cb}} + 1 \right) \cdots \left( \frac{s^2}{\omega_n^2} + \frac{2\zeta_\omega}{\omega_n} s + 1 \right) \cdots}{\left( \frac{s}{\omega_{c1}} + 1 \right) \left( \frac{s}{\omega_{c2}} + 1 \right) \cdots \left( \frac{s^2}{\omega_n^2} + \frac{2\zeta_\omega}{\omega_n} s + 1 \right) \cdots}$$

The corresponding frequency response function, in Bode form, is

$$G(j\omega) = K_0 \frac{\left( \frac{j\omega}{\omega_{ca}} + 1 \right) \left( \frac{j\omega}{\omega_{cb}} + 1 \right) \cdots \left( \frac{(j\omega)^2}{\omega_n^2} + \frac{2\zeta_\omega}{\omega_n} j\omega + 1 \right) \cdots}{\left( \frac{j\omega}{\omega_{c1}} + 1 \right) \left( \frac{j\omega}{\omega_{c2}} + 1 \right) \cdots \left( \frac{(j\omega)^2}{\omega_n^2} + \frac{2\zeta_\omega}{\omega_n} j\omega + 1 \right) \cdots}$$

Putting $G(j\omega)$ into Bode form requires putting each of the first- and second-order factors into Bode form
First-Order Factors in Bode Form

- **First-order frequency-response factors** include:

  \[ G(j\omega) = (j\omega)^n, \ G(j\omega) = j\omega + \sigma, \ G(j\omega) = \frac{1}{j\omega + \sigma} \]

  - For the first factor, \( G(j\omega) = (j\omega)^n \), \( n \) is a positive or negative integer
    - Already in Bode form

  - For the second two, divide through by \( \sigma \), giving

    \[ G(j\omega) = \sigma \left( \frac{j\omega}{\sigma} + 1 \right) \quad \text{and} \quad G(j\omega) = \frac{1}{\sigma \left( \frac{j\omega}{\sigma} + 1 \right)} \]

  - Here, \( \sigma = \omega_c \), the **corner frequency** associated with that zero or pole, so

    \[ G(j\omega) = \omega_c \left( \frac{j\omega}{\omega_c} + 1 \right) \quad \text{and} \quad G(j\omega) = \frac{1}{\omega_c \left( \frac{j\omega}{\omega_c} + 1 \right)} \]
Second-Order Factors in Bode Form

- **Second-order frequency-response factors** include:

  \[ G(j\omega) = (j\omega)^2 + 2\zeta \omega_n (j\omega) + \omega_n^2 \quad \text{and} \quad G(j\omega) = \frac{1}{(j\omega)^2 + 2\zeta \omega_n (j\omega) + \omega_n^2} \]

- Again, normalize the \((j\omega)^0\) coefficient, giving

  \[ G(j\omega) = \omega_n^2 \left[ \left( \frac{j\omega}{\omega_n} \right)^2 + \frac{2\zeta}{\omega_n} (j\omega) + 1 \right] \quad \text{and} \quad G(j\omega) = \frac{1/\omega_n^2}{\left( \frac{j\omega}{\omega_n} \right)^2 + \frac{2\zeta}{\omega_n} (j\omega) + 1} \]

- Putting each factor into its Bode form involves factoring out any DC gain component

- Lump all of **DC gains** together into a single gain constant, \(K_0\)

  \[ G(j\omega) = K_0 \frac{(\frac{j\omega}{\omega_{ca}} + 1)(\frac{j\omega}{\omega_{cb}} + 1)\cdots\left( \frac{(j\omega)^2}{\omega_{na}} + \frac{2\zeta}{\omega_{na}} j\omega + 1 \right)\cdots}{(\frac{j\omega}{\omega_{c1}} + 1)(\frac{j\omega}{\omega_{c2}} + 1)\cdots\left( \frac{(j\omega)^2}{\omega_{n1}} + \frac{2\zeta_1}{\omega_{n1}} j\omega + 1 \right)\cdots} \]
Bode Plot Construction

- Frequency response function in Bode form

\[
G(j\omega) = K_0 \frac{\left(\frac{j\omega}{\omega_{ca}+1}\right)\left(\frac{j\omega}{\omega_{cb}+1}\right)\cdots\left(\frac{j\omega}{\omega_{n_a}+1}\right)^2 + \frac{2\zeta_a}{\omega_{n_a}}j\omega+1\right)\cdots}{\left(\frac{j\omega}{\omega_{c_1}+1}\right)\left(\frac{j\omega}{\omega_{c_2}+1}\right)\cdots\left(\frac{j\omega}{\omega_{n_1}+1}\right)^2 + \frac{2\zeta_1}{\omega_{n_1}}j\omega+1\right)\cdots}
\]

- Product of a constant DC gain factor, \(K_0\), and first- and second-order factors

- Plot the frequency response of each factor individually, then combine graphically

  - Overall response is the product of individual factors
    - Product of gain responses – sum on a dB scale
    - Sum of phase responses
Bode Plot Construction

- **Bode plot construction procedure:**
  1. Put the sinusoidal transfer function into *Bode form*
  2. Draw a *straight-line asymptotic approximation* for the gain and phase response of each individual factor
  3. *Graphically add* all individual response components and sketch the result

- Next, we’ll look at the straight-line asymptotic approximations for the Bode plots for each of the transfer function factors
Bode Plot – Constant Gain Factor

\[ G(j\omega) = K_0 \]

- Constant gain

\[ |G(j\omega)| = K_0 \]

- Constant Phase

\[ \angle G(j\omega) = 0^\circ \]
Bode Plot – Poles/Zeros at the Origin

\[ G(j\omega) = (j\omega)^n \]

- \( n > 0: \)
  - \( n \) zeros at the origin

- \( n < 0: \)
  - \( n \) poles at the origin

- **Gain:**
  - Straight line
  - Slope = \( n \cdot 20 \frac{dB}{dec} = n \cdot 6 \frac{dB}{oct} \)
  - \( 0dB \) at \( \omega = 1 \)

- **Phase:**
  \[ \angle G(j\omega) = n \cdot 90^\circ \]
Bode Plot — First-Order Zero

- **Single real zero at** $s = -\omega_c$

- **Gain:**
  - $0 \text{dB for } \omega < \omega_c$
  - $+20 \frac{dB}{\text{dec}} = +6 \frac{dB}{\text{oct}} \text{ for } \omega > \omega_c$
  - Straight-line asymptotes intersect at $(\omega_c, 0 \text{dB})$

- **Phase:**
  - $0^\circ \text{ for } \omega \leq 0.1\omega_c$
  - $45^\circ \text{ for } \omega = \omega_c$
  - $90^\circ \text{ for } \omega \geq 10\omega_c$
  - $\frac{+45^\circ}{\text{dec}} \text{ for } 0.1\omega_c \leq \omega \leq 10\omega_c$
Bode Plot – First-Order Pole

- Single real pole at \( s = -\omega_c \)

- **Gain:**
  - \( 0 \text{dB} \) for \( \omega < \omega_c \)
  - \(-20 \frac{\text{dB}}{\text{dec}} = -6 \frac{\text{dB}}{\text{oct}} \) for \( \omega > \omega_c \)
  - Straight-line asymptotes intersect at \( (\omega_c, 0 \text{dB}) \)

- **Phase:**
  - \( 0^\circ \) for \( \omega \leq 0.1\omega_c \)
  - \(-45^\circ \) for \( \omega = \omega_c \)
  - \(-90^\circ \) for \( \omega \geq 10\omega_c \)
  - \(-45^\circ \) for \( 0.1\omega_c \leq \omega \leq 10\omega_c \)

\[
G(j\omega) = \frac{1}{\frac{j\omega}{\omega_c} + 1}
\]
Bode Plot – Second-Order Zero

- Complex-conjugate zeros:
  \[ s_{1,2} = -\sigma \pm j\omega_d \]

- **Gain:**
  - 0 dB for \( \omega \ll \omega_n \)
  - +40 \( \frac{dB}{dec} = +12 \frac{dB}{oct} \) for \( \omega \gg \omega_n \)
  - Straight-line asymptotes intersect at \((\omega_n, 0 dB)\)
  - \( \zeta \)-dependent peaking around \( \omega_n \)

- **Phase:**
  - 0° for \( \omega \ll \omega_n \)
  - 90° for \( \omega = \omega_n \)
  - 180° for \( \omega \gg \omega_n \)
  - \( \zeta \)-dependent slope through \( \omega_n \)
  - Sketch as step-change at \( \omega_n \) for low \( \zeta \), 
  + 90°/dec for high \( \zeta \), or in between

\[
G(j\omega) = \left(\frac{j\omega}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1
\]
Bode Plot – Second-Order Pole

- Complex-conjugate poles:
  \[ s_{1,2} = -\sigma \pm j\omega_d \]

- **Gain:**
  - 0 dB for \( \omega \ll \omega_n \)
  - \(-40 \, \text{dB/dec} = -12 \, \text{dB/oct} \) for \( \omega \gg \omega_n \)
  - Straight-line asymptotes intersect at \((\omega_n, 0\, \text{dB})\)
  - \(\zeta\)-dependent peaking around \(\omega_n\)

- **Phase:**
  - 0° for \( \omega \ll \omega_n \)
  - \(-90° \) for \( \omega = \omega_n \)
  - \(-180° \) for \( \omega \gg \omega_n \)
  - \(\zeta\)-dependent slope through \(\omega_n\)
  - Sketch as step-change at \(\omega_n\) for low \(\zeta\), \(-90°/\text{dec}\) for high \(\zeta\), or in between

\[ G(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1} \]
Consider a system with the following transfer function

\[ G(s) = \frac{10(s + 20)}{s(s + 400)} \]

The sinusoidal transfer function:

\[ G(j\omega) = \frac{10(j\omega + 20)}{j\omega(j\omega + 400)} \]

Put it into Bode form

\[ G(j\omega) = \frac{10 \cdot 20 \left(\frac{j\omega}{400} + 1\right)}{j\omega \cdot 400 \left(\frac{j\omega}{400} + 1\right)} = \frac{0.5 \left(\frac{j\omega}{20} + 1\right)}{j\omega \cdot \left(\frac{j\omega}{400} + 1\right)} \]

Represent as a product of factors

\[ G(j\omega) = 0.5 \cdot \left(\frac{j\omega}{20} + 1\right) \cdot \frac{1}{j\omega} \cdot \frac{1}{\left(\frac{j\omega}{400} + 1\right)} \]
Bode Plot Construction – Example
Bode Plot Construction – Example

Bode Phase Plot – Asymptotic Approximation

- $G_1(j\omega) = 0.5$
- $G_2(j\omega) = j\omega/20 + 1$
- $G_3(j\omega) = 1/j\omega$
- $G_4(j\omega) = 1/(j\omega/400 + 1)$
- $G(j\omega)$ – Approx.
- $G(j\omega)$ – Actual

K. Webb

MAE 4421
Polar Frequency Response Plots
Polar Frequency Response Plots

- $G(j\omega)$ is a complex function of frequency
  - Typically plot as Bode plots
    - Magnitude and phase plotted separately
    - Aids visualization of system behavior

- A real and an imaginary part at each value of $\omega$
  - A point in the complex plane at each frequency
  - Defines a curve in the complex plane
  - A polar plot
    - Parametrized by frequency – not as easy to distinguish frequency as on a Bode plot

- Polar plots are not terribly useful as a means of displaying a frequency response
  - However, an important concept later, when we introduce the Nyquist stability criterion
Polar Frequency Response Plots

- Identical frequency responses plotted two ways:
  - Bode plot and polar plot
- Note uneven frequency spacing along polar plot curve
  - Dependent on frequency rates of change of gain and phase
Relationship between Frequency Response and Transient Response
We have seen relationships – some exact, some approximate – between closed-loop pole locations and closed-loop transient response.

Also have relationships between *closed-loop frequency response* and *closed-loop transient responses*.

Applicable to *second-order systems*:

\[
T(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

Also applicable to higher-order systems that are reasonably *approximated as second-order*.

Systems with a pair of dominant second-order poles.
Transient/Frequency Response Relationship

- Qualitative 2nd-order time/freq. response/pole relationships
  - Damping ratio vs. overshoot vs. peaking
  - Natural frequency vs. risetime vs. bandwidth

- Gain Response: System 1 (blue) vs. System 2 (red)
- Step Response: System 1 (solid blue) vs. System 2 (dashed red)

\[ \zeta_1 = 0.9 \]
\[ \omega_n = 3.1 \text{ rad/sec} \]
\[ \zeta_2 = 0.3 \]
\[ \omega_n = 31.4 \text{ rad/sec} \]
For systems with $\zeta < 0.707$, the gain response will exhibit **peaking**

Can relate **peak magnitude** to the damping ratio

$$M_p = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$

Relative to low-frequency gain

And the **peak frequency** to the damping ratio and natural frequency

$$\omega_p = \omega_n \sqrt{1 - 2\zeta^2}$$
Can also relate a system’s **bandwidth** (i.e., -3dB frequency, $\omega_{BW}$) to the speed of its step response.

Bandwidth as a function of $\omega_n$ and $\zeta$:

$$\omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Bandwidth as a function of **1% settling time** and $\zeta$:

$$\omega_{BW} = \frac{4.6}{t_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Bandwidth as a function of **peak time** and $\zeta$:

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$
Steady-State Error from Bode Plots
Static Error Constants

- For unity-feedback systems, open-loop transfer function gives static error constants.
  - Use static error constants to calculate steady-state error:
    \[
    \begin{align*}
    K_p &= \lim_{s \to 0} G(s) \\
    K_v &= \lim_{s \to 0} sG(s) \\
    K_a &= \lim_{s \to 0} s^2 G(s)
    \end{align*}
    \]

- We can also determine static error constants from a system’s open-loop Bode plot.
Static Error Constant – Type 0

- For a type 0 system
  \[K_p = \lim_{s \to 0} G(s)\]

- At low frequency, i.e., below any open-loop poles or zeros
  \[G(s) \approx K_p\]

- Read \(K_p\) directly from the open-loop Bode plot
  - Low-frequency gain

\[G(s) = \frac{100(s + 30)}{(s + 3)(s + 200)}\]
Static Error Constant – Type 1

- For a type 1 system
  \[ K_v = \lim_{s \to 0} sG(s) \]

- At low frequencies, i.e. below any other open-loop poles or zeros
  \[ G(s) \approx \frac{K_v}{s} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_v}{\omega} \]

- A straight line with a slope of \(-20 \text{ dB/dec}\)
- Evaluating this low-frequency asymptote at \(\omega = 1\) yields the velocity constant, \(K_v\)
- On the Bode plot, extend the low-frequency asymptote to \(\omega = 1\)
  - Gain of this line at \(\omega = 1\) is \(K_v\)
Static Error Constant – Type 1

Velocity Constant of a Type 1 System

\[ G(s) = \frac{85(s + 0.1)(s + 50)}{s(s^2 + 10s + 125)} \]

\[ K_v = 10.6 \text{ dB} \rightarrow 3.4 \]
Static Error Constant – Type 2

- For a type 2 system
  \[ K_a = \lim_{s \to 0} s^2 G(s) \]

- At low frequencies, i.e. below any other open-loop poles or zeros
  \[ G(s) \approx \frac{K_a}{s^2} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_a}{\omega^2} \]

- A straight line with a slope of \(-40 \text{ dB/dec}\)

- Evaluating this low-frequency asymptote at \(\omega = 1\) yields the acceleration constant, \(K_a\)

- On the Bode plot, extend the low-frequency asymptote to \(\omega = 1\)
  - Gain of this line at \(\omega = 1\) is \(K_a\)
Static Error Constant – Type 2

$$G(s) = \frac{1600(s + 0.1)(s + 5)}{s^2(s + 100)}$$

**Graph:**
- **Title:** Acceleration Constant of a Type 2 System
- **Axes:**
  - **Y-axis:** Gain [dB]
  - **X-axis:** Frequency [rad/sec]
- **Graph Details:**
  - A blue curve representing the system's behavior.
  - A black dotted line indicating a reference or comparison point.
  - A red dot marking a specific frequency and gain value.

**Equation at Graph Point:**
- **Gain Value:** $K_a = 18.1 \text{ dB} \rightarrow 8.0$
Stability

- Consider the following system

\[ R(s) \rightarrow \sum \rightarrow K \rightarrow \frac{1}{s(s+1)(s+2)} \rightarrow Y(s) \]

- We already have a couple of tools for assessing stability as a function of loop gain, \( K \):
  - Routh Hurwitz
  - Root locus

- Root locus:
  - Stable for some values of \( K \)
  - Unstable for others
Stability

- In this case gain is stable below some value.
- Other systems may be stable for gain above some value.
- Marginal stability point:
  - Closed-loop poles on the imaginary axis at $\pm j\omega_1$
  - For gain $K = K_1$
Open-Loop Frequency Response & Stability

Marginal stability point occurs when closed-loop poles are on the imaginary axis

Angle criterion satisfied at $\pm j\omega_1$

$$|KG(j\omega_1)| = 1 \quad \text{and} \quad \angle KG(j\omega_1) = -180^\circ$$

Note that $-180^\circ = 180^\circ$

$KG(j\omega)$ is the open-loop frequency response

Marginal stability occurs when:

Open-loop gain is: $KG(j\omega) = 0 \text{ dB}$

Open-loop phase is: $\angle KG(j\omega) = -180^\circ$
Stability from Bode Plots

- Varying $K$ simply shifts gain response up or down
- Here, stable for smaller gain values
  - $|KG(j\omega)| < 0 \text{ dB}$ when $\angle KG(j\omega) = 180^\circ$
- Often, stable for larger gain values
  - $|KG(j\omega)| > 0 \text{ dB}$ when $\angle KG(j\omega) = 180^\circ$
- Root locus provides this information
  - Bode plot does not
A method does exist for determining stability from the open-loop frequency response:

- **Nyquist stability criterion**
  - Graphical technique
  - Uses open-loop frequency response
  - Determine system stability
  - Determine gain ranges for stability

Before introducing the Nyquist criterion, we must first introduce the concept of *complex functional mapping*.
Consider a complex function

\[ F(s) = \frac{(s - z_1)(s - z_2)\cdots}{(s - p_1)(s - p_2)\cdots} \]

Takes one complex value, \( s \), and yields a second complex value, \( F(s) \)

In other words, it maps \( s \) to \( F(s) \)
Mapping of Contours

- $F(s)$ provides a mapping of individual points in the s-plane to corresponding points in the F-plane.
- Can also map all points around a contour in the s-plane to another contour in the F-plane.
Mapping of Contours

- Recall how we approached the application of the angle criterion
  - Vector approach to the evaluation of a transfer function at a particular point in the s-plane
    \[|G(s_1)| = \frac{\prod |\text{vectors from zeros to } s_1|}{\prod |\text{vectors from poles to } s_1|}\]
    \[\angle G(s_1) = \Sigma \angle(\text{ from zeros to } s_1) - \Sigma \angle(\text{ from poles to } s_1)\]
- Can take the same approach to evaluating complex functions around \textit{contours} in the s-plane
Mapping Contours – Example 1

- Map contour $A$ by $F(s) = (s - z_1)$ in a $\textit{clockwise}$ direction
  - Contour $A$ does not enclose the zero
- Here, $R = V$, so $|R| = |V|$ and $\angle R = \angle V$

- As $F(s)$ is evaluated around $A$, $\angle V$ never exceeds $0^\circ$ or $180^\circ$
- $R$ does the same:
  - Does not rotate through a full $360^\circ$
  - $\textit{Contour B does not encircle the origin}$
Mapping Contours – Example 2

- Map contour $A$ by $F(s)$ in a \textit{clockwise} direction
  - Contour $A$ does not enclose the pole
- Here, $R = 1/V$, so $|R| = 1/|V|$ and $\angle R = -\angle V$

\[ F(s) = \frac{1}{(s - p_1)} \]

- $\angle V$ oscillates over some range well within $0^\circ$ and $180^\circ$
  - $R$ rotates through the \textit{negative} of the same range
  - \textit{Contour B does not encircle the origin}
Now, **contour A encloses a single zero**

\[ R = V, \text{ so } |R| = |V| \text{ and } \angle R = \angle V \]

\[ F(s) = (s - z_1) \]

- \( V \) rotates through a full 360° in a clockwise direction
- \( R \) does the same:
  - **Contour B encircles the origin in a clockwise direction**
Now, contour A encloses a single pole

\[ R = 1/V, \text{ so } |R| = 1/|V| \text{ and } \angle R = -\angle V \]

\[ F(s) = \frac{1}{(s - p_1)} \]

\[ V \text{ rotates through a full } 360^\circ \text{ in a clockwise direction} \]

\[ R \text{ rotates in the opposite direction} \]

\[ \text{Contour B encircles the origin in a CCW direction} \]
Mapping Contours – Example 5

- Now, **contour A encloses two poles**
- \( R = \frac{1}{V_1V_2} \), so \(|R| = \frac{1}{|V_1||V_2|}\) and \( \angle R = -(\angle V_1 + \angle V_2) \)

\[
F(s) = \frac{1}{(s - p_1)(s - p_2)}
\]

- \( V_1 \) and \( V_2 \) each rotate through a full \( 360^\circ \) in a clockwise direction
  - \( R \) rotates in the **opposite direction**
  - **Contour B encircles the origin twice in a CCW direction**
Now, **contour A encloses one pole and one zero**

\[ R = \frac{V_1}{V_2}, \quad \text{so} \quad |R| = \frac{|V_1|}{|V_2|} \quad \text{and} \quad \angle R = \angle V_1 - \angle V_2 \]

\[ F(s) = \frac{(s - z_1)}{(s - p_1)} \]

\[ \angle V_1 \text{ and } \angle V_2 \text{ rotate through } 360^\circ \text{ in a CW direction} \]

- Their contributions rotate in *opposite* directions
- \( \angle R \) does not rotate through a full 360°
- **Contour B does not encircle the origin**
Some observations regarding complex mapping of contour $A$ in a CW direction to contour $B$:

- If $A$ does not enclose any poles or zeros, $B$ does not encircle the origin
- If $A$ encloses a single pole, $B$ will encircle the origin once in a CCW direction
- If $A$ encloses two poles, $B$ will make two CCW encirclements of the origin
- If $A$ encloses a pole and a zero, $B$ will not encircle the origin

Next, we’ll use these observations to help derive the \textit{Nyquist stability criterion}
Nyquist Stability Criterion

- Our goal is to assess closed-loop stability
  - Determine if there are any closed-loop poles in the RHP
- Consider a generic feedback system:

![Feedback System Diagram]

- Closed-loop transfer function

\[ T(s) = \frac{G(s)}{1 + G(s)H(s)} \]

- Closed-loop poles are roots (zeros) of the closed-loop characteristic polynomial:

\[ 1 + G(s)H(s) \]
Nyquist Stability Criterion

- Can represent the individual transfer functions as
  \[ G(s) = \frac{N_G}{D_G} \quad \text{and} \quad H(s) = \frac{N_H}{D_H} \]

- The closed-loop polynomial becomes
  \[ 1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H} \]

- From this, we can see that:
  - The \textit{poles} of \(1 + G(s)H(s)\) are the poles of \(G(s)H(s)\), the \textit{open-loop poles}
  - The \textit{zeros} of \(1 + G(s)H(s)\) are the poles of \(T(s)\), the \textit{closed-loop poles}
Nyquist Stability Criterion

- To determine stability, look for RHP closed-loop poles
- Evaluate $1 + G(s)H(s)$ CW around a contour that encircles the entire right half-plane
  - Evaluate $1 + G(s)H(s)$ along entire $j\omega$-axis
  - Encircle the entire RHP with an infinite-radius arc
- If $1 + G(s)H(s)$ has one RHP pole, resulting contour will encircle the origin once CCW
- If $1 + G(s)H(s)$ has one RHP zero, resulting contour will encircle the origin once CW
Nyquist Stability Criterion

- Total number of CW encirclements of the origin, $N$, by the resulting contour will be

  \[ N = Z - P \]

  - $P = \# \text{ of RHP poles of } 1 + G(s)H(s)$
  - $Z = \# \text{ of RHP zeros of } 1 + G(s)H(s)$

- Want to detect RHP \textbf{poles} of $T(s)$, \textbf{zeros} of $1 + G(s)H(s)$, so

  \[ Z = N + P \]

  - $Z = \# \text{ of closed-loop RHP poles}$
  - $P = \# \text{ of open-loop RHP poles}$
  - $N = \# \text{ of CW encirclements of the origin}$
Nyquist Stability Criterion

- Basis for detecting closed-loop RHP poles
  - Map contour encircling the entire RHP through closed-loop characteristic polynomial
  - Count number of CW encirclements of the origin by resulting contour
  - Calculate the number of closed-loop RHP poles:
    \[ Z = N + P \]

- Need to know:
  - Closed-loop characteristic polynomial
  - Number of RHP poles of closed-loop characteristic polynomial
Nyquist Stability Criterion

- Instead, map through $G(s)H(s)$
  - Open-loop transfer function
  - Easy to use for mapping – we know poles and zeros
  - Resulting contour shifts left by $1$ – that’s all

- Now, count encirclements of the point $s = -1$
Nyquist Stability Criterion

- **Nyquist stability criterion**

  - If a contour that encloses the entire RHP is mapped through the open-loop transfer function, $G(s)H(s)$, then the number of closed-loop RHP poles, $Z$, is given by

    \[ Z = N + P \]

  where

  - $N = \# \text{ of CW encirclements of } -1$
  - $P = \# \text{ of open-loop RHP poles}$
Nyquist Stability Criterion

- Want to detect *net clockwise encirclements*
  \[ N = \# \text{CW encirclements} - \# \text{CCW encirclements} \]

- Draw a line from \( s = -1 \) in any direction
- Count number of times contour crosses the line in each direction

![Graph showing Nyquist plot with contour and line]
Nyquist Diagram

- The contour that results from mapping the perimeter of the entire RHP is a **Nyquist diagram**
- Consider four segments of the contour:
  1. Along positive $j\omega$-axis, we’re evaluating $G(j\omega)H(j\omega)$
     - Open-loop frequency response
  2. Here, $s \to C^{\infty}$
     - Maps to zero for any physical system
  3. Here, evaluating $G(-j\omega)H(-j\omega)$
     - Complex conjugate of segment ①
     - Mirror ① about the real axis
  4. The origin
     - Sometimes a special case – more later
Nyquist Criterion – Example 1

- Apply the Nyquist criterion to determine stability for the following system

- First evaluate along segment ①, +jω-axis
  - This is the frequency response
  - Read values off of the Bode plot
Nyquist Criterion – Example 1

- Segment ① is a polar plot of the frequency response
- All of segment ②, arc at $C^\infty$, maps to the origin
Segment ③ is the complex conjugate of segment ①
Mirror about the real axis
Nyquist Criterion – Example 1

- Count CW encirclements of $s = -1$
  - Draw a line from $s = -1$ in any direction
- Here, $N = 2$
- Closed-loop RHP poles given by:
  $$Z = N + P$$
- No open-loop RHP poles, so $P = 0$
  $$Z = 2 + 0 = 2$$
- Two RHP poles, so system is $\text{unstable}$
Nyquist Criterion – Example 2

- This system is open-loop stable
  - Stable for low enough $K$
  - Nyquist plot will not encircle $s = -1$

- Three poles and no zeros
  - Unstable for $K$ above some value
  - Nyquist plot will encircle $s = -1$
For $K = 30$, $N = 0$, and the system is stable.

Modifying $K$ simply scales the magnitude of the Nyquist plot.
Nyquist Criterion – Example 2

- Here, the Nyquist plot crosses the negative real axis at $s = -0.5$
- As gain increases real-axis crossing moves to the left
- Increasing $K$ by 2x or more results in two encirclements of $s = -1$
  - Unstable for $K > 60$
  - More later ...
We evaluate the open-loop transfer function along a contour including the $j\omega$-axis.

$G(j\omega)$ is **undefined** at the pole.
- Must **detour around the pole**

Consider the common case of a pole at the origin.

This is the special case for segment ④.

$$G(s) = \frac{1}{s(s + 2)}$$
Nyquist Diagram – Poles at the Origin

- Segment ④ contour: \( s = \rho e^{j\theta} \) for \( 0^\circ \leq \theta \leq 90^\circ \)
- Evaluate \( G(s) \) around segment ④ as \( \rho \to 0 \)

\[
G(s) \bigg|_{s=\rho e^{j\theta}} = \frac{1}{\rho e^{j\theta}(\rho e^{j\theta} + 2)}
\]

- **Magnitude:**

\[
|G(\rho e^{j\theta})| = \frac{1}{\rho|\rho e^{j\theta} + 2|} = \frac{1}{2\rho}
\]

- As \( \rho \to 0 \)

\[
\lim_{\rho \to 0} |G(\rho e^{j\theta})| = \infty
\]

- Maps to an arc at \( C^\infty \)
Nyquist Diagram – Poles at the Origin

- Segment ④ traversed in a CCW direction
  - $\theta$ varies from $0^\circ$ ... $+90^\circ$
- **Phase** of the resulting contour:
  \[ \angle G(\rho e^{i\theta}) = -\theta^+ \]
  - Negative because it is angle from a pole
  - Extra phase from additional pole
- $G(s)$ maps segment ④ to:
  - An arc at $C^\infty$
  - Rotating CW from $0^\circ$ to $-90^\circ^+$
Nyquist Criterion – Example 3

- Apply the Nyquist criterion to determine stability for the following system

\[
R(s) \sum \frac{1}{s(s+2)} Y(s)
\]

- Use Bode plot to map segment ①
  - Infinite DC gain
  - Starts at \(-90^\circ\) at \(C^\infty\) for \(\omega = 0\)
Nyquist Criterion – Example 3

- Segment ① starts at $C^\infty$ at $-90^\circ$
- Heads to the origin at $-180^\circ$
- All of segment ②, arc at $C^\infty$, maps to the origin
Nyquist Criterion – Example 3

- Segment ③ is the complex conjugate of segment ①
- Mirror about the real axis
Nyquist Criterion – Example 3

- Segment 4 maps to a CW arc at $C^\infty$
  - CW, so it does not encircle $-1$
  - Can’t draw to scale
- Here, $N = 0$
- No open-loop RHP poles, so $P = 0$
  - $Z = 0$
- No RHP poles, so system is **stable**
Stability Margins
Stability Margins

- Recall a previous example

- According to the Nyquist plot, the system is stable
  - How stable?

- Two stability metrics
  - Both are measures of how close the Nyquist plot is to encircling the point $s = -1$
  - **Gain margin** and **phase margin**
Crossover Frequencies

- Two important frequencies when assessing stability:
  - **Gain crossover frequency**
    - The frequency at which the open-loop gain crosses 0 dB
  - **Phase crossover frequency**
    - The frequency at which the open-loop phase crosses $-180^\circ$
Gain Margin

- An open-loop-stable system will be closed-loop stable as long as its gain is less than unity at the phase crossover frequency.

- **Gain margin, GM**
  - The change in open-loop gain at the phase crossover frequency required to make the closed-loop system unstable.

![Nyquist Diagram]
Phase Margin

- An open-loop-stable system will be closed-loop stable as long as its phase has not fallen below $-180^\circ$ at the gain crossover frequency

- **Phase margin, PM**
  - The change in open-loop phase at the gain crossover frequency required to make the closed-loop system unstable
Gain and Phase Margins from Bode Plots

GM and PM from Bode Plots

Gain [dB]

Phase [deg]

Frequency [rad/sec]

GM

PM

ΩPM

ΩGM
Phase Margin and Damping Ratio, $\zeta$

- PM can be expressed as a function of damping ratio, $\zeta$, as
  $$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right)$$

- For $PM \leq 65^\circ$ or so, we can approximate:
  $$PM \approx 100\zeta \quad \text{or} \quad \zeta \approx \frac{PM}{100}$$
Frequency Response Analysis in MATLAB
bode.m

```matlab
[mag, phase] = bode(sys, w)
```

- **sys**: system model – state-space, transfer function, or other
- **w**: *optional* frequency vector – in rad/sec
- **mag**: system gain response vector
- **phase**: system phase response vector – in degrees

- If no outputs are specified, bode response is automatically plotted – preferable to plot yourself
- Frequency vector input is optional
  - If not specified, MATLAB will generate automatically
- May need to do: `squeeze(mag)` and `squeeze(phase)` to eliminate singleton dimensions of output matrices
nyquist.m

`nyquist(sys, w)`

- `sys`: system model – state-space, transfer function, or other
- `w`: optional frequency vector – in rad/sec

- MATLAB generates a Nyquist plot automatically
- Can also specify outputs, if desired:
  
  \[ [\text{Re}, \text{Im}] = \text{nyquist}(\text{sys}, \text{w}) \]

- Plot is not be generated in this case
margin.m

\[
[\text{GM, PM, wgm, wpm}] = \text{margin}(\text{sys})
\]

- **sys**: system model – state-space, transfer function, or other
- **GM**: gain margin
- **PM**: phase margin – in degrees
- **wgm**: frequency at which GM is measured, the phase crossover frequency – in rad/sec
- **wpm**: frequency at which PM is measured, the gain crossover frequency

- If no outputs are specified, a Bode plot with GM and PM indicated is automatically generated