Theorem 2. Let $t$ be a degenerate tensor that occurs in a linear tensor field $T(x, y, z) = T_0 + xT_1 + yT_1 + zT_2$, and $v$ be its dominant eigenvector. Then $v^T T v = 0$.

Proof. Since $t$ is degenerate, it has the form $t = k(vv^T - \frac{1}{3}I)$ for some $k \in \mathbb{R}$. For $t$ to occur in the field, we need $u = \Sigma^{-1}(t) = 0$. Therefore,

$$u = \langle T, t \rangle = 0 \quad \text{(22)}$$

This is equivalent to

$$\text{trace}[k(vv^T - \frac{1}{3}I)] = 0 \quad \text{(23)}$$

or

$$\text{trace}(vv^T) = \frac{1}{3} \text{trace}(T) \quad \text{(24)}$$

Since $T$ is traceless, the above equation becomes

$$\text{trace}(vv^T) = 0 \quad \text{(25)}$$

Applying the cyclic property of trace [7], we have

$$\text{trace}(v^T T v) = \text{trace}(vv^T) = 0 \quad \text{(26)}$$

Since $v^T T v$ is a $1 \times 1$ matrix, we have

$$v^T T v = \text{trace}(v^T T v) = 0 \quad \text{(27)}$$

Theorem 3. Given a linear tensor field $T(x, y, z) = T_0 + xT_1 + yT_1 + zT_2$, and a unit vector $v$ that satisfies $v^T T v = 0$, there exist $x_0, y_0, z_0 \in \mathbb{R}$ such that $T(x_0, y_0, z_0)$ is a degenerate tensor and $v$ is a dominant eigenvector of $T(x_0, y_0, z_0)$. The dominant eigenvalue is given by $k = \frac{1}{v^T T_0 v}$.

Proof. Note that $v$ is the dominant eigenvector of the tensor $t = k(vv^T - \frac{1}{3}I)$, for all $k \neq 0$. We must choose $k$ so that $t$ occurs in the field. Since $v^T T v = 0$, we have $0 = kv^T T v = \text{trace}(kv^T T v) = \text{trace}(kvv^T T) = \text{trace}(k vv^T T) - \text{trace}(k \frac{1}{3}I T) = \text{trace}(k vv^T - \frac{1}{3}I T) = \langle T, t \rangle$. This implies that $\Sigma^{-1}(t)$ gives $u = 0$. Next, we must ensure that $w = 1$.

$$1 = w = \langle T_0, t \rangle \quad \text{(29)}$$

$$= \text{trace}[k(vv^T - \frac{1}{3}I) T_0] \quad \text{(30)}$$

$$= k \text{trace}(vv^T T_0) - \frac{k}{3} \text{trace}(T_0) \quad \text{(31)}$$

$$= k \text{trace}(v^T T_0) \quad \text{(32)}$$

$$= k v^T T_0^T v \quad \text{(33)}$$

Therefore,

$$k = \frac{1}{v^T T_0 v} \quad \text{(34)}$$

With $k$ set to this value, $t$ occurs in the field. This implies that $k$ is unique.

Theorem 4. Let $v_1, v_2, v_3$ be respectively the major, medium, and minor eigenvectors of a neutral tensor $t = wT_0 + xT_1 + yT_1 + zT_2$. Then $v_1^T T v_1 = v_2^T T v_2 = v_3^T T v_3$.

Proof. Recall that $T$ has a zero dot product with $t = wT_0 + xT_1 + yT_1 + zT_2$, i.e., $\text{trace}(iT) = 0$. Since $t$ is neutral, it has the form $t = k(v_1 v_1^T - v_3 v_3^T)$ for some $k$. Consequently,

$$\text{trace}[(v_1 v_1^T - v_3 v_3^T)T] = 0 \quad \text{(35)}$$

This is equivalent to

$$\text{trace}(v_1^T T v_1) = \text{trace}(v_3 v_3^T T) \quad \text{(36)}$$

Reusing the cyclic property of trace, we obtain

$$\text{trace}(v_1^T T v_1) = \text{trace}(v_3 v_3^T T) \quad \text{(37)}$$

Again, both $v_1^T T v_1$ and $v_3^T T v_3$ are $1 \times 1$ matrices, we have $v_1^T T v_1 = v_3^T T v_3$.

Theorem 5. Given a medium eigenvector $v_2$ which resides on the $k$-th level set of $v^T T v$, the corresponding major and minor eigenvectors must reside on the $\frac{1}{2}$-th level set of $v^T T v$.

Proof. Notice that $v_1^T T v_1 + v_2^T T v_2 + v_3^T T v_3 = \text{trace}(v^T T v)$ where $V = (v_1, v_2, v_3)$. We have $\text{trace}(v^T T v) = \text{trace}(T) = 0$ since $T$ is traceless. Consequently,

$$v_1^T T v_1 + v_2^T T v_2 + v_3^T T v_3 = 0 \quad \text{(38)}$$

Since $v_1^T T v_1 = v_3^T T v_3$ (Theorem 4), we have that $2v_2^T T v_2 = 0$, i.e., $v_2^T T v_1 = v_2^T T v_3 = \frac{1}{2} v_2^T T v_2$.

Theorem 6. Under the structurally stable condition that $T$ is non-degenerate, a level set of $v^T T v$ on the unit sphere must be two non-intersecting non-circular spherical ellipses, except one situation where it is the union of two great circles, residing in two intersecting planes.

Proof. Since the unit sphere remains the same under any orthonormal change of basis, we can find such a basis under which $T$ is diagonal, i.e.,

$$\begin{pmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & -a-b
\end{pmatrix}$$

where $a \geq b \geq 0$ and $a > 0$.

Under this basis, it is straightforward to verify that a level set of $v^T T v$ on the unit sphere is the intersection of

$$a\alpha^2 + b\beta^2 - (a+b)\gamma^2 = k \quad \text{(39)}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{(40)}$$

which is equivalent to

$$(2a+b)\alpha^2 + (a+2b)\beta^2 = k + (a+b) \quad \text{(41)}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{(42)}$$

which is the union of two ellipses (thus planar) with inverse symmetry. Additionally, we get the equations

$$(a-b)\alpha^2 - (a+2b)\beta^2 = k - b \quad \text{(43)}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{(44)}$$

When $k = b$ the level set satisfies either

$$\alpha = \sqrt{\frac{a+2b}{a-b}} \gamma \quad \text{(45)}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{(46)}$$
where $M$ is the matrix $[T_0 \nu \ T_0 \nu \ T_0 \nu]$. 

**Proof.** A traceless tensor $T(x, y, z)$ is neutral and has medium eigenvector $v$ if and only if

$$T(x, y, z)v = 0.$$  

(50)

Substituting the field, we get

$$(T_0 + xT_x + yT_y + zT_z)\nu = 0$$

(51)

$$xT_{x\nu} + yT_{y\nu} + zT_{z\nu} = -T_0\nu.$$  

(52)

which can be rewritten as Equation 49. 

**Theorem 8.** Given a unit vector $v$ where the projection of $T$ onto the plane orthogonal to $v$ is a degenerate two-dimensional tensor, the set of points on the neutral surface that have $v$ as their medium eigenvector is a line.

**Proof.** Usually Equation 49 gives a unique point $(x, y, z)$ for each $v$. However, if $M$ is singular then this fails. If $M$ is singular and $T_0\nu$ is not in its image then this gives an infinity point of the neutral surface. $T_0\nu$ is in its image so there is a line of possible $v$. 

**B SINGULARITY LINE FORMULA**

In this section we provide the detail of finding the 3D coordinates of a neutral point corresponding to the singularities in the medium eigenvector manifold (Section 5.2).

Let $O$ be a singularity of a linear tensor field $T_0 + xT_x + yT_y + zT_z$, whose corresponding medium eigenvector is $s$. Since $O$ corresponds to a topological circle (a straight line in $\mathbb{R}^3$), we need an additional parameter $a$, a unit vector perpendicular to $s$, to identify individual points on the line.

From Proposition 7, we know that the 3D coordinates $(x, y, z)$ of a neutral point can be found from a given medium eigenvector $\nu$ by

$$[x \ y \ z] = -[T_0 \nu \ T_0 \nu \ T_0 \nu]^{-1} T_0\nu.$$  

(53)

Thus, the point on the infinite line that corresponds to the vector $a$ is the following limit:

$$-\lim_{\eta \to 0} [T_0(s + \eta u) \ T_0(s + \eta u) \ T_0(s + \eta u)]^{-1} T_0(s + \eta u)$$  

(54)

For convenience, we name the following variables:

$$M_s = [T_s \nu \ T_s \nu \ T_s \nu]$$  

(55)

$$M_u = [T_u \nu \ T_u \nu \ T_u \nu]$$  

(56)

$$v_s = T_0s$$  

(57)

$$v_u = T_0u$$  

(58)

Consequently, the limit in Equation 54 can be rewritten as

$$-\lim_{\eta \to 0} (M_s + \eta M_u)^{-1}(v_s + \eta v_u)$$  

(59)

This is equivalent to

$$-\lim_{\eta \to 0} \frac{adj(M_s + \eta M_u)(v_s + \eta v_u)}{|M_s + \eta M_u|}$$  

(60)

where $adj(M)$ and $|M|$ are the adjoint matrix and the determinant of the matrix $M$, respectively. It can be verified that

$$\lim_{\eta \to 0} adj(M_s + \eta M_u)(v_s + \eta v_u) = adj(M_s)v_s = 0$$  

(61)

and that

$$\lim_{\eta \to 0} |M_s + \eta M_u| = |M_s| = 0$$  

(62)

Consequently, to evaluate the limit in Equation 60 we apply L’Hospital’s rule, i.e.

$$\lim_{\eta \to 0} \frac{\frac{d}{d\eta}[adj(M_s + \eta M_u)(v_s + \eta v_u)]}{\frac{d}{d\eta} [M_s + \eta M_u]}$$  

(63)

From classical matrix calculus results, we have

$$\lim_{\eta \to 0} \frac{d}{d\eta} |M_s + \eta M_u| = \text{trace}(adj(M_s)M_u)$$  

(64)

and

$$\lim_{\eta \to 0} \frac{d}{d\eta}[adj(M_s + \eta M_u)(v_s + \eta v_u)]$$

$$= \Gamma v_s + adj(M_s)v_u$$  

(65)

where

$$\Gamma = \lim_{\eta \to 0} \frac{d}{d\eta}[adj(M_s + \eta M_u)]$$

$$= M_s^{-1}[\text{trace}(adj(M_s)M_u)]^T - M_u adj(M_s)|v_s + adj(M_s)v_u$$  

(66)

**C HIGHER-RESOLUTION IMAGES**

In Figure 12 we show higher-resolution images of Figure 11 (a-d).
Fig. 12: Higher-resolution images of Figure 11 (a-d).