\( 7.6 \)
\( \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \), \( \chi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \chi \)

(a) In this particular case we can establish by inspection that eigenvalue \( \lambda = 2 \) is uncontrollable. This can be verified by computing the rank of \([2I-A; B]\), i.e.,

\[
\text{rank} \begin{bmatrix} 3 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 2 \Rightarrow \lambda = 2 \text{ is uncontrollable.}
\]

Likewise, \( \lambda = 2 \) is also unobservable.

(b) \( H(t) = C e^{A_t} B = [C, 10] \begin{bmatrix} e^{A_t} \\ 0 \\ 1 e^{2t} \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \)

\[
= C e^{A_t} B_1,
\]

\[
= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & e^t \\ e^t \\ 0 \end{bmatrix}
\]

\[
H(s) = \mathcal{L}[H(t)] = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & 0 \end{bmatrix}
\]

(c) Since \( \lambda_3 = 2 \), the system is not internally stable. However, it is BIBO stable because \( H(s) \) has a pole at \( s = -1 \).

\( 8.3 \)

\[
H(s) = \frac{s+1}{s^2 + 2} = \frac{s+1}{(s+j\sqrt{2})(s-j\sqrt{2})}
\]

Since there are no pole-zero cancellations, we can directly obtain a minimal realization from \( H(s) \), e.g., in controllable
canonical form
\[
\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}
\]

An unobservable realization can be obtained by adding a pole and a zero that can be cancelled, i.e.,
\[
H(s) = \frac{(s+1)(s-1)}{(s^2+2)(s-1)} = \frac{s^2-1}{s^3-s^2+2s-2}.
\]

Again, in controllable canonical form,
\[
\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \mathbf{x}
\]

The dual of the last system is the triple \((A^T, C^T, B^T)\), i.e.
\[
\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x},
\]

is unobservable.

Finally, the augmented system
\[
\dot{\mathbf{x}} = \begin{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & A_1 \\\n0 & A_2
\end{bmatrix} \mathbf{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \mathbf{x}
\]

with \(A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}\), and \(A_2\) an arbitrary square matrix yields an uncontrollable and unobservable realization of \(H(s)\).

\[
\begin{array}{ccc}
\text{H}(s) & \xrightarrow{\text{S}} & \text{H}(s) \\
G(s) & \xrightarrow{\text{G}(s)} & \text{G}(s)
\end{array}
\]

\[
H(s) = \frac{s+1}{s(s+3)}, \quad G(s) = \frac{k}{s+a}, \quad a, k \in \mathbb{R}.
\]
(a) Implementation of \( G(s) = \frac{s+1}{s(s+3)} + \frac{s+1}{s^2+3s} \):

In controllable canonical form,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & -3 & 0 \\
k & k & -a
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} r,
\quad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}
\]

Implementation of \( H(s) = \frac{k}{s+a} \):

\[
\begin{align*}
\dot{x}_2 &= -a x_2 + k u_2, \quad y_2 = x_2, \quad x_2 = x_3 \\
\dot{x}_2 &= -3 x_2 + u_1 = -3 x_2 + r - x_3 \\
\dot{x}_3 &= -a x_3 + k[x_1 + x_2] = k x_1 + k x_2 - a x_3
\end{align*}
\]

(iii) The closed-loop transfer function is given by

\[
G(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s^2 + (1+a)s + a}{s^3 + (3a)s^2 + (3at+k)s + k} = \frac{b_3 + b_2 s + b_1 s^2}{s^3 + a_3 s^2 + a_2 s + a_1}
\]

If \(a\) and \(k\) are such that no pole-zero cancellations occur, then (in controllable canonical form)

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-k & -(3a+k) & -(3a)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} r,
\quad y = \begin{bmatrix} a & (1+a) & 1 \end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

- 3 -
(b) In the first implementation,

\[
Q = \begin{bmatrix}
0 & 1 & -3 \\
1 & -3 & 9 - k \\
0 & k & -k(a+2)
\end{bmatrix}, \quad \Phi = \begin{bmatrix}
1 & 1 & 0 \\
0 & -2 & -1 \\
-k & 6 - k & 2 + a
\end{bmatrix}
\]

if \( k = 0 \),

\[
Q = \begin{bmatrix}
0 & 1 & -3 \\
1 & -3 & 9 \\
0 & 0 & 0
\end{bmatrix}, \quad \Phi = \begin{bmatrix}
1 & 1 & 0 \\
0 & -2 & -1 \\
0 & 0 & 2 + a
\end{bmatrix}
\]

system is uncontrollable and observable.

If \( a = +1 \), the system is also unobservable (when \( k = 0 \) at the same time).

\[ q_{0.1}(a) \]

\[
\dot{x} = \begin{bmatrix}
-0.01 & 0 \\
0 & -0.02
\end{bmatrix} x + \begin{bmatrix}
1 & 1 \\
-\frac{1}{4} & \frac{3}{4}
\end{bmatrix} u, \quad u = F x
\]

Let \( F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \), then

\[
\dot{x} = (A + B F) x = \begin{bmatrix}
-0.01 + k_{11} & k_{12} \\
k_{21} & -0.02 + k_{22}
\end{bmatrix} x,
\]

where \( k_{11} = f_{11} + f_{21}, \quad k_{12} = f_{12} + f_{22}, \quad k_{21} = -\frac{1}{4} (f_{11} - 3 f_{21}), \quad k_{22} = -\frac{1}{4} (f_{12} - 3 f_{22}). \)

Using Matlab we can verify that the closed-loop eigenvalues are \(-0.1025 \pm j0.04944\).
9.3(a)

\[ x(k+1) = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} u(k). \]

Now,

\[ Q = \begin{bmatrix} B | AB | A^2B \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

Let \( Q_1 = \begin{bmatrix} b_1 | Ab_1 | Ab_2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 1 & -1 & 9 \\ -1 & -1 & -1 \end{bmatrix} \), then \( \text{rank}(Q_1) = 3 \)

\( \Rightarrow \) system is controllable through \( u_1 \).

Let \( u_1 = [f_1, f_2, f_3] \) and \( u_2 = 0 \), then the closed-loop system is described by

\[ x(k+1) = \begin{bmatrix} 1 & 4 & 0 \\ 2+f_1 & -1+f_2 & f_3 \\ -f_1 & -f_2 & 1-f_3 \end{bmatrix} x(k) - 5 - \]
\[\Sigma(A+BF) = 90, 0, 0, 0^2.\]

Now,
\[
\det (\lambda I - (A+BF)) = \det \begin{bmatrix} \lambda - 1 & -4 & 0 \\ -2 - f_1 & \lambda + f_2 & -f_3 \\ f_1 & f_2 & \lambda - 1 + f_3 \end{bmatrix}
\]
\[
= (\lambda - 1) [(2 + f_2)(\lambda + f_3) + f_2 f_3] + 4 [-(2 + f_1)(\lambda + f_3) + f_1 f_3]
\]
\[
= \lambda^3
\]
\[
= \lambda^3 + (f_3 - f_2 - 1) \lambda^2 + (f_2 - 4 f_1) \lambda + 9 - 9 f_3 + 4 f_1
\]

Equating coefficients of equal powers, yields

\[
f_3 - f_2 - 1 = 0, \quad f_2 - 4 f_1 = 0, \quad -9 f_3 + 4 f_1 + 9 = 0
\]

Solution of the 3 simultaneous equations results in

\[
F = \left[ -\frac{81}{32}, -\frac{9}{8}, -\frac{1}{8} \right]
\]