\[ A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \]  
\[ \pi_A(\lambda) = \det(\lambda I - A) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3. \]

Using Leverrier's algorithm (can also use MATLAB to compute the eigenvalues of \( A \)),

\[ N_1 = I_3, \ a_1 = -\text{tr}(A) = -2 = -2 \]

\[ N_2 = N_1 A + a_1 I_3 = A - 2 I_3 = \begin{bmatrix} 0 & -2 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix} \]

\[ a_2 = -\frac{1}{2} \text{tr}(N_2 A) = -\frac{1}{2} \text{tr} \begin{bmatrix} 1 & 7 & -5 \\ 2 & 0 & 1 \\ 2 & -8 & 9 \end{bmatrix} = -\frac{1}{2} (10) = -5 \]

\[ N_3 = N_2 A + a_2 I_3 = \begin{bmatrix} 1 & 7 & -5 \\ 2 & 0 & 1 \\ 2 & -8 & 9 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} -4 & 7 & -5 \\ 2 & -5 & 1 \\ 2 & -8 & 4 \end{bmatrix} \]

\[ a_3 = -\frac{1}{3} \text{tr}(N_3 A) = -\frac{1}{3} \text{tr} \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} = -\frac{1}{3} (-18) = 6 \]

\[ \pi_A(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6 \]

\[ \mathcal{A} = \{1, -2, 3\} = \{\lambda_1, \lambda_2, \lambda_3\} \Rightarrow \text{eigenvalues are distinct.} \]

Eigenvectors:  \((\lambda_i I - A)\vec{e}_i = \begin{bmatrix} \lambda_i - 2 & 2 & -3 \\ -1 & \lambda_i - 1 & -1 \\ -1 & -3 & \lambda_i + 1 \end{bmatrix} \begin{bmatrix} \vec{e}_{i1} \\ \vec{e}_{i2} \\ \vec{e}_{i3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

\(\lambda_1 = 1:\)

\[ \begin{bmatrix} -1 & 2 & -3 \\ -1 & 0 & -1 \\ -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} \vec{e}_{11} \\ \vec{e}_{12} \\ \vec{e}_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -e_{11} + 2e_{12} - 3e_{13} = 0 \\ -e_{11} - e_{13} = 0 \Rightarrow e_{11} = -e_{13} \\ -e_{11} - 3e_{12} + 2e_{13} = 0 \end{cases} \]

Let \( e_{13} = 1 \), then \( e_{11} = -1 \) and \( 2e_{12} = e_{11} + 3e_{13} = 2 \Rightarrow e_{12} = 1 \)

\[ \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]
\( \lambda_2 = -2 \):

\[ (\lambda_2 I - A) \xi = \begin{bmatrix} -1 & 2 & -3 \\ -1 & -3 & -1 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0 \Rightarrow -e_{21} - 3 e_{22} - e_{23} = 0 \\
-4 e_{21} + 2 e_{22} - 3 e_{23} = 0 \\
e_{21} - 3 e_{22} - e_{23} = 0 \]

Let \( e_{23} = 1 \), then \(-4 e_{21} + 2 e_{22} = 3\) \(\Rightarrow -4 e_{21} + 2 e_{22} = 3\)

\[ -e_{21} - 3 e_{22} = 1 \] \[ 4 e_{21} + 12 e_{22} = -4 \]

\[ 14 e_{22} = -1 \Rightarrow e_{22} = -\frac{1}{14} \]

and \( e_{21} = -(1 + 3 e_{22}) = -\frac{13}{14} \)

\[ \hat{\xi}_2 = \begin{bmatrix} -\frac{13}{14} \\ -\frac{1}{14} \\ 1 \end{bmatrix} \text{ or its scaled version } \hat{\xi}_2 = \begin{bmatrix} -13 \\ -1 \\ 14 \end{bmatrix}. \]

\( \lambda_3 = 3 \):

\[ (\lambda_3 I - A) \xi = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & -1 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0 \Rightarrow -e_{31} + 2 e_{32} - e_{33} = 0 \\
e_{31} + 2 e_{32} - 3 e_{33} = 0 \\
e_{31} - 3 e_{32} + 4 e_{33} = 0 \]

Let \( e_{33} = 1 \), then \( e_{31} + 2 e_{32} = 3 \)

\[ -e_{31} - 3 e_{32} = -4 \]

\[ -e_{32} = -1 \Rightarrow e_{32} = 1 \text{ and } e_{31} = 1 \]

\[ \hat{\xi}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

Let \( T = [\hat{\xi}_1 \hat{\xi}_2 \hat{\xi}_3] \), then \( \hat{A} = T^T A T = \text{diag}(1, -2, 3) \).

\[ A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \hline \hline \hline O & A_{22} \end{bmatrix}, \quad A_{11} = 2, \ A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ A_{22} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Now, \( \lambda I_y = A = \begin{bmatrix} \lambda I_1 - A_{11} \\ \hline \hline \hline \hline 0 \\ \lambda I_2 - A_{22} \end{bmatrix} \), \( \lambda_1 = 1 \)

\( \Pi_A(\lambda) = \det(\lambda I_y - A) = \det(\lambda I_1 - A_{11}) \cdot \det(\lambda I_2 - A_{22}) \)

\( = (\lambda - 2) \det(\lambda I_2 - A_{22}) \)

\( -2 - \)
\[
\det(\lambda I_3 - A_{22}) = \det \begin{bmatrix}
\lambda & 0 & -1 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix} = \lambda^3
\]

\[\therefore \ \Pi_A(\lambda) = (\lambda - 2)^3 \Rightarrow \ \mathcal{J}(A) = \{2, 0, 0, 0\} \Rightarrow \text{2 distinct eigenvalues and eigenvalue } 0 \text{ has multiplicity 3.}
\]

\underline{Eigenvectors}:

\[\lambda_1 = 2:\]
\[
(\lambda_1 I - A) e_1 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 2 & 0 & -1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix} \begin{bmatrix}
e_{11} \\
e_{12} \\
e_{13} \\
e_{14}
\end{bmatrix} = \begin{bmatrix}
-e_{11} = 0 \\
e_{12} - e_{14} = 0 \\
e_{13} = 0 \\
e_{14} = 0
\end{bmatrix}
\]

\[\therefore \text{the all-zero vector! However, an eigenvector cannot be zero. Since the first column of } \lambda_1 I - A \text{ is all zeros, the eigenvector } [1 \ 0 \ 0 \ 0]^T \text{ will satisfy } (\lambda_1 I - A) e_1 = 0. \text{ Hence, } e_1 = [1 \ 0 \ 0 \ 0]^T.
\]

\[\lambda_2 = 0: \text{ Because } \lambda_2 = 0 \text{ has multiplicity } n_2 = 3, \text{ let's compute the geometric multiplicity of } \lambda_2, \text{ i.e.}
\]
\[q_2 = n - \text{rank}(\lambda_2 I - A) = 4 - \text{rank}(-A) = 4 - \text{rank}(A) = 4 - 2 = 2,
\]

\[\text{since } A \text{ has only 2 linearly independent rows. Thus, } \exists 2 \text{ eigenvectors that satisfy } (\lambda_2 I - A) e_i = 0, \ i = 2, 3.
\]

\[\text{Now,}
(\lambda_2 I - A) e_i = -A e_i = \begin{bmatrix}
-2 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} e_i = 0, \ i = 2, 3.
\]

\[\text{For example, } e_2 = [0 \ 1 \ 0 \ 0]^T \text{ and } e_3 = [1 \ 0 \ -2 \ 0]^T \text{ will do the job.}
\]

\[\text{So, need to compute 1 generalized eigenvector, which could be associated with either } e_2 \text{ or } e_3, \text{ i.e.,}
\]
\[-A e_2^2 = e_2 \text{ or } -A e_3^2 = e_3.
\]

\[-3-\]
Clearly, there is no $e_3^2$ that satisfies $Ae_3^2 = e_3^2$, only $e_2^2$ that satisfies $-Ae_2^2 = e_2^2$, namely,

$$
-Ae_2^2 = \begin{bmatrix}
-2 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
e_{21}^2 \\
e_{22}^2 \\
e_{23}^2 \\
e_{24}^2
\end{bmatrix} = \begin{bmatrix}0 \\
0 \\
0 \\
0
\end{bmatrix} \Rightarrow e_{22}^2 = -2e_{21}^2,
$$

$$e_{24}^2 = -1.$$

Let $e_{21}^2 = 1$, then $e_{22}^2 = -2$. Thus, $e_2^2 = [1 \ 0 \ -2 \ -1]^T$.

The non-singular similarity transformation is given by

$$T = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -2 & -2 \\
0 & 0 & -1 & 0
\end{bmatrix} = [e_1^2; e_2^2; e_3^2; e_3^2].$$

Finally,

$$\hat{A} = T'AT = \begin{bmatrix}2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

8.25

$$A = \begin{bmatrix}
-1 & 2 & 0 \\
1 & 1 & 0 \\
2 & -1 & 2
\end{bmatrix} \quad \text{Want } \hat{A} \text{ using Cayley-Hamilton theorem.}$$

From C-H theorem, $A^3 + a_1A^2 + a_2A + a_3I = 0$.

Thus, $\bar{A} [A^3 + a_1A^2 + a_2A + a_3I = 0] \Rightarrow A^2 + a_1A + a_2I + a_3A^{-1}I = 0$.

If $a_3 \neq 0$, then $\hat{A}^{-1} = -\frac{1}{a_3} [a_2I + a_1A + A^2]$.

From Leverrier, $a_1 = -\text{tr}(A) = -(2) = -2$.

$$N_2 = A + a_1I = A - 2I = \begin{bmatrix}
-3 & 2 & 0 \\
1 & -1 & 0 \\
2 & -1 & 0
\end{bmatrix}$$

$$-4-$$
\[
N_2 A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 0 \\ -2 & 1 & 0 \\ -3 & 3 & 0 \end{bmatrix}
\]

\[
a_2 = -\frac{1}{2} \text{tr}(N_2 A) = -\frac{1}{2} (6) = -3
\]

\[
N_3 = N_2 A + a_2 I = \begin{bmatrix} 2 & -4 & 0 \\ -2 & -2 & 0 \\ -3 & 3 & -3 \end{bmatrix}
\]

\[
N_3 A = \begin{bmatrix} 2 & -4 & 0 \\ -2 & -2 & 0 \\ -3 & 3 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}
\]

\[
a_3 = -\frac{1}{3} \text{tr}(N_3 A) = -\frac{1}{3} (18) = 6 \neq 0
\]

\[
A' = -\frac{1}{6} \left[ -3 I - 2 A + A^2 \right]
\]

\[
= -\frac{1}{6} \left\{ \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 0 \\ -2 & -2 & 0 \\ -4 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right\}
\]

\[
= \frac{1}{6} \begin{bmatrix} -2 & 4 & 0 \\ 2 & 2 & 0 \\ 3 & -3 & 3 \end{bmatrix}
\]

\[\text{(8.31)}\]

\[
A = \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} A_11 & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad A_{11} = \frac{1}{2}, \quad A_{12} = \left[ \begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right], \quad A_{22} = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 0 \end{bmatrix}
\]

Want \( A^k \).

\[
\pi_A(\lambda) = \det (\lambda I - A) = \det (\lambda I - A_{11}) \cdot \det (\lambda I - A_{22})
\]

\[
= (\lambda - \frac{1}{2}) \cdot (\lambda - \frac{1}{2}) = (\lambda - \frac{1}{2})^3 \quad \Rightarrow \quad \sigma(A) = \\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \}
\]

\[
\lambda^3 - \frac{3}{2} \lambda^2 + \frac{3}{4} \lambda = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3
\]

\[\text{5-} \]
Now, \( A^k = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} \).

\( \begin{array}{c}
\beta(3I - A) = \beta \left[ \frac{N_1 \beta^3 + N_2 \beta + N_3}{\pi_A(\beta)} \right] = N_1 \frac{\beta^3}{\pi_A(\beta)} + N_2 \frac{\beta^2}{\pi_A(\beta)} + N_3 \frac{\beta}{\pi_A(\beta)},
\end{array} \)

where \( N_1, N_2 \) and \( N_3 \) are obtained applying the Leverrier algorithm.

\( N_1 = I_3 \)

\( N_2 = N_1 A + a_1 I_3 = A - \frac{3}{2} I_3 = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \)

\( N_3 = N_2 A + a_2 I_3 = N_2 A + \frac{3}{4} I_3 \)

\( N_2 A = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{2} \\ 0 & -\frac{1}{2} & -1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{3}{2} \\ 0 & \frac{1}{4} & -1 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \)

Now,

\( \frac{\beta^3}{\pi_A(\beta)} = \frac{\beta^3}{(\beta - \frac{1}{2})^3} = \frac{1}{(1 - \frac{1}{2} \beta)^3} \Rightarrow \beta^{k+1} \begin{pmatrix} 1 \\ \frac{1}{(1 - \frac{1}{2} \beta)^3} \end{pmatrix} = \begin{pmatrix} (k+1)(k+2) (\frac{1}{2})^k \end{pmatrix}, \quad k = 0, 1, \ldots \)

Because

\( \begin{array}{c}
\beta \begin{pmatrix} \frac{(n+1)(n+2)}{2} \end{pmatrix} u(n) = \frac{\beta^3}{(3 - \alpha)^3} \\
\beta \begin{pmatrix} \frac{\beta^2}{\pi_A(\beta)} \end{pmatrix} \end{array} \)

\( \begin{array}{c}
\beta \begin{pmatrix} \frac{\beta^2}{\pi_A(\beta)} \end{pmatrix} = \beta \begin{pmatrix} \frac{\beta^2}{(3 - \frac{1}{2} \beta)^3} \end{pmatrix} = k (k+1) (\frac{1}{2})^k, \quad k = 0, 1, \ldots \\
\begin{pmatrix} n(n+1) \end{pmatrix} u(n) \end{array} \)

Because

\( \begin{array}{c}
\beta \begin{pmatrix} \frac{n(n+1)}{2} \end{pmatrix} u(n) = \frac{\beta^3}{(3 - \alpha)^3} \\
\begin{pmatrix} \frac{n(n+1)}{2} \end{pmatrix} u(n) \end{array} \)

- 6 -
Finally,
\[
\tilde{Z} \left\{ \frac{3}{\pi \nu_0} \right\} = \tilde{Z} \left\{ \frac{8}{(3-\frac{1}{2})^3} \right\} = k(k-1)(\frac{1}{2})^{k-1}, \quad k=0,1,\ldots
\]
because \[
\tilde{Z} \left\{ \frac{n(n-1)}{2} a^{n-2} u(n) \right\} = \frac{3}{(3-a)^3},
\]
Hence,
\[
A^k = \tilde{Z} \left\{ 3(3I-A)^{-1} \right\} = N_1 \tilde{Z} \left\{ \frac{3^3}{(3-\frac{1}{2})^3} \right\} + N_2 \tilde{Z} \left\{ \frac{3^2}{(3-\frac{1}{2})^3} \right\} + N_3 \tilde{Z} \left\{ \frac{3}{(3-\frac{1}{2})^3} \right\}
\]
\[
= \begin{bmatrix}
(c_{1/2})^k & -k(c_{1/2})^{k-1} & (2-k)(c_{1/2})^{k-1} \\
0 & (c_{1/2})^k & k(c_{1/2})^{k-2} \\
0 & 0 & (c_{1/2})^k
\end{bmatrix}
\]

(8.33) \[A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \] Want \( e^{At} \).

We have 3 choices to compute \( e^{At} \), namely,
1) \( e^{At} = I + At + \frac{1}{2!} (At)^2 + \frac{1}{3!} (At)^3 + \ldots \)
2) \( e^{At} = \tilde{Z} \left\{ (3I-A)^{-1} \right\} \)
3) \( e^{At} = \beta_0(t)I + \beta_1(t)A + \beta_2(t)A^2 \), using Cayley-Hamilton theorem.

Let's use option 3). By inspection, \( \pi_A(\lambda) = \lambda - 3\lambda^2 + 3\lambda - 1 \)
\( \Rightarrow \sigma(A) = \{1, 1, 1\} \) \( \Rightarrow \) eigenvalue \( \lambda = 1 \) has multiplicity 3.

Now,\[
e^{\lambda t} = \beta_0(t)I + \beta_1(t)\lambda + \beta_2(t)\lambda^2.
\]
With \( \lambda = 1 \),
\[e^t = \beta_0(t) + \beta_1(t) + \beta_2(t) \quad (1)\]

- 7 -
To get a unique solution, we need 3 equations. Since eigenvalue \( \lambda = 1 \) has multiplicity 3, we proceed as follows:

\[
\begin{align*}
\frac{d}{d\lambda} e^{\lambda t} \bigg|_{\lambda = 1} &= t e^{\lambda t} \\
\frac{d^2}{d\lambda^2} e^{\lambda t} \bigg|_{\lambda = 1} &= t^2 e^{\lambda t} = 2 \beta_2(t) \\
\end{align*}
\]

(1)

From (3), \( \beta_2(t) = \frac{1}{2} t^2 e^t \). From (2), \( \beta_1(t) = t e^t - 2 \beta_2(t) = t e^t - t^2 e^t \). From (1), \( \beta_0(t) = e^t \beta_1(t) - \beta_2(t) = e^t - t e^t + \frac{1}{2} t^2 e^t \).

Furthermore,

\[
A = \begin{bmatrix}
0 & 0 & 1 \\
1 & -3 & 3 \\
3 & -8 & 6
\end{bmatrix}
\]

Thus,

\[
e^{At} = \beta_0(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta_1(t) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} + \beta_2(t) \begin{bmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{bmatrix}
\]

\[
= \begin{bmatrix}
\beta_0(t) \\
\beta_1(t) \\
\beta_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
\beta_0(t) - 3 \beta_2(t) \\
-3 \beta_1(t) - 8 \beta_2(t)
\end{bmatrix} + \begin{bmatrix}
\beta_0(t) \\
\beta_1(t) + 3 \beta_2(t) \\
\beta_0(t) + 3 \beta_1(t) + 6 \beta_2(t)
\end{bmatrix}
\]

(9.30)

\[
\begin{align*}
\dot{y}_1(t) + 2y_1(t) + 2y_2(t) &= u_1(t), \quad y_1(0) = 1 \\
\dot{y}_2(t) + 4y_2(t) + 3y_2(t) &= u_2(t), \quad y_2(0) = 2, \quad \frac{d}{dt} y_2(t) \bigg|_{t=0} = 0
\end{align*}
\]

Taking the Laplace transform of (2), yields

\[
S^2 Y_2(S) - SY_2(0) + 4[SY_2(S) - Y_2(0)] + 3Y_2(S) = U_2(S) = 1, \quad \mathcal{L}\{y_2(t)\} = 1.
\]

or \((S^2 + 4S + 3)Y_2(S) - 2S - 8 = 1 \Rightarrow Y_2(S) = \frac{2S + 9}{S^2 + 4S + 3} = \frac{2S + 9}{(S + 1)(S + 3)}\)

\[
\text{Re} \{s\} > -1
\]

\[
-8-
\]
Taking the Laplace transform of (1), yields

\[ sY_1(s) - y_1(0) + 2Y_1(s) + 2Y_2(s) = U_1(s) = 0 \]
\[ (s+2)Y_1(s) = y_1(0) - 2y_2(s) = 1 - e^{-\left(\frac{2s + 9}{s^2 + 4s + 3}\right)} = \frac{s^2 - 15}{s^2 + 4s + 3} \]
\[ \Rightarrow Y_1(s) = \frac{s^2 - 15}{(s+2)(s+1)(s+3)} \] \( \text{Re} \{s\} > -1 \)
\[ = \frac{-7}{s+1} + \frac{11}{s+2} - \frac{3}{s+3}, \text{ using partial fractions} \]

Thus,
\[ y_1(t) = \mathcal{L}^{-1}\left\{Y_1(s)\right\} = \left[ 7e^{-t} + 11e^{-2t} - 3e^{-3t} \right] u(t) \]

Also,
\[ Y_2(s) = \frac{7/2}{s+1} - \frac{3/2}{s+3}, \text{ Re} \{s\} > -1 \]
\[ \Rightarrow y_2(t) = \left[ \frac{7}{2}e^{-t} - \frac{3}{2}e^{-3t} \right] u(t) \]

**9.37**
\[ x_1(k+1) = x_1(k) - x_2(k) + x_3(k) \]
\[ x_2(k+1) = x_2(k) + x_3(k) \]
\[ x_3(k+1) = x_3(k) \]
\[ x_1(0) = 2, \ x_2(0) = 5, \ x_3(0) = 10. \]

Want \( X(k) \), \( X(k) = [x_1(k) \ x_2(k) \ x_3(k)]^T \).
We know \( X(0) = A^k X(0) \).

Now, \( \mathcal{Z}\{X_3(k+1) = x_3(k)\} \Rightarrow \frac{3}{3-1} X_3(\hat{s}) - \frac{3}{3-1} X_3(0) = X_3(\hat{s}) \)
or \( X_3(\hat{s}) = \frac{3}{3-1} X_3(0) = 10 \frac{3}{3-1} \Rightarrow X_3(k) = \mathcal{Z}^{-1}\left\{\frac{3}{3-1}X_3(\hat{s})\right\} = 10 u(k) \)
\( \mathcal{Z}\{x_2(k+1) = x_2(k) + x_3(k)\} \Rightarrow \frac{3}{3-1} X_2(\hat{s}) - \frac{3}{3-1} X_2(0) = X_2(\hat{s}) + X_3(\hat{s}) \)
\[ X_2(3) = x_2(0) \frac{3}{3-1} + \frac{1}{3-1} x_3(3) = 5 \frac{3}{3-1} + \frac{10}{(3-1)^2} \]

\[ \Rightarrow x_2(k) = 2^{-1} \{ x_2(3) \}^2 = 5 u(k) + 10 k u(k) \]

\[ Z^2 x_1(k+1) = x_1(k) - x_2(k) + x_3(k) \]

\[ \Rightarrow 3 X_1(3) - 3 X_1(0) = X_1(3) - X_2(3) + X_3(3) \]

or

\[ X_1(3) = x_1(0) \frac{3}{3-1} - \left[ 5 \frac{3}{(3-1)^2} + 10 \frac{3}{(3-1)^3} \right] + 10 \frac{3}{(3-1)^2} \]

\[ = 2 \frac{3}{3-1} + 5 \frac{3}{(3-1)^2} - 10 \frac{3}{(3-1)^3} \]

\[ \Rightarrow x_1(k) = 2^{-1} \{ X_1(3) \}^2 = 2 u(k) + 5 k u(k) - 10 \frac{k(k-1)}{2} u(k) \]

\[ = \left[ 2 + 10 k - 5 k^2 \right] u(k) \]