Computational Framework of Perceptual Grouping
Perceptual Grouping: Image Segmentation

- Find contiguous clusters of pixels, so pixels within each cluster are
  - closer
  - more similar in terms of
    - color
    - texture
  - Than pixels from different clusters
Example: N-Cuts Segmentation

results of the Ncuts algorithm

regions or segments
Graph-based Clustering
Graph-based Clustering

- Image elements are represented by a graph
Graph-based Clustering

- Image elements are represented by a graph
- Clustering = Graph partitioning into subgraphs
Major Goal: Graph Partitioning
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Find a cut, such that in one cluster all pixels are:

- neighbors
- more similar to one another than to pixels in the other cluster
Graph Terminology

- Clique
- Adjacency matrix
- Node degree
- Volume
- Cut
- Association
Clique

• A subgraph with all nodes fully connected
• Maximal clique
• Maximum weighted clique
Adjacency Matrix

\[ A = \begin{bmatrix}
  & j \\
  i & \ldots & a_{ij} \\
  & \vdots \\
  & \ddots \\
\end{bmatrix} \]

\[ a_{ij} = \exp \left( -\frac{1}{2\sigma^2} \| \text{SIFT}_i - \text{SIFT}_j \|_2^2 \right) \]
Adjacency Matrix

\[ A = \begin{bmatrix}
  & \cdots & \\
  & a_{ij} & \\
  \vdots & & \ddots & \ddots & \ddots \\
  \end{bmatrix} \]

- Affinities between pairs of nodes (i,j) in the graph

\[ a_{ij} = \exp \left( -\frac{1}{2\sigma^2} \| \text{SIFT}_i - \text{SIFT}_j \|^2 \right) \]
Adjacency Matrix

\[ A = \begin{bmatrix}
  a_{ij} \\
  \vdots \\
  a_{ij}
\end{bmatrix} \]

• Affinities between pairs of nodes \((i,j)\) in the graph

• Example: Nodes = Pixels \(\Rightarrow\)

\[ a_{ij} = \exp\left(-\frac{1}{2\sigma^2}\| \text{SIFT}_i - \text{SIFT}_j \|^2\right) \]
Node Degree

\[ d_i = \sum_j a_{ij} \]

\[ d_i = 6 \]
Node Degree

\[ D = \begin{bmatrix}
    d_1 & 0 & 0 & \ldots \\
    0 & d_2 & 0 \\
    \vdots \\
    \ldots & 0 & 0 & d_N \\
\end{bmatrix} \]

\[ 1_i = \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    1 \\
    \vdots \\
    0 \\
\end{bmatrix} \leftarrow i \]

\[\Rightarrow \quad d_i = 1_i^T D 1_i\]
Volume of a Subgraph

\[ \text{vol}(G_1) = \sum_{i \in G_1} d_i \]
Volume of a Subgraph

\[ \text{vol}(G_1) = 1^T_{G_1} \begin{bmatrix} d_1 & 0 & 0 & \ldots \\ 0 & d_2 & 0 \\ \vdots \\ \ldots & 0 & 0 & d_N \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

\[ \Rightarrow \text{vol}(G_1) = 1^T_{G_1} D 1_{G_1} \]
Association within a Subgraph

\[ \text{assoc}(G_1) = \sum_{i,j \in G_1} a_{ij} \]
Association within a Subgraph

adjacency matrix

\[ A = \begin{bmatrix}
    & \cdots & a_{ij} \\
    i & \cdots
\end{bmatrix} \]

\[ \text{assoc}(G_1) = \sum_{i, j \in G_1} a_{ij} = \mathbf{1}_{G_1}^T A \mathbf{1}_{G_1} \]
Cut between Two Graph Partitions

\[ G_2 \subset G \quad G_1 \subset G \]

\[ G_1 \cap G_2 = \emptyset \]

\[ \text{cut}(G_1, G_2) = \sum_{i \in G_1, j \in G_2} a_{i,j} \]
Cut between Two Graph Partitions

$G_2 \subset G$

$G_1 \subset G$

$G_1 \cap G_2 = \emptyset$

\[
\text{cut}(G_2, G_1) = \sum_{i \in G_2, j \in G_1} a_{ij}
\]
Cut between Two Graph Partitions

\[ G_2 \subset G \quad \text{and} \quad G_1 \subset G \]

\[ G_1 \cap G_2 = \emptyset \]

\[ \text{cut}(G_1, G_2) \neq \text{cut}(G_2, G_1) \]

in general
Cut between Two Graph Partitions

\[
cut(G_1, G_2) = \text{vol}(G_1) - \text{assoc}(G_1)
\]

\[
= \mathbf{1}_{G_1}^T D \mathbf{1}_{G_1} - \mathbf{1}_{G_1}^T A \mathbf{1}_{G_1}
\]

\[
= \mathbf{1}_{G_1}^T (D - A) \mathbf{1}_{G_1}
\]
Cut between Two Graph Partitions

\[
\text{cut}(G_2, G_1) = \mathbf{1}_{G_2}^T (D - A) \mathbf{1}_{G_2}
\]

\[
= (\mathbf{1} - \mathbf{1}_{G_1})^T (D - A) (\mathbf{1} - \mathbf{1}_{G_1})
\]