Wavelets and Multiresolution Processing

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Multiresolution Analysis (MRA)

- A scaling function is used to create a series of approximations of a function or image, each differing by a factor of 2 from its neighboring approximations.

- Additional functions called wavelets are then used to encode the difference in information between adjacent approximations.
Series Expansions

Express a signal $f(x)$ as

$$f(x) = \sum_{k} \alpha_k \varphi_k(x)$$

If the expansion is unique, the $\varphi_k(x)$ are called basis functions, and the expansion set $\{\varphi_k(x)\}$ is called a basis.
Series Expansions

- All the functions expressible with this basis form a **function space** which is referred to as the **closed span** of the expansion set

\[
V = \text{Span}\{\varphi_k(x)\}
\]

- If \( f(x) \in V \), then \( f(x) \) is in the closed span of \( \{\varphi_k(x)\} \) and can be expressed as

\[
f(x) = \sum_{k} \alpha_k \varphi_k(x)
\]
The expansion functions form an orthonormal basis for $V$

$$\left\langle \varphi_j(x), \varphi_k(x) \right\rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

The basis and its dual are equivalent, i.e.,

$$\varphi_k(x) = \tilde{\varphi}_k(x)$$

and

$$\alpha_k = \left\langle \varphi_k(x), f(x) \right\rangle = \int \varphi_k^*(x) f(x) \, dx$$
Scaling Functions

- Consider the set of expansion functions composed of integer translations and binary scalings of the real square-integrable function $\varphi(x)$ defined by

$$\left\{ \varphi_{j,k}(x) \right\} = \left\{ 2^{j/2} \varphi(2^j x - k) \right\}$$

for all $j, k \in \mathbb{Z}$ and $\varphi(x) \in L^2(\mathbb{R})$

- By choosing the scaling function $\varphi(x)$ wisely, $\left\{ \varphi_{j,k}(x) \right\}$ can be made to span $L^2(\mathbb{R})$
\[ \left\{ \varphi_{j,k}(x) \right\} = \left\{ 2^{j/2} \varphi(2^j x - k) \right\} \]

- Index \( k \) determines the position of \( \varphi_{j,k}(x) \) along the x-axis, index \( j \) determines its width; \( 2^{j/2} \) controls its height or amplitude.

- By restricting \( j \) to a specific value \( j = j_o \) the resulting expansion set \( \left\{ \varphi_{j_o,k}(x) \right\} \) is a subset of \( \left\{ \varphi_{j,k}(x) \right\} \)

- One can write \( V_{j_o} = \text{Span}_{k} \left\{ \varphi_{j_o,k}(x) \right\} \)
Example: The Haar Scaling Function

\[ \varphi(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & \text{otherwise} 
\end{cases} \]

\[ f(x) = 0.5\varphi_{1,0}(x) + \varphi_{1,1}(x) - 0.25\varphi_{1,4}(x) \]

\[ \varphi_{0,k}(x) = \frac{1}{\sqrt{2}} \varphi_{1,2k}(x) + \frac{1}{\sqrt{2}} \varphi_{1,2k+1}(x) \]

\[ V_0 \subset V_1 \]
MRA Requirements

1. *The scaling function is orthogonal to its integer translates*

2. *The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales:*

\[
V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty
\]
Wavelet Functions

- Given a scaling function which satisfies the MRA requirements, one can define a wavelet function $\psi(x)$ which, together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces $V_j$ and $V_{j+1}$.
Wavelet Functions

- Define the wavelet set

\[ \{ \psi_{j,k}(x) \} = \left\{ 2^{j/2} \psi(2^j x - k) \right\} \]

for all \( k \in \mathbb{Z} \) that spans the \( W_j \) spaces

- We write

\[ W_j = \text{Span} \left\{ \psi_{j,k}(x) \right\} \]

and, if

\[ f(x) \in W_j \]

\[ f(x) = \sum_{k} \alpha_k \psi_{j,k}(x) \]
Orthogonality: 

$$V_{j+1} = V_j \oplus W_j$$

- This implies that
  $$\langle \varphi_{j,k}(x), \psi_{j,l}(x) \rangle = 0$$
  for all appropriate $j, k, l \in \mathbb{N}$

- We can write
  $$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$$
  and also
  $$L^2(\mathbb{R}) = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$
  (no need for scaling functions, only wavelets!)
Example: Haar Wavelet Functions in $W_0$ and $W_1$

\[ \psi(x) = \psi_{0,0}(x) \]
\[ \psi_{0,2}(x) = \psi(x - 2) \]
\[ \psi_{1,0}(x) = \sqrt{2} \psi(2x) \]
\[ f(x) \in V_1 = V_0 \oplus W_0 \]
\[ f_a(x) \in V_0 \]
\[ f_d(x) \in W_0 \]

\[ f(x) = f_a(x) + f_d(x) \]

**FIGURE 7.12** Haar wavelet functions in $W_0$ and $W_1$.  

low frequencies  

high frequencies
Wavelet Series Expansions

- A function $f(x) \in L^2(\mathbb{R})$ can be expressed as

$$f(x) = \sum_{j=0}^{\infty} \sum_{k} c_{j_0}(k) \varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_{k} d_{j}(k) \psi_{j,k}(x)$$

approximation or scaling coefficients

c_{j_0}(k) = \left\langle f(x), \varphi_{j_0,k}(x) \right\rangle$

d_{j}(k) = \left\langle f(x), \psi_{j,k}(x) \right\rangle$

detail or wavelet coefficients

$L^2(\mathbb{R}) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \cdots$
Example: The Haar Wavelet Series
Expansion of $y=x^2$

- Consider $y = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$

- If $j_0 = 0$, the expansion coefficients are

  $$c_0(0) = \int_0^1 x^2 \varphi_{0,0}(x) \, dx = \frac{1}{3} \quad d_0(0) = \int_0^1 x^2 \psi_{0,0}(x) \, dx = -\frac{1}{4}$$

  $$d_1(0) = \int_0^1 x^2 \psi_{1,0}(x) \, dx = -\frac{\sqrt{2}}{32} \quad d_1(1) = \int_0^1 x^2 \psi_{1,1}(x) \, dx = -\frac{3\sqrt{2}}{32}$$

  $$y = \frac{1}{3} \varphi_{0,0}(x) + \left[ \frac{1}{4} \psi_{0,0}(x) \right] + \left[ -\frac{\sqrt{2}}{32} \psi_{1,0}(x) - \frac{3\sqrt{2}}{32} \psi_{1,1}(x) \right] + \cdots$$
Example: The Haar Wavelet Series
Expansion of $y = x^2$
The Discrete Wavelet Transform (DWT)

- Let \( f(x) \), \( x = 0,1,\ldots,M - 1 \) denote a discrete function

- Its DWT is defined as

  \[
  W_\varphi(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \varphi_{j_0,k}(x)
  \]

  \[
  W_\psi(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \psi_{j,k}(x) \quad j \geq j_0
  \]

- \( f(x) = \frac{1}{\sqrt{M}} \sum_k W_\varphi(j_0, k) \varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j, k) \psi_{j,k}(x) \)

- Let \( j_0 = 0 \) and \( M = 2^J \) so that \( \begin{cases} x = 0,1,\ldots,M - 1, \\ j = 0,1,\ldots,J - 1, \\ k = 0,1,\ldots,2^j - 1 \end{cases} \)
Example: Computing the DWT

Consider the discrete function

\[ f(0) = 1, \ f(1) = 4, \ f(2) = -3, \ f(3) = 0 \]

It is \( M = 4 = 2^2 \rightarrow J = 2 \)

The summations are performed over

\[ x = 0,1,2,3 \quad \text{and} \quad k = 0 \text{ for } j = 0 \text{ and} \]
\[ k = 0,1 \text{ for } j = 1 \]

Use the Haar scaling and wavelet functions
Example: Computing the DWT

\[
W_\varphi(0,0) = \frac{1}{2} \sum_{x=0}^{3} f(x) \varphi_{0,0}(x) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 3 \cdot 1 + 0 \cdot 1 \right] = 1
\]

\[
W_\psi(0,0) = \frac{1}{2} \sum_{x=0}^{3} f(x) \psi_{0,0}(x) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 3 \cdot (-1) + 0 \cdot (-1) \right] = 4
\]

\[
W_\psi(1,0) = \frac{1}{2} \sum_{x=0}^{3} f(x) \psi_{1,0}(x) = \frac{1}{2} \left[ 1 \cdot \sqrt{2} + 4 \cdot (-\sqrt{2}) - 3 \cdot 0 + 0 \cdot 0 \right] = -1.5\sqrt{2}
\]

\[
W_\psi(1,1) = \frac{1}{2} \sum_{x=0}^{3} f(x) \psi_{1,1}(x) = \frac{1}{2} \left[ 1 \cdot 0 + 4 \cdot 0 - 3 \cdot \sqrt{2} + 0 \cdot (-\sqrt{2}) \right] = -1.5\sqrt{2}
\]
Example: Computing the DWT

- The DWT of the 4-sample function relative to the Haar wavelet and scaling functions thus is $\left\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\right\}$

- The original function can be reconstructed as

$$f(x) = \frac{1}{2} \left[ W_\phi (0, 0) \phi_{0,0}(x) + W_\psi (0, 0) \psi_{0,0}(x) + W_\psi (1, 0) \psi_{1,0}(x) + W_\psi (1, 1) \psi_{1,1}(x) \right]$$

for $x = 0, 1, 2, 3$
In 2-D, one needs one scaling function

\[ \varphi(x, y) = \varphi(x)\varphi(y) \]

and three wavelets

\[
\begin{align*}
\psi^H(x, y) &= \psi(x)\varphi(y) & \text{• detects horizontal details} \\
\psi^V(x, y) &= \varphi(x)\psi(y) & \text{• detects vertical details} \\
\psi^D(x, y) &= \psi(x)\psi(y) & \text{• detects diagonal details}
\end{align*}
\]

\( \varphi(.) \) is a 1-D scaling function and

is its corresponding wavelet \( \psi(.) \)
2-D DWT: Definition

- Define the scaled and translated basis functions

\[ \varphi_{j,m,n}(x, y) = 2^{j/2} \varphi(2^j x - m, 2^j y - n) \]
\[ \psi^i_{j,m,n}(x, y) = 2^{j/2} \psi^i(2^j x - m, 2^j y - n), \quad i = \{H, V, D\} \]

- Then

\[ W_{\varphi}(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \varphi_{j_0,m,n}(x, y) \]
\[ W_{\psi}^i(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi^i_{j,m,n}(x, y), \quad i = \{H, V, D\} \]

\[ f(x, y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n W_{\varphi}(j_0, m, n) \varphi_{j_0,m,n}(x, y) \]
\[ + \frac{1}{\sqrt{MN}} \sum_{i=H, V, D} \sum_{j=j_0}^{\infty} \sum_m \sum_n W_{\psi}^i(j, m, n) \psi^i_{j,m,n}(x, y) \]
Filter bank implementation of 2-D wavelet analysis FB:

\[ f(x, y) \]

\[ W_\psi(j + 1, m, n) \]

Resulting decomposition:

\[ (L^2, H^2) \]

Synthesis FB:

\[ W_\psi(j + 1, m, n) \]
Example: A Three-Scale FWT

FIGURE 7.23 A three-scale FWT.
Analysis and Synthesis Filters

**FIGURE 7.24**
Fourth-order symlets:
(a)–(b) decomposition filters;
(c)–(d) reconstruction filters;
(e) the one-dimensional wavelet; (f) the one-dimensional scaling function; and (g) one of three two-dimensional wavelets, \( \psi^H(x, y) \).
Want to Learn More About Wavelets?

- “An Introduction to Wavelets,” by Amara Graps
- Amara’s Wavelet Page (with many links to other resources) [http://www.amara.com/current/wavelet.html](http://www.amara.com/current/wavelet.html)
- Gilbert Strang’s tutorial papers from his MIT webpage [http://www-math.mit.edu/~gs/](http://www-math.mit.edu/~gs/)