Dijkstra's algorithm: Correctness by induction

We prove that Dijkstra's algorithm (given below for reference) is correct by induction. In the following, G is the input graph, s is the source vertex, $\ell(uv)$ is the length of an edge from u to v, and V is the set of vertices.

Dijkstra(G, s)
  for all $u \in V \setminus \{s\}$, $d(u) = \infty$
  $d(s) = 0$
  $R = \{\}$
  while $R \neq V$
    pick $u \notin R$ with smallest $d(u)$
    $R = R \cup \{u\}$
    for all vertices v adjacent to u
      if $d(v) > d(u) + \ell(u, v)$
        $d(v) = d(u) + \ell(u, v)$

Let $d(v)$ be the label found by the algorithm and let $\delta(v)$ be the shortest path distance from s-to-v. We want to show that $d(v) = \delta(v)$ for every vertex v at the end of the algorithm, showing that the algorithm correctly computes the distances. We prove this by induction on $|R|$ via the following lemma:

Lemma: For each $x \in R$, $d(x) = \delta(x)$.

Proof by Induction: *Base case ($|R| = 1$):* Since R only grows in size, the only time $|R| = 1$ is when $R = \{s\}$ and $d(s) = 0 = \delta(s)$, which is correct.

Inductive hypothesis: Let u be the last vertex added to R. Let $R' = R \cup \{u\}$. Our I.H. is: for each $x \in R'$, $d(x) = \delta(x)$.

Using the I.H.: By the inductive hypothesis, for every vertex in R' that isn't u, we have the correct distance label. We need only show that $d(u) = \delta(u)$ to complete the proof.

Suppose for a contradiction that the shortest path from s-to-u is Q and has length

$$\ell(Q) < d(u).$$

Q starts in R' and at some leaves R' (to get to u which is not in R'). Let xy be the first edge along Q that leaves R'. Let Q_x be the s-to-x subpath of Q. Clearly:

$$\ell(Q_x) + \ell(xy) \leq \ell(Q).$$

Since $d(x)$ is the length of the shortest s-to-x path by the I.H., $d(x) \leq \ell(Q_x)$, giving us

$$d(x) + \ell(xy) \leq \ell(Q_x).$$

Since y is adjacent to x, $d(y)$ must have been updated by the algorithm, so

$$d(y) \leq d(x) + \ell(xy).$$

Finally, since u was picked by the algorithm, u must have the smallest distance label:

$$d(u) \leq d(y).$$

Combining these inequalities in reverse order gives us the contradiction that $d(x) < d(x)$. Therefore, no such shorter path Q must exist and so $d(u) = \delta(u)$. □

This lemma shows the algorithm is correct by "applying" the lemma for $R = V$.