Review: Recent Development in the Weak Noise Theory for the KPZ Equation by Li-Cheng Tsai

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Summary In this essay, we review a presentation given by Professor Li-Cheng Tsai (University of Utah) detailing his recent work on the Weak Noise Theory for the KPZ Equation on November 4, 2022 at the Pacific Northwest Integrable Probability Conference at Oregon State University, Corvallis, OR. We begin with an overview of the KPZ Equation, the Large Deviation Principle, the Most Probable Shape, and the Narrow Wedge Initial Condition. Finally, we detail the main results of Professor Tsai's most recent work in the field and a brief overview of related, open problems.

Background, Motivation, & Definitions

KPZ Equation

The Kadar-Parisi-Zhang (KPZ) Equation is a stochastic PDE often used to describe the stochastic evolution (random growth) of an interface[1]. The equation takes the form

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \xi \tag{1}$$

where h is the space-time function representing the interface at time t and spatial coordinate x, while $\xi(x,t)$ is a random forcing representing space-time white noise.

[1] Examples of random growth of interfaces include the growth frontier of bacterial colonies in a petri dish and ballistic deposition.

The term $\frac{1}{2}\partial_{xx}h$ represents a smoothing factor, while $\frac{1}{2}(\partial_x h)^2$ is the slope dependent growth velocity.

Freidlin-Wentzell LDP

The Freidlin-Wentzell Large Deviation Property is a result in Large Deviation Theory, the subfield of probability theory which focuses on the asymptotic behavior of probability distribution tails. With regard to the KPZ Equation, this means introducing a small parameter ϵ to the equation and observing the atypical behavior as $\epsilon \to 0$. This gives the form[2]

$$\partial_t h_\epsilon = \tfrac{1}{2}\partial_{xx} h_\epsilon + \tfrac{1}{2}(\partial_x h_\epsilon)^2 + \sqrt{\epsilon}\xi \tag{2}$$

[2] Equation 2 with $(t,x) \in [0,2] \times \mathbb{R}$ can be transformed by change of variable $h(t,x) = h_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x)$ to the same form as Equation 1 but with $(x,t) \in [0, 2\epsilon^2] \times \mathbb{R}$

It has been shown that, with high probability, h_ϵ approximates h_0. However, there is a small probability that $h_\epsilon(2,0) \approx \lambda \neq h_0(2,0)$. This characterizes the atypical asymptotic behavior of the KPZ Equation. This LDP can be further broken down into two subcategories: the One-Point LDP, given in Equation 3, and the Process Level LDP, given in Equation 4.

$$\mathbb{P}[h_\epsilon(2,0) \approx \lambda] \approx \exp\big(-\epsilon^{-1} I_{(2,0)}(\lambda)\big) \tag{3}$$

This form is a short hand for the notation:

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \epsilon\mathbb{P}[|h_\epsilon(2,0) - \lambda| \le \delta] = -I_{(2,0)}(\lambda)$$

$$\mathbb{P}[h_\epsilon \approx g] \approx \exp\big(-\epsilon^{-1} I(g)\big) \tag{4}$$

Here I is the "rate function" and g is a generic function. ϵ^{-1} can be seen as the "speed of deviation".

Most Probable Shape

The Most Probable Shape is the function encoding the development of the interface h after an atypical event. To borrow an analogy from Professor Tsai's presentation: the Most Probable Shape is not about finding the probabality of winning the lottery, but about finding what your life is likely to be after winning the lottery. The Most Probable Shape is given by

$$h = \arg\min_{g}\{I(g) : g(2,0) = \lambda\}, \tag{5}$$

conditioned on $\{h_\epsilon(2,0) \approx \lambda > h_0(2,0)\}$. Finding the Most Probable Shape is the main concern of Weak Noise Theory (WNT) for the KZP Equation. In particular, using the Process Level I, WNT seeks to find the Most Probable Shape, as well as the One-Point I. This is accomplished by first fixing the initial condition $h_\epsilon(\cdot,0) = f_{ic}$ and conditioning on $\{h_\epsilon(2,0) \approx \lambda\}$. Then the Most Probable Shape is found by minimizing

$$\inf\{I(g) : g(0,\cdot) = f_{ic},\ g(2,0) = \lambda)\} \tag{6}$$

$$= \inf\{\tfrac{1}{2}\|\rho\|_2^2 : H[\rho](0,\cdot) = f_{ic},\ H[\rho](2,0) = \lambda)\} \tag{7}$$

In the general case, we defined ρ to be the generic function that satisfies

$$\mathbb{P}[\sqrt{\epsilon}\xi \approx \rho] \approx \exp\left(-\epsilon^{-1}\|\rho\|_2^2\right)$$

then

$$I(g) = \inf\{\tfrac{1}{2}\|\rho\|_2^2 : H[\rho] = g\}$$

and

$$\partial_t H = \tfrac{1}{2}\partial_{xx}H + \tfrac{1}{2}(\partial_x H)^2 + \rho,$$

$$H = H(\rho) = H(\rho)[t,x].$$

This objective function leads to the obvious questions of existence and uniqueness of a solution. This is addressed by Propositions 1 and 2, respectively.

Proposition 1. *The minimizer* $(H[\rho_m],\rho_m])$ *exists.*

Proposition 2. *For some initial conditions*[3]*, the minimizer is not unique*[4].

[3] Specifically, Professor Tsai mentions Brownian inital conditions here.

[4] This non-uniqueness is shown in the physics literature by demonstrating that symmetry breaking occurs. The details of symmetry breaking are outside the scope of this review.

Narrow Wedge Initial Condition

One particular initial condition of interest for the KPZ Equation is the Narrow Wedge Initial Condition. Under this condition, we take $Z_\epsilon(\cdot,0) = \delta_0$ and let $h_\epsilon = \log Z_\epsilon$. This gives

$$h_\epsilon(t,x) = -\frac{x}{2t}\sqrt{2\pi t}$$

as $t \to 0$. The main results of Professor Tsai's presentation relate to the Large Deviation Properties of the KPZ Eqauation under the Narrow Wedge Initial Condition.

Under this formulation, the original KPZ Equation can be linearized using the Hopf-Cole Transformation shown below

$$\partial_t Z_\epsilon = \tfrac{1}{2}\partial_{xx}Z_\epsilon + \sqrt{\epsilon}\xi Z_\epsilon.$$

This is known as the Stochastic Heat Equation (SHE).

Main Results

$\lambda \to -\infty$ limits of the Most Probable Shape

Let h_ϵ start at the Narrow Wedge Initial Condition. Then Proposition 3 holds.

Proposition 3. $\forall\lambda \in (-\infty,\lambda_0]$*, the minimizer* $(h,w) = (H[\rho_m],\rho_m)$ *exists.*

where h is the Most Probable Shape. Futhermore, consider the scaled Most Probable Shape h_λ. Then, as proved by Professor Tsai, Theorem 1 holds.

$h_\lambda := \frac{1}{|\lambda|} h(t, \frac{x}{|\lambda|^{1/2}})$

Theorem 1. *$h_\lambda \to h_*^-$ as $\lambda \to -\infty$ uniformly over compact subsets of* $(0, 2] \times \mathbb{R}$.

Importantly, h_*^- has an explicit form which was previously predicted in various works in physics. This form was derived by forming an Euler-Langrian problem from the Hamilton equations of Narrow Wedge Condition. While the details of this process are outside the scope of this review, it is important to note that the value found by solving the Euler-Lagrangian, w_λ can be considered as the potential energy. With this, they form an "energy-entropy competition" equation of the form[5]

[5] γ can be interpreted as the Most Probable Shape of the path from (t, x) backward to $(0, 0)$.

$$h_*(t,x) = -\inf\left\{ \int_0^t \underbrace{\frac{1}{2}(\gamma')(t-s)}_{entropy} + \underbrace{(w_\lambda)(s,\gamma(t-s))}_{potential} ds \right\}$$

Proposition 4. $\forall (t,x) \in (0,2) \times \mathbb{R}$, *the infimum $h_*(t,x)$ is acheived by some γ_*, which we call a geodesic.*

$\lambda \to \infty$ limits of the Most Probable Shape

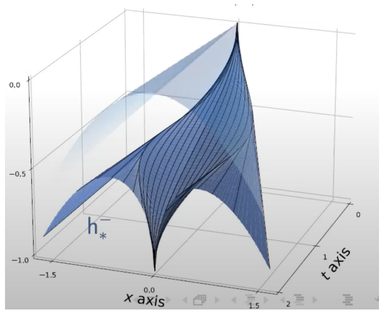

Figure 1: Geodesic of the Most Probable Shape of the KPZ Equation given the Narrow Wedge Initial Condition as $\lambda \to -\infty$. (Screen grab from Professor Tsai's presentation)

In addition to the asymptotic behavior of the KPZ Equation under the Narrow Wedge Initial Condition as $\lambda \to -\infty$, Professor Tsai also addresses the asymptotic behavior as $\lambda \to \infty$. Similar to Theorem 1, Professor Tsai finds that, for the Narrow Wedge Initial Condition, Theorem holds.

Theorem 2. *$h_\lambda \to h_*^+$ as $\lambda \to \infty$ on compact subsets of* $(0, 2] \times \mathbb{R}$ *uniformly over minimizers.*

Importantly though, there are differences between the $-\infty$ and $+\infty$ cases. The main difference is related to the potential energy w_λ:

$$\lambda \to -\infty \Rightarrow w_\lambda \to w_* \tag{8}$$

$$\lambda \to +\infty \Rightarrow w_\lambda \to 0. \tag{9}$$

This implies that for the $-\infty$ case, w_λ spans the entire region of space-time, while in the case of $+\infty$, w_λ concentrates around $[0, 2] \times \{x = 0\}$.75cm[0]

[0] Expect in the case $x = 0$.

Open Problems

Professor Tsai concluded his presentation with three main open problems in the field of WNT for the KPZ Equation:

1. Removing the uniqueness assumption
2. Extracting the $\lambda \to -\infty$ limit from the solution formula
3. Extension to other initial conditions

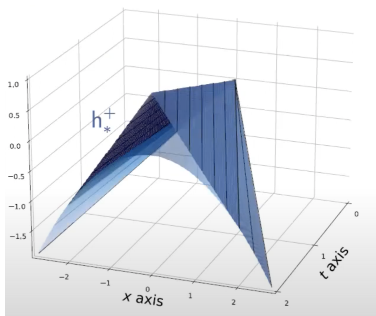

Figure 2: Geodesic of the Most Probable Shape of the KPZ Equation given the Narrow Wedge Initial Condition as $\lambda \to +\infty$. (Screen grab from Professor Tsai's presentation)

References

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